Duality Theory of Weak Interaction

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Abstract. The main objective of this article is to study the duality of weak interaction, and to derive the long overdue weak potential and force formula. The new field equations for weak interaction is derived by applying the principle of interaction dynamics (PID) and the principle of representation invariance (PRI) to a standard $SU(2)$ gauge action coupled with the section for the Dirac spinors for fermions. The new field equations establish a natural duality between weak gauge fields $\{W_\mu^a\}$, representing the $W^\pm$ and Z intermediate vector bosons, and three bosonic scalar fields ϕ^a. An important $SU(2)$ constant vector $\{\alpha_a^w\}$, representing the portions distributed to the gauge potentials W_μ^a by the weak charge, is then induced naturally by the representation invariance of the new model (PRI). With this constant distribution vector, we are able to formulate the total force exerted on a particle with N weak charges, and to derive, for the first time, the weak force formula based on the field equations.

Contents

1.	Introduction	2
2.	Principle of Representation Invariance (PRI)	4
3.	Duality Theory of Weak Interactions	5
3.1.	Field equations for weak interactions	5
3.2.	Two types of duality	6
3.3.	Non-coexistence of charged and neutral particles	6
3.4.	Scaling relation	8
4.	Weak Interaction Potentials	11
4.1.	Weak interaction potentials	11
4.2.	Dual field potential	12
4.3.	Weak decay conditions	14
4.4.	Weak interaction potential	16
5.	Consistency with GWS Electroweak Theory	18
5.1.	GWS action	18
5.2.	Weinberg-Salam electroweak theory	20
5.3.	Electroweak theory obeying PRI	23
References		26

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1. Introduction

Weak force/interaction is one of the four fundamental forces/interactions in nature, and the other three are gravitational force, electromagnetic force and strong force. Both gravitational force and electromagnetic force are long-range forces. The gravitational force is described first by the Newton inversely proportional to distance force formula, and then by Einstein's general theory of relativity in the cosmic scale, and by recently modified gravitational field equations taking in consideration of the presence of dark matter and dark energy. Electromagnetic force obeys the inverse square law as well.

Both strong and weak forces are very short range forces. The Yukawa potential gave a partial information of strong force, as the corresponding Yukawa force is always attractive. As part of a research program to establish a unified field theory for interactions, we derive in [1] a more complete description of strong force consisting three levels of strong interaction potentials: the quark potential S_q, the nucleon/hadron potential S_n and the atom/molecule potential S_a. Among many other features, these potentials offer a clear mechanism for both quark confinement and asymptotic freedom. It is worth mentioning that the classical Yukawa potential is phenomenological and cannot be derived from quantum chromodynamics (QCD), while the new potentials are derived directly from the new field equations based on QCD, PRI and PID.

Unfortunately, there is simply no such force/potential formula for the weak interaction. The main objective of this article is to study the duality of weak interaction, and to derive such weak potential and force formula. Here are the main ingredients of the paper.

First, one key point of the study is the new field equations for weak interaction decoupled from the unified field model developed by the authors in [2, 3]. The model is derived by applying the principle of interaction dynamics (PID) and principle of representation invariance (PRI) to a standard $SU(2)$ gauge action functional. Intuitively, PID amounts to taking the variation of the action functional under energy-momentum conservation constraint. PRI requires that physical laws be independent of representations of the gauge groups.

The new field equations establish a natural duality between weak gauge fields $\{W_\mu^a\}$, representing the $W^\pm$ and Z intermediate vector bosons, and three bosonic scalar fields ϕ^a. A possible duality is the degenerate case where the three scalar fields ϕ^a are a constant vector ζ_a times a single scalar field ϕ, and the duality reduces to the duality between $\{W_\mu^a\}$ and one neutral Higgs boson field ϕ.

Second, another key point of the study is that the field equations must satisfy PRI, which induces an important $SU(2)$ constant vector $\{\alpha_a^w\}$. The components of this vector represent the portions distributed to the gauge potentials W_μ^a by the weak charge g_w. Consequently, in the same spirit as electromagnetism, the time-components W_0^a of the gauge potentials represent the weak-charge potentials, and the total force exerted on a particle with N weak charges Ng_w is

$$F_{WE} = -Ng_w\alpha_a^w\nabla W_0^a. \tag{1.1}$$

It is the weak charge distribution vector α_a^w, due to PRI, that allows us to formulate the total weak potential/force as a field exerted on a particle. It is clear that F_{WE}

is a representation invariant scalar, obeying PRI. This clearly overcomes one of the main difficulties encountered in classical theories.

In the same token, the spatial components $\vec{W}^a = (W_1^a, W_2^a, W_3^a)$ represent the weak-rotation potentials, yielding the following total weak-rotation force

$$F_{WM} = g_w \varepsilon^{abc} \alpha_a^w \vec{J}^b \times \text{ curl}\vec{W}^c, \tag{1.2}$$

where $\{\vec{J}^b\} = \{J_1^b, J_2^b, J_3^b\}$ is the weak charge current density, and ε^{abc} is the structural constants using the Pauli matrices as generators for $SU(2)$. Also, F_{WM} is a representation invariant scalar, obeying PRI.

Third, with the above physical meaning of the gauge potentials and the associated forces, for the first time, we derive the weak potential and weak force formula given by

$$\begin{aligned} W &= g_w e^{-k_1 r}\left[\frac{1}{r} - e^{-k_0 r}\psi(r)\right], \\ F &= g_w^2 e^{-k_1 r}\left[\frac{k_1}{r} + \frac{1}{r^2} - (K_1\psi - \psi')e^{-k_0 r}\right], \end{aligned} \tag{1.3}$$

where $K_1 = k_0 + k_1$, $k_0 = m_H c/\hbar$, $k_1 = m_W c/\hbar$, m_H and m_W are the masses of the Higgs and W bosons, and $\psi(r) = \psi_1(r) + \psi_2(r)\ln r$ with $\psi_i(r)$ being polynomials; see (4.46). This force formula is consistent with observations: there is a radius $r_0 > 0$ such that F is repelling for $r < r_0$, and attractive for $r_0 < r < r_1$. In addition, F is a short-range force. Namely, F diminishes for $r \geq 10^{-16} cm$.

Fourth, with the duality, our analysis shows that the charged gauge bosons $W^\pm$ do not appear simultaneously with the neutral boson Z in one physical situation. The same non-existence holds true for the neutral and charged Higgs particles as well.

Fifth, the new duality model for weak interaction not only produces consistent physical conclusions as the classical GWS electroweak theory, but also leads to new insights and predictions for weak interaction. Here are a few similarities and distinctions between these two models:

- Both theories produces the right intermediate vector bosons $W^\pm$ and Z, the neutral Higgs, the neutral current, and the scaling relation, consistent with experimental observations.
- The GWS model mixes transformations of different representations of $U(1)$ and $SU(2)$, and utilizes both the electromagnetic gauge potential and the weak gauge potentials to define the intermediate vector bosons. This gauge mixing causes the decoupling of the model to electromagnetic and weak components difficult, if not impossible. This mixing also violates PRI. The duality model used in this paper can be easily decoupled to study individual interactions involved, and satisfies PRI.
- In the GWS model, the Higgs mechanism of mass generation and energy creation is achieved by introducing the Higgs sector with a Higgs scalar field in the Lagrangian action functional. The mass generation and energy creation mechanism is achieved in a completely different and much simpler fashion in the duality model by using energy-momentum conservation constraint variation (PID) to the standard $SU(2)$ gauge functional.

- Due partially to mixing the gauge fields for electromagnetic and weak interactions, it is difficult to use the classical theory to derive any force/potential formulas for weak interaction. However, as mentioned earlier, the new duality model leads naturally to a long overdue force formula for weak interaction.

The paper is organized as follows. The PRI is briefly introduced in Section 2. Section 3 addresses various features of the new duality model for weak interaction. Section 4 derives the weak potential and force formulas. Section 5 is devoted to the comparison between the classical GWS and the new electroweak theory.

2. Principle of Representation Invariance (PRI)

The Yang-Mills theory for $SU(N)$ gauge fields possesses $K = N^2-1$ vector fields A^a_μ $(1 \le a \le K)$ and N fermion wave functions given by

$$\{A^a_\mu\} = \{A^1_\mu, \cdots, A^K_\mu\}, \qquad \Psi = \begin{pmatrix} \psi^1 \\ \vdots \\ \psi^N \end{pmatrix}. \tag{2.1}$$

The Yang-Mills action of A^a_μ and Ψ is given by

$$L_{YM} = \int (\mathcal{L}_G + \mathcal{L}_F) dx, \tag{2.2}$$

where $\mathcal{L}_G$ is the gauge field sector and $\mathcal{L}_F$ is the fermion action density:

$$\begin{aligned} \mathcal{L}_G &= g^{\mu\alpha} g^{\nu\beta} F^a_{\mu\nu} F^a_{\alpha\beta}, \\ \mathcal{L}_F &= \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi, \end{aligned} \tag{2.3}$$

and

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\lambda^a_{bc} A^b_\mu A^c_\nu, \tag{2.4}$$

$$D_\mu = \partial_\mu + igA^a_\mu \tau_a, \tag{2.5}$$

where $g^{\mu\nu}$ is the Minkowski metric, $\bar{\Psi} = \Psi^\dagger \gamma^0$, γ^μ $(\mu = 0, 1, 2, 3)$ are the Dirac matrices, m is the mass matrix, g is the gauge coupling constant, λ^a_{bc} are the $SU(N)$ structural constants, and τ_a are the generators of the representation of $SU(N)$.

The gauge invariance amounts to saying that the action (2.2) with (2.3)-(2.5) is invariant under the gauge transformation

$$\begin{aligned} \widetilde{\Psi} &= e^{i\theta^a \tau_a} \Psi, \\ \widetilde{A}^a_\mu &= A^a_\mu - \frac{1}{g}\partial_\mu \theta^a + \lambda^a_{bc} \theta^b A^c_\mu, \end{aligned} \tag{2.6}$$

where $\theta^a = \theta^a(x)$ $(1 \le a \le K)$ are real functions of space-time points x.

Mathematically, the $SU(N)$ generators τ_a $(1 \le a \le K)$ constitute a basis of the tangent space $T_e SU(N)$ of $SU(N)$ at the unit matrix $e = I$. There are infinite number of bases in $T_e SU(N)$, which transform under the complex general linear group $GL(\mathbb{C}^{N^2-1})$:

$$\widetilde{\tau}_a = x^b_a \tau_b, \tag{2.7}$$

where (x_a^b) are $(N^2-1)\times(N^2-1)$ nondegenerate matrices. Under the transformation (2.7), the quantities $\theta^a, A_\mu^a, F_{\mu\nu}^a, \lambda_{bc}^a$ in (2.4)-(2.6) will change as tensors:

$$\begin{aligned} \widetilde{\theta}^a &= \widetilde{x}_b^a \theta^b, & \widetilde{A}_\mu^a &= \widetilde{x}_b^a A_\mu^b, \\ F_{\mu\nu}^a &= \widetilde{x}_b^a F_{\mu\nu}^b, & \widetilde{\lambda}_{bc}^a &= x_b^t x_c^s \widetilde{x}_d^a \lambda_{ts}^d, \end{aligned} \tag{2.8}$$

where $(\widetilde{x}_b^a) = (x_b^a)^{-1}$ is the inverse matrix of (x_b^a).

In the same spirit as the Einstein principle of general relativity, physical laws should be independent of the choice of the bases $\{\tau_a\}$ in $T_e SU(N)$, which implies that the Yang-Mills action (2.2) should be invariant under the transformations (2.7) and (2.8). However, we see that the density $\mathcal{L}_G$ in (2.3) does not satisfy this symmetry. To solve this problem, we find in [3] the tensor

$$G_{ab} = \frac{1}{4N}\lambda_{ac}^d \lambda_{db}^c, \tag{2.9}$$

are symmetric and positive, which can be regarded as a Riemannian metric of $SU(N)$. Hence we need to modified the Yang-Mills action density as follows:

$$\mathcal{L}_G = g^{\mu\alpha} g^{\nu\beta} G_{ab} F_{\mu\nu}^a F_{\alpha\beta}^b. \tag{2.10}$$

Consequently, the action (2.2) with (2.10) is invariant under the transformations (2.7) and (2.8).

Furthermore, we proposed in [3] the following principle.

Principle of Representation Invariance (PRI). *Any $SU(N)$ gauge field theory is independent of the choice of $SU(N)$ generators $\{\tau_a\}$. Namely, the action (2.2) of gauge fields is invariant and the associated gauge field equations are covariant under the transformation (2.7).*

It is easy to see that any gauge theory with the classical Higgs mechanism violates PRI, and can only be considered as an approximation to the reality. The unified field model based on PID derived in [2, 3] not only provides the same outcomes the standard model with the classical Higgs mechanism, but also obeys PRI.

3. Duality Theory of Weak Interactions

3.1. **Field equations for weak interactions.** We derive now the field equations for weak interaction using an $SU(2)$ gauge theory based on PID and PRI. The action functional is

$$L_W = \int \mathcal{L}_W \, dx, \tag{3.1}$$

where

$$\mathcal{L}_W = -\frac{1}{4} G_{ab}^w g^{\mu\alpha} g^{\nu\beta} W_{\mu\nu}^a W_{\alpha\beta}^b + \bar{L}(i\gamma^\mu D_\mu - m^l)L. \tag{3.2}$$

Here G_{ab}^w is the metric defined by (2.9), $L = (L_1, L_2)^T$ are the wave functions of left-hand lepton and quark pairs (each has 3 generations), and

$$W_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + \frac{g_w}{\hbar c}\lambda_{bc}^a W_\mu^b W_\nu^c. \tag{3.3}$$

Here W_μ^a $(1 \le a \le 3)$ are the $SU(2)$ gauge fields for the weak interaction, and

$$D_\mu L = (\partial_\mu + i g_w W_\mu^a \tau_a)L, \tag{3.4}$$

where τ_a $(1 \le a \le 3)$ are the generators of $SU(2)$.

Using PID [2, 3], the weak interaction field equations are given by

$$G^w_{ab}\left[\partial^\mu W^b_{\mu\nu} - \frac{g_w}{2\hbar c}\lambda^b_{cd} g^{\alpha\beta} W^c_{\alpha\nu} W^d_\beta\right] - g_w J_{\nu a} = \left[\partial_\mu - \frac{g_w}{\hbar c}\alpha^w_b W^b_\nu + \frac{k_0^2}{4}x_\nu\right]\phi_a, \tag{3.5}$$

$$(i\gamma^\mu \widetilde{D}_\mu - m^l)L = 0, \tag{3.6}$$

where $J_{\nu a} = \bar{L}\gamma_\nu \tau_a L$, k_0 is a constant, $k_0^2 x_\mu/4$ is a mass potential, g is the weak charge, and α^w_a is the $SU(2)$ vector representing the portions distributed to the gauge potentials by the weak charge.

3.2. **Two types of duality.** The above field equations readily lead to a natural duality:

$$\{W^a_\mu\} \quad \longleftrightarrow \quad \{\phi_a\}. \tag{3.7}$$

The left side of this duality induces the intermediate vector bosons $W^\pm$ and Z, and the right had side gives rise to three Higgs bosons: one neutral and two charged.

Another possible duality is the degenerate case where the three scalar fields ϕ_a are a constant vector ζ_a times a single scalar field ϕ: $\phi_a = \frac{g_w}{\sqrt{\hbar c}}\zeta_a \phi$. In this case, the duality reduces to

$$\{W^a_\mu\} \quad \longleftrightarrow \quad \{\phi\}. \tag{3.8}$$

Again, the left side of this duality induces the intermediate vector bosons $W^\pm$ and Z. However, the right had side gives rise to one neutral Higgs boson.

For the duality (3.8), if we take the Pauli matrices as the generators of $SU(2)$, then (3.5) can be rewritten as

$$\partial^\mu W^b_{\mu\nu} - \frac{g_w}{2\hbar c}\varepsilon^{abc} g^{\alpha\beta} W^b_{\alpha\nu} W^c_\beta - g_w J^a_\nu = \frac{g_w}{\sqrt{\hbar c}}\zeta^a\left[\partial_\mu - \frac{g_w}{\hbar c}\alpha^w_b W^b_\nu + \frac{k_0^2}{4}x_\nu\right]\phi. \tag{3.9}$$

3.3. **Non-coexistence of charged and neutral particles.** In Section 5, we will discuss the mass generation mechanism for the field equations (3.5) and (3.6).We focus here on charged Higgs particles and the non-coexistence of weak interaction intermediate vector bosons using these field equations.

Equation (3.5) need to be supplemented with coupling gauge equations to complement the adjoint fields ϕ_a created, which are taken as

$$\partial^\mu W^a_\mu = \beta^a \qquad \text{for } a = 1, 2, 3, \tag{3.10}$$

where β^a are parameters which may vary for different physical situations.

For simplicity we take the Pauli matrices σ_a $(1 \le a \le 3)$ as the generators of $SU(2)$. Then make the transformation

$$\begin{pmatrix} \widetilde{\sigma}_1 \\ \widetilde{\sigma}_2 \\ \widetilde{\sigma}_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix}. \tag{3.11}$$

Under this transformation, $(W^1_\mu, W^2_\mu, W^3_\mu)$ and (ϕ_1, ϕ_2, ϕ_3) are transformed to

$$(W^\pm_\mu, Z_\mu) = (W^1_\mu \pm iW^2_\mu, W^3_\mu),$$
$$(\phi^\pm, \phi^0) = (\phi_1 \pm i\phi_2, \phi_3).$$

Then by PRI, equations (3.5) become

$$\partial^\nu W^\pm_{\nu\mu} \pm \frac{ig_w}{2\hbar c} g^{\nu\nu}(W^\pm_{\nu\mu} Z_\nu - Z_{\nu\mu} W^\pm_\nu) - g_w J^\pm_\mu = \left[\partial_\mu - k_W^2 W^\pm_\mu - k_Z^2 Z_\mu + \frac{k_0^2}{4} x_\mu\right] \phi^\pm, \tag{3.12}$$

$$\partial^\nu Z_{\nu\mu} - \frac{ig_w}{2\hbar c} g^{\nu\nu}(W^+_{\nu\mu} W^-_\nu - W^-_{\nu\mu} W^+_\nu) - g_w J^0_\mu = \left[\partial_\mu - k_W^2 W^\pm_\mu - k_Z^2 Z_\mu + \frac{k_0^2}{4} x_\mu\right] \phi^0, \tag{3.13}$$

where

$$\begin{aligned} J^\pm_\mu &= \frac{1}{\sqrt{2}}(J^1_\mu \pm i J^2_\mu), \qquad J^{NC}_\mu = J^3_\mu, \\ W^\pm_{\nu\mu} &= \partial_\nu W^\pm_\mu - \partial_\mu W^\pm_\nu \pm \frac{ig_w}{\hbar c}(Z_\mu W^\pm_\nu - Z_\nu W^\pm_\mu), \\ Z_{\nu\mu} &= \partial_\nu Z_\mu - \partial_\mu Z_\nu + \frac{ig_w}{\hbar c}(W^+_\mu W^-_\nu - W^-_\mu W^+_\nu). \end{aligned} \tag{3.14}$$

Here

$$k_W^2 = \frac{g_w \alpha_1^w}{\sqrt{2}\hbar c}, \quad k_Z^2 = \frac{g_w \alpha_3^w}{\hbar c}, \quad \begin{pmatrix} k_w^2 \\ k_w^2 \\ k_Z^2 \end{pmatrix} = \frac{g_w}{\sqrt{\hbar c}} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1^w \\ \alpha_2^w \\ \alpha_3^w \end{pmatrix}, \tag{3.15}$$

where $\{\alpha_b^w\} = (\alpha_1^w, \alpha_2^w, \alpha_3^w)$ is as in (3.5), and the second component $\alpha_2^w = 0$ when we use the Pauli representation.

It is easy to see that (3.12) for W^+_μ and W^-_μ are complex conjugate to each other. Here are two important solutions, leading to two different weak interactions:

First, if

$$W^\pm_\mu = 0, \qquad \phi^0 = 1, \qquad \beta^a = 0, \tag{3.16}$$

then Z_μ satisfies the equation

$$\Box Z_\mu + k_Z^2 Z_\mu - g J^0_\mu - \frac{k_0^2}{4} x_\mu = 0 \tag{3.17}$$

where $\Box$ is the wave operator given by

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2.$$

This is the case where the weak interaction involves the neutral Higgs boson ϕ^0 and the neutral intermediate vector boson Z with mass parameter k_Z^2.

Second, if

$$Z_\mu = 0, \qquad \phi^\pm = 1, \qquad \beta^a = 0, \tag{3.18}$$

then $W^\pm_\mu$ satisfy

$$\Box W^\pm_\mu + k_W^2 W^\pm_\mu - g J^\pm_\mu - \frac{k_0^2}{4} x_\mu = o(W^\pm) \tag{3.19}$$

This is the case the weak interaction occurs through the two charged intermediate vector bosons $W^\pm$, with mass parameter k_W^2, and the two charged Higgs bosons $\phi^\pm$.

This two solution cases suggest that the charged gauge bosons $W^{\pm}$ cannot appear simultaneously with the neutral boson Z in one physical situation.

Now we consider the adjoint fields $\phi^{\pm}$ and ϕ^0. If

$$Z_\mu = 0, \quad \beta^1 < 0, \quad \beta^2 = 0, \tag{3.20}$$

taking divergence on both sides of (3.12) we get

$$\Box\phi^{\pm} + (k_0^2 + k_W^2|\beta^1|)\phi^{\pm} + g_w\partial^\mu J_\mu^{\pm} = o(W^{\pm}, \phi^{\pm}). \tag{3.21}$$

Also, if

$$W_\mu^{\pm} = 0, \qquad \beta^3 < 0, \tag{3.22}$$

then we obtain from (3.13) that

$$\Box\phi^0 + (k_0^2 + k_Z^2|\beta^3|)\phi^0 + g\partial^\mu J_\mu^0 = o(Z, \phi^0). \tag{3.23}$$

Hence these two cases suggest also that there exist charged and neutral Higgs particles $\phi^{\pm}$ and ϕ^0, and the charged Higgs $\phi^{\pm}$ cannot coexist with the neutral Higgs ϕ^0.

In summary, from the above discussion we deduce the following physical conclusions:

1) Existence of charged and neutral Higgs particles $\phi^{\pm}$ and ϕ^0, satisfying equations (3.21) and (3.23) respectively.
2) Non-coexistence of charged and neutral weak interaction particles. Namely, $W^{\pm}$ and $\phi^{\pm}$ cannot coexist with Z or ϕ^0.
3) Finally, the two parameters $k_0^2 + k_W^2|\beta^1|$ and $k_o^2 + k_Z^2|\beta^3|$ define the masses m_H^c and m_H^0 of the Higgs bosons $\phi^{\pm}$ and ϕ^0. We have

$$\frac{(m_H^c)^2}{(m_H^0)^2} = \frac{k_0^2 + k_W^2}{k_0^2 + k_Z^2}\frac{|\beta^1|}{|\beta^3|} = \frac{m_0^2 + m_W^2|\beta^1|}{m_0^2 + m_Z^2|\beta^3|},$$

where m_0 is the mass associated with the mass potential k_0. We conjecture that the masses m_H^c and m_H^0 of $\phi^{\pm}$ and ϕ^0 also satisfy the scale relation (3.28), i.e.

$$\frac{m_H^c}{m_H^0} = \frac{m_W}{m_Z} = \frac{|j^{\pm}|}{|j_\mu^{NC}|} = \cos\theta_W. \tag{3.24}$$

where θ_W is the Weinberg angle.

We remark that Conclusions 1) and 2) above cannot be derived from the classical weak interaction theories.

3.4. **Scaling relation.** We know from (5.36) that

$$\frac{m_W}{m_Z} = \cos\theta_W. \tag{3.25}$$

According to the IVB theory for weak interaction, the charged and the neutral currents are

$$j_\mu^{\pm} = \frac{g_w}{\sqrt{2}}J_\mu^{\pm}, \qquad j_\mu^{NC} = \frac{g_w}{\cos\theta_W}J_\mu^{NC}. \tag{3.26}$$

After proper scaling for $J_\mu^{\pm}$, i.e. taking $\frac{1}{\sqrt{2}}J_\mu^{\pm}$ as $J_\mu^{\pm}$, (3.26) can be rewritten as

$$j_\mu^{\pm} = g_w J_\mu^{\pm}, \quad j_\mu^{NC} = \frac{g_w}{\cos\theta_W}J_\mu^{NC},$$

Hence we can consider g_w and $g_w/\cos\theta_W$ as the intensities of the currents $j_\mu^\pm$ and j_μ^{NC} respectively, denoted by

$$|j_\mu^\pm| = g_w, \qquad |j_\mu^{NC}| = \frac{g_w}{\cos\theta_W}. \tag{3.27}$$

Therefore, from (3.25) and (3.27) we get the scale relation between masses and intensities of currents as

$$\frac{|j_\mu^\pm|}{|j_\mu^{NC}|} = \frac{m_{W^\pm}}{m_Z} = \cos\theta_W. \tag{3.28}$$

By PRI, the weak interaction can be decoupled with other interactions. If we use the Pauli matrices σ_1, σ_2 and σ_3 as the generators for $SU(2)$, then G_{ab} is Euclidean. However, the corresponding action density $\mathcal{L}_W$ does not lead to the scaling relation (3.28). To solve this problem, we take another $SU(2)$ representation with the following generators:

$$\tau_1 = \sigma_1, \qquad \tau_2 = \sigma_2, \qquad \tau_3 = \sqrt{\cos\theta_W}\sigma_3. \tag{3.29}$$

In this case, the metric G_{ab} defined by (2.9) is

$$G_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \cos\theta_W \end{pmatrix}.$$

By (2.10) the action density corresponding to the representation (3.29) is given by

$$\begin{aligned} \mathcal{L}_W = &-\frac{1}{4}[W^1_{\mu\nu}W^{1\mu\nu} + W^2_{\mu\nu}W^{2\mu\nu} + \cos\theta_W W^3_{\mu\nu}W^{3\mu\nu}] \\ &+ \bar{L}[i\gamma^\mu(\partial_\mu - ig_w W^a_{\mu\nu}\tau_a) - m^L]L, \end{aligned} \tag{3.30}$$

where

$$W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + \frac{g_w}{\hbar c}\lambda^a_{bc}W^b_\mu W^c_\nu, \tag{3.31}$$

and λ^a_{bc} are the structural constants with respect to (3.29), which are antisymmetric for all indices a, b, c.

Thus, under the div_A-free constraint associated with

$$\begin{aligned} D_{\mu 1} &= \partial_\mu - \left(\frac{m_W c}{\hbar}\right)^2 W^1_\mu + \frac{k_0^2}{4}x_\mu, \\ D_{\mu 2} &= \partial_\mu - \left(\frac{m_W c}{\hbar}\right)^2 W^2_\mu + \frac{k_0^2}{4}x_\mu, \\ D_{\mu 3} &= \cos\theta_W\partial_\mu - \frac{1}{\cos\theta_W}\left(\frac{m_W c}{\hbar}\right)^2 W^3_\mu + \cos\theta_W\frac{k_0^2}{4}x_\mu, \end{aligned}$$

the Euler-Lagrangian equations of (3.30)-(3.31) are as follows

$$
\begin{aligned}
&\partial^\nu W^1_{\nu\mu} + k^2\phi W^1_\mu - \frac{g_w g^{\nu\nu}}{2\hbar c}(W^2_{\nu\mu}W^3_\nu - W^3_{\nu\mu}W^2_\nu) \\
&\qquad - g_w J^1_\mu = \left[\partial_\mu + \frac{k_0^2}{4}x_\mu\right]\phi, \\
&\partial^\nu W^2_{\nu\mu} + k^2\phi W^2_\mu - \frac{g_w g^{\nu\nu}}{2\hbar c}(W^3_{\nu\mu}W^1_\nu - W^1_{\nu\mu}W^3_{\nu 7}) \\
&\qquad - g_w J^2_\mu = \left[\partial_\mu + \frac{k_0^2}{4}x_\mu\right]\phi, \\
&\partial^\nu W^3_{\nu\mu} + \frac{k^2}{\cos^2\theta_W}\phi W^3_\mu - \frac{g_w}{2\hbar c\cos\theta_W}g^{\nu\nu}(W^1_{\nu\mu}W^2_\nu - W^2_{\nu\mu}W^1_\nu) \\
&\qquad - \frac{g_w}{\cos\theta_W}J^3_\mu = \left[\partial_\mu + \frac{k_0^2}{4}x_\mu\right]\phi,
\end{aligned}
\tag{3.32}
$$

where $k = m_W c/\hbar$. Under the unitary rotation transformation

$$
\begin{pmatrix} W^+_\mu \\ W^-_\mu \\ Z_\mu \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} W^1_\mu \\ W^2_\mu \\ W^3_\mu \end{pmatrix}. \tag{3.33}
$$

By PRI, the equations (3.32) becomes

$$
\begin{aligned}
&\partial^\nu W^\pm_{\nu\mu} + k^2\phi W^\pm_\mu \pm \frac{i g_w}{2\hbar c}g^{\nu\nu}(W^\pm_{\nu\mu}Z_\nu - Z_{\nu\mu}W^\pm_\nu) \\
&\qquad - g_w J^\pm_\mu = \eta^\pm\left[\partial_\mu + \frac{k_0^2}{4}x_\mu\right]\phi, \\
&\partial^\nu Z_{\nu\mu} + \frac{k^2}{\cos^2\theta_W}\phi Z_\mu - \frac{i g_w}{2\hbar c\cos\theta_W}g^{\nu\nu}(W^+_{\nu\mu}W^-_\nu - W^-_{\nu\mu}W^+_\nu) \\
&\qquad - \frac{g_w}{\cos\theta_W}J^{NC}_\mu = \left[\partial_\mu + \frac{k_0^2}{4}x_\mu\right]\phi,
\end{aligned}
\tag{3.34}
$$

where $J^\pm_\mu$, $J^{NC}_\mu = J^3_\mu$ and $W^\pm_{\nu\mu}$, $Z_{\nu\mu}$ are defined (3.14). It is clear that for (3.34), scaling relation (3.28) holds true.

Note here that the 2nd-order $SU(2)$ tensor diag $(k^2, k^2, k^2/\cos^2\theta_W)$ are invariant for the transformation (3.33). Namely

$$
\begin{pmatrix} k^2 & 0 & 0 \\ 0 & k^2 & 0 \\ 0 & 0 & \frac{k^2}{\cos^2\theta_W} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} k^2 & 0 & 0 \\ 0 & k^2 & 0 \\ 0 & 0 & \frac{k^2}{\cos^2\theta_W} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}^\dagger.
$$

Hence from (3.32) to (3.34) we have

$$
\left(k^2\phi W^1_\mu, k^2\phi W^2_\mu, \frac{k^2}{\cos^2\theta_W}\phi W^3_\mu\right) \longrightarrow \left(k^2\phi W^+_\mu, k^2\phi W^-_\mu, \frac{k^2\phi}{\cos^2\theta_W}Z_\mu\right).
$$

Remark 3.1. In (3.28), the ratio between the mass loss and intensity loss of the charged bosons $W^\pm$ and charged currents $j^\pm$ is the same as the ratio between those of Z and j^{NC}. The parts lost can be considered as being transformed into electromagnetic energy.

4. Weak Interaction Potentials

4.1. **Weak interaction potentials.** We now consider the duality between $\{W_\mu^a\}$ and a single neutral Higgs field given by (3.8). It is clear that both the weak gauge fields W_μ^a and the adjoint scalar field ϕ carry rich physical information, as the electromagnetic potential A_μ in QED. For example, the electric field E and magnetic field H are written as

$$E = -\left(\frac{\partial \vec{A}}{\partial x^0} + \nabla A_0\right), \qquad H = \mathrm{curl}\vec{A},$$

where $\vec{A} = (A_1, A_2, A_3), \nabla = \left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right)$, the electromagnetic energy density ε is

$$\varepsilon = \frac{1}{8\pi}(E^2 + H^2),$$

and the photon γ is expressed by A_μ satisfying

$$\Box A_\mu = 0.$$

So far, very little information has been extrapolated from the weak gauge fields. For example, we know that Z_μ and $W_\mu^\pm$ satisfying (3.17) and (3.19) represent the neutral and charged bosons, and $\phi^\pm, \phi^0$ satisfying (3.21) and (3.23) represent the neutral and charged Higgs particles.

In the same spirit as electromagnetism, we introduce below two physical quantities associated with the weak gauge potentials W_μ^a.

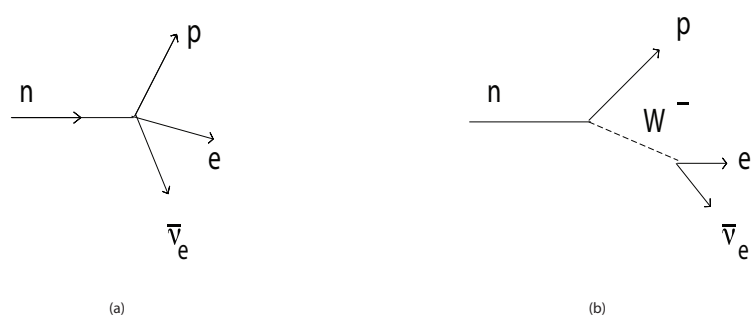

Figure 4.1

First, the β-decay is a weak process, as illustrated by Figure 4.1 (a) and (b). Physically, the process in Figure 4.1 (a) is regarded as an exchange of a massive

vector meson W^-, as shown in (b). The force range is about $r = 10^{-16}$cm. Before the β-decay, the neutron n is an energy pack bound by the potential energy ϕ in the radius $r = 10^{-16}$, and when the momentum energy in the interior of a neutron is greater than the bounding energy, the neutron is split into a proton p and an intermediate vector boson W^-, and the β-decay occurs. The interior momentum energy is characterized by

$$M = \int G_{ab} \nabla W_\mu^a \nabla W_\mu^b dy, \qquad y \in \mathbb{R}^3, \tag{4.1}$$

where $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$. Obviously, the right-hand side of (4.1) obeys PRI. Since ε is the momentum energy, it is not Lorentz invariant.

Second, the weak gauge potential W_μ^a have three constituents:

$$\{W_\mu^1, W_\mu^2, W_\mu^3\}.$$

The time-components W_0^a represent the weak-charge potentials with corresponding forces exerted by a particle with one weak charge g_w on another with N weak charges given by:

$$F_{WE}^a = -N g_w \nabla W_0^a \qquad \text{for } a = 1, 2, 3.$$

The total force exerted on the particle is

$$F_{WE} = -N g_w \alpha_a^w \nabla W_0^a, \tag{4.2}$$

where α_a^w is as in (3.5).

The spatial components $\vec{W}^a = (W_1^a, W_2^a, W_3^a)$ represent the weak-rotational potentials, yielding the following weak-rotational forces:

$$\begin{aligned} F_{WM}^a &= g_w \varepsilon^{abc} \vec{J}^b \times \ \mathrm{curl} \vec{W}^c, \\ F_{WM} &= g_w \varepsilon^{abc} \alpha_a^w \vec{J}^b \times \ \mathrm{curl} \vec{W}^c, \end{aligned} \tag{4.3}$$

where $\{\vec{J}^b\} = \{J_1^b, J_2^b, J_3^b\}$ is the weak current density. Obviously, F_{WE} and F_{WM} are gauge group representation invariant, i.e. they obey PRI.

4.2. **Dual field potential.** We take the Pauli matrices σ_a as the generators of an $SU(2)$ representation. Thus, $G_{ab} = \delta_{ab}$ and we derive from (3.9) the following equation

$$\begin{aligned} &\Box\phi + \left(\frac{m_H c}{\hbar}\right)^2 \phi + \frac{\xi^a}{\sqrt{\hbar c}} \alpha_b^w W_\mu^b D^\mu \phi \\ &\qquad = -\sqrt{\hbar c} \xi^a \left[\partial^\mu J_{\mu a} + \frac{g_w}{2\hbar c} \lambda_{cd}^b g^{\alpha\beta} \partial^\mu (W_{\alpha\nu}^c W_\beta^d)\right], \end{aligned} \tag{4.4}$$

where $\xi^a = \zeta^a / |\zeta|^2$ ($\zeta^a = \zeta_a$).

Assume that ϕ and W_μ^a are small and are independent of time variable $x_0 = ct$. Ignoring the higher order terms, equation (4.4) becomes

$$\nabla^2 \phi - \left(\frac{m_H c}{\hbar}\right)^2 \phi = -\sqrt{\hbar c} \xi^a \partial^\mu J_{\mu a}. \tag{4.5}$$

Equation (4.5) provides a model describing the scalar potential ϕ of weak force, which holds energy to form a particle.

By definition, we have

$$\partial^\mu J_{\mu a} = \partial_\mu \bar{L} \gamma^\mu \sigma_a L + \bar{L} \gamma^\mu \sigma_a \partial_\mu L.$$

By the Dirac equation (3.6).

$$\partial_\mu \bar{L}\gamma^\mu \sigma_a L = -ig_w W^b_\mu \bar{L}\gamma^\mu \sigma_b \sigma_a L + im\bar{L}\sigma_a L,$$
$$\bar{L}\gamma^\mu \sigma_a \partial_\mu L = ig_w W^b_\mu \bar{L}\gamma^\mu \sigma_a \sigma_b L - im\bar{L}\sigma_a L.$$

Thus we obtain

$$\partial^\mu J_{\mu a} = ig_w W^b_\mu \bar{L}\gamma^\mu [\sigma_a, \sigma_b] L = -2g_w \varepsilon_{abc} W^b_\mu J^\mu_c.$$

Noting that

$$\varepsilon_{abc}\xi^a W^b_\mu = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \xi^1 & \xi^2 & \xi^3 \\ W^1_\mu & W^2_\mu & W^3_\mu \end{vmatrix} = \vec{\xi} \times \vec{W}_\mu$$

where $\vec{\xi} = (\xi^1, \xi^2, \xi^3), \vec{W}_\mu = (W^1_\mu, W^2_\mu, W^3_\mu)$. Hence we obtain that

$$\xi^a \partial^\mu J_{\mu a} = 2g_w(\vec{\xi} \times \vec{W}_\mu) \cdot \vec{J}^\mu, \qquad \vec{J}^\mu = (J^\mu_1, J^\mu_2, J^\mu_3). \tag{4.6}$$

The weak charge densities $J^a_0 = J^0_a$ are

$$J^a_0 = \alpha^a_w \delta(x) \qquad \text{for } a = 1, 2, 3, \tag{4.7}$$

where $\vec{\alpha}_w = (\alpha^1_w, \alpha^2_w, \alpha^3_w)$ is as in (3.5). Therefore it follows from (4.6) that

$$\xi^a \partial^\mu J_{\mu a} = 2g_w \vec{\alpha}_w \cdot (\vec{\xi} \times \vec{W}_\mu)\delta(x), \tag{4.8}$$

where $\vec{\omega} = \vec{W}_0(0), \vec{\xi} = (\zeta_1, \zeta_2, \zeta_3)/|\zeta|^2$, and

$$\kappa = 2\vec{\alpha}_w \cdot (\vec{\xi} \times \vec{\omega}) = 2\vec{\xi} \cdot (\vec{\omega} \times \vec{\alpha}_w) = \begin{vmatrix} \xi^1 & \xi^2 & \xi^3 \\ \omega^1 & \omega^2 & \omega^3 \\ \alpha^1_w & \alpha^2_w & \alpha^3_w \end{vmatrix}. \tag{4.9}$$

Thus, by (4.8) and (4.9) the equation (4.5) is rewritten as

$$\nabla^2 \phi - \left(\frac{m_H c}{\hbar}\right)^2 \phi = -\kappa g_w \delta(x). \tag{4.10}$$

Let

$$k_H = \frac{m_H c}{\hbar}, \tag{4.11}$$

where m_H is the mass of a Higgs particle.

By (4.10) we derive the dual field potential ϕ:

$$\phi = \frac{\kappa g_w}{r} e^{-k_H r}. \tag{4.12}$$

Formula (4.12) leads to a few physical conclusions for weak interaction as follows:

1). The masses m_H and m_π of the Higgs and π meson are

$$m_H \cong 125\ GeV/c^2, \qquad m_\pi = 0.135\ GeV/c^2,$$

which implies that

$$m_H/m_\pi \cong 10^3.$$

By (4.12) we have

$$\frac{r_W}{r_S} = \frac{m_\pi}{m_H} = 10^{-3},$$

where r_W and r_S are the force ranges of weak and strong interactions. Hence the weak force range is $r_W = 10^{-16}$cm, consistent with experimental data.

2). By (4.12), the $SU(2)$ coupling constant g_w in (3.31) is endowed with a new physical meaning as the weak charge, reminiscent of the electric charge e.

3). The weak force parameter κ given by (4.9) is an $SU(2)$ pseudo-scalar. In addition, since the quantities $\omega^a = W_0^a(0)$ and θ^a defined by (4.7) characterize the interior properties of weak interaction particles such as the electron e, the neutron n and the proton p, the parameter κ reflects the interior structure of e, n, p.

4). For a particle, e.g. for the neutron n, we conjecture that the condition for decay depends on if the interior momentum M defined by (4.1) satisfies the following condition

$$M \geq \int_{|y|<r_0} \frac{g_w^2}{(\hbar c)^{3/2}} \zeta^a \alpha_w^b W_\mu^a W_\mu^b \phi dy, \ y \in \mathbb{R}^3, \ r_0 = k_0^{-1}. \tag{4.13}$$

In this case, n decays as

$$n \to p + e + \bar{\nu}_e.$$

Otherwise, if

$$M < \int_{|y|<r_0} \frac{g_w^2}{(\hbar c)^{3/2}} \zeta^a \alpha_w^b W_\mu^a W_\mu^b \phi dy, \tag{4.14}$$

the neutron n does not decay. Hence this explains why neutrons can spontaneously undergo a β-decay under proper conditions.

5). By (4.9), κ can be expressed as

$$\kappa = |\vec{\theta}| \cdot |\vec{\omega}| \cos\Phi \sin\varphi,$$

where Φ=the angle between $\vec{\xi}$ and $(\vec{\omega} \times \vec{\theta})$, and $\varphi =$ the angle between $\vec{\omega}$ and $\vec{\theta}$.

6). The parameter κ may be related with weak decay coupling constants, or equivalently with the Cabibbo-Kobayashi-Maskawa angles. Hence κ influences decay types.

4.3. **Weak decay conditions.** When a weak process is coupled with some external fields, energy exchange occurs. In general, gravity is much weaker than electromagnetic and strong interactions. Hence, ignoring the gravitational terms, the weak interacting field equations coupling external forces can be written as

$$\begin{aligned} &\partial^\nu W_{\nu\mu}^a - \frac{g_w}{2\hbar c} \varepsilon^{abc} g^{\nu\nu} W_{\nu\mu}^b W_\nu^c - g_w J_\mu^a \\ &= \frac{g_w}{\sqrt{\hbar c}} \zeta^a \left[\partial_\mu + \frac{e}{\hbar c} A_\mu + \frac{g_w \alpha_w^b}{\hbar c} W_\mu^b + \frac{g_s \alpha_s^k}{\hbar c} S_\mu^k + \frac{k_H^2}{4} x_\mu \right] \phi, \end{aligned} \tag{4.15}$$

where $\{\alpha_k^s\} = (\alpha_1^s, \cdots, \alpha_8^s)$ is the $SU(3)$ tensor, S_μ^k $(1 \leq k \leq 8)$ is the gauge potential of strong interaction, e is the electric charge whose sign is undetermined, and g_s is the strong charge.

As W_μ^a are the weak potential in the interior of a particle, ϕ is given by (4.12), and

$$W_\mu^a = 0 \quad \text{at} \quad r > r_0 = \frac{1}{k_H} \cong 10^{-16}\text{cm}. \tag{4.16}$$

Take the gauge

$$\partial^\mu W_\mu^a = \text{const}. \tag{4.17}$$

and assume that W_μ^a are independent of t. Equations (4.15) are rewritten as

$$
\begin{aligned}
(4.18)\qquad & -\nabla^2 W_\mu^a - \frac{g_w^2}{(\hbar c)^{3/2}}\zeta^a\alpha_w^b\phi = g_w J_\mu^a + \frac{g_w}{2\hbar c}\varepsilon^{abc}g^{\nu\nu}W_{\nu\mu}^b W_\nu^c \\
& + \frac{g_w}{\sqrt{\hbar c}}\zeta^a\left[\frac{e}{\hbar c}A_\mu + \frac{g_s\alpha_s^k}{\hbar c}S_\mu^k + \frac{k_H^2}{4}x_\mu\right]\phi + \frac{g_w}{\sqrt{\hbar c}}\zeta^a\partial_\mu\phi.
\end{aligned}
$$

Multiplying both sides of (4.18) by W_μ^a and integrating the sum in $y \in \mathbb{R}^3$ with $|y| < r_0 = k_0^{-1}$, by (4.7), (4.16) and (4.17) we deduce that

$$
\begin{aligned}
(4.19)\qquad & \int_{B_{r_0}} |\nabla W|^2 dy - \int_{B_{r_0}} \frac{g_w^2}{(\hbar c)^{3/2}}\zeta^a\alpha_w^b W_\mu^b\cdot W_\mu^a\phi dy \\
& = g_w\alpha_w^a\omega^a + \frac{g_w}{2\hbar c}\int_{B_{r_0}} \varepsilon^{abc}g^{\nu\nu}W_\mu^a W_{\nu\mu}^b W_\nu^c dy \\
& \quad + \int_{B_{r_0}} \frac{g_w}{\sqrt{\hbar c}}\left[\frac{e}{\hbar c}A_\mu + \frac{g_s\alpha_k^2}{\hbar c}S_\mu + \frac{k_H^2}{4}x_\mu\right]W_\mu^a\zeta^a\phi dy,
\end{aligned}
$$

where α_w^a and ω^a are as in (4.7) and (4.9), ϕ as (4.12), and $B_{r_0} = \{y \in \mathbb{R}^3 |\ |y| < r_0\}$.

Approximatively, taking the spheric coordinates we have

$$
(4.20)\qquad A = \int_{B_{r_0}} \frac{e^{-k_0 r}}{r}A_\mu W_\mu^a\zeta^a dy = \frac{r_0^2}{2}|\Omega|\mathcal{A}_\mu\mathcal{W}_\mu,
$$

$$
(4.21)\qquad S = \int_{B_{r_0}} \frac{e^{-k_0 r}}{r}S_\mu W_\mu^a\zeta^a dy = \frac{r_0^2}{2}|\Omega|S_\mu\mathcal{W}_\mu,
$$

where $|\Omega|$ is the area of the unit sphere, and

$$
\mathcal{A}_\mu = A_\mu(0), \qquad \mathcal{S}_\mu = \alpha_s^k S_\mu^k(0), \qquad \mathcal{W}_\mu = W_\mu^a(0)\zeta^a.
$$

Let

$$
\begin{aligned}
& M = \int_{B_{r_0}} |\nabla W|^2 dy, \qquad\qquad V = \int_{B_{r_0}} \frac{g_w^2}{(\hbar c)^{3/2}}\zeta^a\alpha_w^b W_\mu^b\cdot W_\mu^a\phi dy, \\
(4.22)\qquad & I = \int_{B_{r_0}} \varepsilon^{abc}g^{\nu\nu}W_\mu^a W_{\nu\mu}^b W_\nu^c dy, \quad \Phi = \alpha_w^a\omega^a, \\
& H = \frac{1}{4}\int_{B_{r_0}} \frac{g_w}{\sqrt{\hbar c}}x_\mu W_\mu^a\zeta^a\phi dy.
\end{aligned}
$$

Then (4.18) is rewritten as

$$
(4.23)\qquad M - V = g_w\Phi + \frac{g_w}{2\hbar c}I + \frac{\kappa e g_w^2}{\hbar c}A + \frac{\kappa g_s g^2}{\hbar c}S + \kappa g_w^2 H.
$$

Therefore, based on the criterion (4.13) and (4.14), we derive from (4.23) that for a particle under an external electromagnetic and strong fields A_μ and S_μ^k, the condition that it can decay is

$$
(4.24)\qquad \left[\Phi + \frac{I}{2}\right] + kg\left[\frac{e}{\hbar c}A + \frac{g_s}{\hbar c}S + H\right] \geq 0.
$$

By (4.20)-(4.22), the first part in the right-hand side of (4.24) represents the weak field energy generated by the weak charge g_w, and the second part is the energy generated by external fields.

4.4. **Weak interaction potential.** By (4.2), the time-components W_0^a of W_μ^a $(a = 1, 2, 3)$ represent the weak charge potentials generated by the weak charge g_w. We now derive an approximate formula for the total potential:

$$W = \alpha_w^a W_0^a, \qquad \text{where } |\alpha_w|^2 = \alpha_w^a \alpha_w^a = 1. \tag{4.25}$$

Assuming that W_μ^a are independent of time and taking linear approximation, from (3.9) we have

$$-\nabla^2 W = g_w \alpha_w^a J_0^a + \frac{g_w}{\sqrt{\hbar c}} \alpha_w^a \zeta^a \left(\frac{k_0^2}{4} c\tau - \frac{g_w}{\hbar c} W \right) \phi, \tag{4.26}$$

where τ is the lifetime of the Higgs, and

$$\phi = \theta + \phi_0, \tag{4.27}$$

where ϕ_0 is given by (4.12) and θ is a constant. Taking a translation

$$W \quad \longrightarrow W + \frac{k_0^2 \hbar c^2 \tau}{4 g_w},$$

and by $J_0^a = \alpha_w^a \delta(x)$, equations (4.26) and (4.27) become

$$-\nabla^2 W + k_1^2 W = g_w \delta(x) - \frac{g_w^2}{(\hbar c)^{3/2}} K W \phi_0, \tag{4.28}$$

where

$$k_1^2 = \frac{g_w^2}{(\hbar c)^{3/2}} \alpha_w^a \zeta^a \theta, \qquad K = \alpha_w^a \zeta^a. \tag{4.29}$$

Solutions of (4.28) and (4.29) can be expressed as

$$W = W_0 + W_1 + \cdots, \tag{4.30}$$

where W_n satisfy

$$-\nabla^2 W_0 + k_1^2 W_0 = g_w \delta(x), \tag{4.31}$$

$$-\nabla^2 W_n + k_1^2 W_n = -\left(\frac{g_w^2}{\hbar c} \right)^{3/2} \frac{A}{r} e^{-k_1 r} W_{n-1} \qquad \text{for } n = 1, 2, \cdots, \tag{4.32}$$

and $A = K\kappa$. The solution of (4.31) is

$$W_0 = \frac{g_w}{r} e^{-k_1 r}. \tag{4.33}$$

When $n = 1$, (4.32) is given by

$$\nabla^2 W_1 - k_1^2 W_1 = A \left(\frac{g_w^2}{\hbar c} \right)^{3/2} \frac{g_w}{r^2} e^{-(k_0 + k_1) r}. \tag{4.34}$$

Let W_1 be radial symmetric and in the form

$$W_1 = A \left(\frac{g_w^2}{\hbar c} \right)^{3/2} g_w e^{-(k_0 + k_1) r} \varphi_1(r). \tag{4.35}$$

Then (4.34) implies that

$$\varphi_1'' + 2 \left(\frac{1}{r} - K_1 \right) \varphi_1' - \frac{2K_1}{r} \varphi_1 + (K_1^2 - k_1^2) \varphi_1 = \frac{1}{r^2}, \tag{4.36}$$

where $K_1 = k_0 + k_1$.

Let φ_1 be expanded as

$$\varphi_1 = \sum_{k=0}^{\infty} p_k r^k \ln r + \sum_{k=0}^{\infty} q_k r^k. \tag{4.37}$$

Inserting (4.37) into (4.36) and comparing coefficients, we deduce that

$$\begin{aligned} &p_0 = 1, \qquad p_1 = \frac{2}{3}(1+q_0), \qquad p_2 = \frac{5}{18}K_1^2 + \frac{1}{6}k_1^2, \qquad \cdots, \\ &q_0 \text{ and } q_1 \text{ are free }, \qquad q_2 = -\frac{1}{36}K_1^2 - \frac{5}{12} + K_1\beta_1, \qquad \cdots. \end{aligned}$$

Then we infer from (4.32) that

$$W_n = A^n \left(\frac{g_w^2}{\hbar c}\right)^{3n/2} g_w e^{-K_n r}\varphi_1(r), \qquad K_n = k_0 + nk_1 \qquad \text{for } n \geq 1, \tag{4.38}$$

and φ_n satisfies

$$\varphi_n'' + 2\left(\frac{1}{r} - K_n\right)\varphi_n' - \frac{2K_n}{r}\varphi_n + (K_n^2 - k_1^2)\varphi_n = \frac{\varphi_{n-1}}{r}. \tag{4.39}$$

The solution of this equation is in the form

$$\varphi_n = \sum_{k=n-1}^{\infty} p_k^n r^k \ln r + \sum_{k=n-1}^{\infty} q_k^n r^k, \tag{4.40}$$

where p_k^n and q_k^n depend on the free parameters q_0 and q_1.

Hence by (4.30) and (4.38), the solution W of (4.28) can be expressed as

$$W = g_w e^{-k_1 r}\left[\frac{1}{r} - \sum_{n=1}^{\infty} A^n \left(\frac{g_w^2}{\hbar c}\right)^{3n/2} e^{-(k_0+(n-1)k_1)r}\psi_n\right], \tag{4.41}$$

where $\psi_n = -\varphi_n$ and φ_n is given by (4.40).

The function $\Psi = e^{-k_0 r}\psi$ with

$$\psi = \sum_{n=1}^{\infty} A^n \left(\frac{g_w^2}{\hbar c}\right)^{3n/2} g_w e^{-nk_1 r}\psi_n$$

is the solution of

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)\Psi - k_1^2\Psi = -\left(\frac{g_w^2}{\hbar c}\right)^{3/2}\frac{A}{r}e^{-k_1 r}\Psi - \left(\frac{g_w^2}{\hbar c}\right)^{3/2}\frac{Ag_w}{r^2}e^{-(k_0+k_1)r}. \tag{4.42}$$

Now we supply ψ with the following initial conditions:

$$\psi(r_1) = a_0, \qquad \psi'(r_1) = a_1 \qquad \text{at } r_1 > 0. \tag{4.43}$$

In summary, we have derived the weak potential and weak force formula given by

$$W = g_w e^{-k_1 r}\left[\frac{1}{r} - e^{-k_0 r}\psi(r)\right], \tag{4.44}$$

$$F = g_w^2 e^{-k_1 r}\left[\frac{k_1}{r} + \frac{1}{r^2} - (K_1\psi - \psi')e^{-k_0 r}\right], \tag{4.45}$$

where $K_1 = k_0 + k_1$, $k_0 = m_H c/\hbar$, $k_1 = m_W c/\hbar$, m_H and m_W are the masses of the Higgs and $W^\pm$ or Z bosons, and by (4.43), $\psi(r)$ can be approximately written as

$$\psi(r) = a_0 + a_1 r \qquad \text{near } r = r_1 = \frac{1}{k_0}. \tag{4.46}$$

Thus the weak force becomes

$$F = g_w^2 e^{-k_1 r}\left[\frac{k_1}{r} + \frac{1}{r^2} - (a_0 - a_1 + a_1 r)e^{-k_0 r}\right]. \tag{4.47}$$

Based on known physical facts, we have

$$a_0 > 0, \qquad a_1 \le 0, \qquad a_0 - a_1 \gg k_1^2. \tag{4.48}$$

Hence we have derived the following physical conclusions:

(1) A particle with weak charge g_w will generate a weak force F exerted on another with weak charge g_w, and the force F is given by (4.45) or (4.47).
(2) By (4.41 and (4.45), there is a radius $r_0 > 0$ such that F is repelling for $r < r_0$, and
$$F \to \infty \quad \text{as } r \to 0.$$
(3) By (4.47) and (4.48), F has an attractive region: $r_0 < r < r_1$.
(4) The weak interaction force F is of short-range:
$$F \sim 0 \quad \text{for } r > \frac{1}{k_1} \sim 10^{-16} cm.$$

5. Consistency with GWS Electroweak Theory

The main objective of this section is to study the consistency of the new electroweak theory based on PID and PRI with the classical GWS electroweak theory.

5.1. **GWS action.** For comparison, we first introduce the classical Glashow-Weinberg-Salam electroweak theory, which is a $U(1) \otimes SU(2)$ gauge theory. We adopt here the classical notations. The action is given by

$$L_{GWS} = \int [\mathcal{L}_G + \mathcal{L}_F + \mathcal{L}_H] dx. \tag{5.1}$$

Here $\mathcal{L}_G$ is the gauge part, $\mathcal{L}_F$ is the fermionic part, and $\mathcal{L}_H$ is the Higgs sector:

$$\begin{aligned}
\mathcal{L}_G &= -\frac{1}{4} W^a_{\mu\nu} W^{\mu\nu a} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}, \\
\mathcal{L}_F &= i\bar{L}\gamma^\mu D_\mu L + i\bar{e}^R \gamma^\mu D_\mu e^R, \\
\mathcal{L}_H &= D_\mu \phi^\dagger D^\mu \phi + \lambda(\phi^\dagger \phi - a^2)^2 + G_e(\bar{L}\phi e^R + \bar{e}^R \phi^\dagger L),
\end{aligned} \tag{5.2}$$

where G_e and $a > 0$ are constants, $L = (\nu_e, e^L)$, e^R is the wave function of right-hand electron, ϕ is the Higgs scalar field, and

$$\begin{aligned}
W^a_{\mu\nu} &= \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g_1 \varepsilon^{abc} W^b_\mu W^c_\nu, \\
B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu, \\
D_\mu e^R &= (\partial_\mu + i g_2 B_\mu) e^R, \\
D_\mu L &= (\partial_\mu + i\frac{g_2}{2} B_\mu - i\frac{g_1}{2} W^a_\mu \sigma_a) L, \\
D_\mu \phi &= (\partial_\mu - i\frac{g_2}{2} B_\mu - i\frac{g_1}{2} W^a_\mu \sigma_a) \phi,
\end{aligned}$$

Here g_1 and g_2 are coupling constants, ε^{kij} $(1 \le k, i, j \le 3)$ are the structural constants of $SU(2)$, σ_k $(1 \le k \le 3)$ are the Pauli matrices, $\{W_\mu^a\}$ is the Yang-Mills gauge field corresponding to the k-th generator of $SU(2)$, and $\{B_\mu\}$ is the gauge field with respect to $U(1)$.

We note that B_μ does not represent the electromagnetic potential A_μ, and the Higgs field ϕ is a complex doublet given by

$$\phi = (\phi^+, \phi^0)^T,$$

which has charge (1,0).

The action (5.1) is invariant under the $SU(2)$ gauge transformation

$$\begin{aligned} &L \to e^{\frac{i}{2}\theta^a\sigma_a} L, \\ &\phi \to e^{-\frac{i}{2}\theta^a\sigma_a}\phi, \\ &e^R \to e^R, \\ &W_\mu^a \to W_\mu^a - \frac{2}{g_1}\partial_\mu\theta^a + \varepsilon^{abc}\theta^b W_\mu^c, \end{aligned} \tag{5.3}$$

and the $U(1)$ gauge transformation

$$\begin{aligned} &L \to e^{\frac{i}{2}\beta} L, \\ &\phi \to e^{-\frac{i}{2}\beta}\phi, \\ &e^R \to e^{i\beta} e^R, \\ &W_\mu^a \to W_\mu^a - \frac{2}{g_2}\partial_\mu\beta, \\ &B_\mu \to B_\mu + \frac{2}{g_2}\partial_\mu\beta. \end{aligned} \tag{5.4}$$

We notice from (5.2) and (5.3) that $\mathcal{L}_F$ contains the following terms:

$$W_a^\mu J_\mu^a, \qquad J_\mu^a = \bar{L}\gamma_\mu\sigma_a L. \tag{5.5}$$

These terms are crucial in the weak interaction theory because under a unitary transformation

$$\begin{pmatrix} \sigma^+ \\ \sigma^- \\ \sigma^0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix}, \tag{5.6}$$

these terms become

$$\begin{aligned} &W^{\mu\pm} J_\mu^\pm, \qquad W^{\mu 3} J_\mu^3, \qquad W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \pm i W_\mu^2), \\ &J^{\mu\pm} = \bar{L}\gamma^\mu\sigma_\pm L, \quad J_\mu^3 = \bar{L}\gamma_\mu\sigma_3 L, \end{aligned} \tag{5.7}$$

where $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$, and $J^{\mu\pm}$ are the charged currents consistent with the classical V-A theory and the intermediate vector boson (IVB) theory, which are two successful models at low energies.

Physically, $W_\mu^\pm$ particles are vector intermediate bosons having mass m_W, which should satisfy the Klein-Gordon equation

$$\partial^\mu\partial_\mu W_\nu^\pm + k^2 W_\nu^\pm = o(W^\pm),$$

where $o(W^{\pm})$ stands for the higher order terms of $W^{\pm}$, and $k = m_W c/\hbar$. However, we find that the variational equations of the action (5.1) have the form

$$\frac{\delta L_{GWS}}{\delta W^a_\mu} = \partial^\mu W^a_{\mu\nu} + o(W) = 0,$$

which implies that $W^{\pm}_\mu$ in (5.7) would be massless, contradicting with the fact that $W^{\pm}$ are massive.

Higgs mechanism provides a resolution. We see from (5.2) and (5.3) that $\phi_0 = (0, a)^T$ is an extremum point of (5.1), i.e. for $\Phi = (W, B, L, R, \phi)$, $\Phi_0 = (0, 0, 0, 0, \phi_0)$ is a solution of

$$\delta L_{GWS} = 0.$$

Consider the translation

$$\Phi = \Phi' + \Phi_0, \qquad \Phi' = (W', B', L', R', \phi'). \tag{5.8}$$

Then the variational equations of L_{GWS} for Φ' are given by (for simplicity, omitting the primes)

$$\begin{pmatrix} \frac{\delta}{\delta W} L_{GWS} \\ \frac{\delta}{\delta B} L_{GWS} \end{pmatrix} = \begin{pmatrix} \partial^\mu W_{\mu\nu} \\ \partial^\mu B_{\mu\nu} \end{pmatrix} + M \begin{pmatrix} W_\nu \\ B_\nu \end{pmatrix} + o(W, B) = 0, \tag{5.9}$$

where M is the mass matrix induced by Φ_0. It is clear that (5.9) is no longer covariant, or equivalently L_{GWS} breaks the symmetry for Φ' for (5.3)-(5.3). But the particles described by (W', B') receive masses due to the symmetry breaking.

5.2. **Weinberg-Salam electroweak theory.** We now recapitulate the WS electroweak theory. In (5.1)-(5.3), replace $\phi = (\phi^+, \phi^0)^T$ by $\phi = (0, \varphi)$, then the system is simplified and still invariant under the transformations (5.3) and (5.4), and avoids the difficulty that there exists a charged and massless bosonic field ϕ^+ in the classical GWS model. In this case, the Higgs action in (5.2) becomes

$$\begin{aligned} \mathcal{L}_H =& \partial^\mu \varphi \partial_\mu \varphi + \varphi^2 \left[\frac{g_1^2}{4} W^a_\mu W^{\mu a} + \frac{g_2^2}{4} B_\mu B^\mu - \frac{g_1 g_2}{2} B^\mu W^3_\mu \right] \\ & - \lambda(\varphi^2 - a^2)^2 + G_e \varphi(\bar{e}^L e^R + \bar{e}^R e^L). \end{aligned} \tag{5.10}$$

Then the Euler-Lagrange equations of the action (5.1) are given by

$$\partial^\nu W^1_{\nu\mu} - \frac{g_1}{2} g^{\nu\nu}(W^2_{\nu\mu} W^3_\mu - W^3_{\nu\mu} W^2_\mu) + \frac{g_1}{2} J^1_\mu + \frac{g_1^2}{2} \varphi^2 W^1_\mu = 0, \tag{5.11}$$

$$\partial^\nu W^2_{\nu\mu} - \frac{g_1}{2} g^{\nu\nu}(W^3_{\nu\mu} W^1_\nu - W^1_{\nu\mu} W^3_\nu) + \frac{g_1}{2} J^2_\mu + \frac{g_1^2}{2} \varphi^2 W^2_\mu = 0, \tag{5.12}$$

$$\partial^\nu W^3_{\nu\mu} - \frac{g_1}{2} g^{\nu\nu}(W^1_{\nu\mu} W^2_\nu - W^2_{\nu\mu} W^1_\nu) + \frac{g_1}{2} J^3_\mu + \frac{g_1}{2} \varphi^2 (g_1 w^3_\mu - g_2 B_\mu) = 0, \tag{5.13}$$

$$\partial^\nu B_{\nu\mu} - \frac{g_2}{2} J^L_\mu - g_2 J^R_\mu + \frac{g_2}{2} \varphi^2 (g_2 B_\mu - g_1 W^3_\mu) = 0, \tag{5.14}$$

$$i\gamma^\mu (\partial_\mu + i\frac{g_2}{2} B_\mu - i\frac{g_1}{2} W^a_\mu \sigma_a) \begin{pmatrix} \nu \\ e^L \end{pmatrix} + G_e e^R \begin{pmatrix} 0 \\ \varphi \end{pmatrix} = 0, \tag{5.15}$$

$$i\gamma^\mu (\partial_\mu + i g_2 B_\mu) e^R + G_e \varphi e^L = 0, \tag{5.16}$$

$$\begin{aligned} & \partial^\mu \partial_\mu \varphi - \frac{1}{2} \varphi (g_1^2 W^a_\mu W^{\mu a} + g_2^2 B_\mu B^\mu - 2 g_1 g_2 W^3_\mu B^\mu) \\ & + 4\lambda a^2 \varphi - 4\lambda \varphi^3 - G_e(\bar{e}^L e^R + \bar{e}^R e^L) = 0. \end{aligned} \tag{5.17}$$

Under the translation (5.8), or equivalently inserting

$$\varphi = \phi_0 + a \tag{5.18}$$

into (5.11)-(5.16) we obtain

$$\partial^\nu \widetilde{W}^1_{\nu\mu} + \frac{g_1^2 a^2}{2} W^1_\mu + \frac{g_1}{2} J^1_\mu = o(W, \phi_0), \tag{5.19}$$

$$\partial^\nu \widetilde{W}^2_{\nu\mu} + \frac{g_1^2 a^2}{2} W^2_\mu + \frac{g_1}{2} J^2_\mu = o(W, \phi_0), \tag{5.20}$$

$$\partial^\nu \widetilde{W}^3_{\nu\mu} + \frac{g_1^2 a^2}{2} W^3_\mu - \frac{g_1 g_2 a^2}{2} B_\mu + \frac{g_1}{2} J^3_\mu = o(W, B, \phi_0), \tag{5.21}$$

$$\partial^\nu B_{\nu\mu} + \frac{g_2^2 a^2}{2} B_\mu - \frac{g_1 g_2 a^2}{2} W^3_\mu - g_2 J^R_\mu - \frac{g_2}{2} J^L_\mu = o(W, B, \phi_0), \tag{5.22}$$

$$i\gamma^\mu D_\mu \nu_e = 0, \tag{5.23}$$

$$i\gamma^\mu D_\mu e^L + a G_e e^R + G_e \phi_0 e^R = 0, \tag{5.24}$$

$$i\gamma^\mu D_\mu e^R + a G_e e^L + G_e \phi_0 e^L = 0, \tag{5.25}$$

where

$$\widetilde{W}^a_{\nu\mu} = \partial_\nu W^a_\mu - \partial_\mu W^a_\nu, \quad B_{\nu\mu} = \partial_\nu B_\mu - \partial_\mu B_\nu. \tag{5.26}$$

From (5.19)-(5.22) we can find the mass terms

$$M \begin{pmatrix} W^1_\mu \\ W^2_\mu \\ W^3_\mu \\ B_\mu \end{pmatrix} = \begin{pmatrix} \frac{g_1^2 a^2}{2} & 0 & 0 & 0 \\ 0 & \frac{g_1^2 a^2}{2} & 0 & 0 \\ 0 & 0 & \frac{g_1^2 a^2}{2} & -\frac{g_1 g_2 a^2}{2} \\ 0 & 0 & -\frac{g_1 g_2 a^2}{2} & \frac{g_2^2 a^2}{2} \end{pmatrix} \begin{pmatrix} W^1_\mu \\ W^2_\mu \\ W^3_\mu \\ B_\mu \end{pmatrix} \tag{5.27}$$

and the current terms

$$J_\mu = \left(\frac{g_1}{2} J^1_\mu, \quad \frac{g_1}{2} J^2_\mu, \quad \frac{g_1}{2} J^3_\mu, \quad -g_2 J^R_\mu - \frac{g_1}{2} J^L_\mu \right)^T. \tag{5.28}$$

In order to generate masses, we have to diagonalize the matrix of (5.27), by a rotating transformation for (W^3_μ, B_μ) as

$$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} \frac{g_1}{|g|} & \frac{g_2}{|g|} \\ -\frac{g_2}{|g|} & \frac{g_1}{|g|} \end{pmatrix} \begin{pmatrix} W^3_\mu \\ B_\mu \end{pmatrix}, \tag{5.29}$$

where $|g| = (g_1^2 + g_2^2)^{1/2}$. On the other hand, the charged currents $J^\pm_\mu$ are given by (5.7) which are derived by the transformation (5.6), or equivalently by the unitary rotation of (W^1_μ, W^2_μ) as

$$\begin{pmatrix} W^+_\mu \\ W^-_\mu \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} W^1_\mu \\ W^2_\mu \end{pmatrix}. \tag{5.30}$$

Hence, under the following unitary transformation

$$\begin{pmatrix} W^+_\mu \\ W^-_\mu \\ Z_\mu \\ A_\mu \end{pmatrix} = U \begin{pmatrix} W^1_\mu \\ W^2_\mu \\ W^3_\mu \\ B_\mu \end{pmatrix}, \qquad U = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{g_1}{|g|} & \frac{g_2}{|g|} \\ 0 & 0 & -\frac{g_2}{|g|} & \frac{g_1}{|g|} \end{pmatrix}, \tag{5.31}$$

the mass matrix M in (5.27) becomes

$$UMU^\dagger = \frac{c^2}{\hbar^2}\begin{pmatrix} m_W^2 & 0 & 0 & 0 \\ 0 & m_W^2 & 0 & 0 \\ 0 & 0 & m_Z^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{5.32}$$

and the current J_μ in (5.28) is as

$$\left(\frac{g_1}{\sqrt{2}}J_\mu^+, \quad \frac{g_1}{\sqrt{2}}J_\mu^-, \quad |g|J_\mu^{NC}, \quad eJ_\mu^{em}\right)^T = UJ_\mu. \tag{5.33}$$

Also, equations (5.19)-(5.22) become

$$\begin{aligned} &\partial^\nu(\partial_\nu W_\mu^+ - \partial_\mu W_\mu^+) + \left(\frac{cm_W}{\hbar}\right)^2 W_\mu^+ + \frac{g_1}{\sqrt{2}}J_\mu^+ = o(\Phi), \\ &\partial^\nu(\partial_\nu W_\mu^- - \partial_\mu W_\mu^-) + \left(\frac{cm_W}{\hbar}\right)^2 W_\mu^- + \frac{g_1}{\sqrt{2}}J_\mu^- = o(\Phi), \\ &\partial^\nu(\partial_\nu Z_\mu - \partial_\mu Z_\nu) + \left(\frac{cm_Z}{\hbar}\right)^2 Z_\mu + |g|J_\mu^{NC} = o(\Phi), \\ &\partial^\nu(\partial_\nu A_\mu - \partial_\mu A_\nu) - eJ_\mu^{em} = o(\Phi), \end{aligned} \tag{5.34}$$

where $\Phi = (W^+, W^-, Z, A, \phi_0)$.

The above field equations (5.31)-(5.34) lead to the following physical conclusions as part of the classical electroweak theory:

1). When the Higgs field φ possesses a nonzero vacuum state as (5.18), the gauge symmetry breaks, and the fields W_μ^a and B_μ are recombined to yield a changed doublet of massive vector intermediate bosons $W_\mu^\pm$, a neutral massive vector boson Z_μ, and a massless photon field A_μ:

$$\begin{aligned} W_\mu^\pm &= \frac{1}{\sqrt{2}}(W_\mu^1 \pm W_\mu^2), \\ Z_\mu &= \frac{1}{|g|}(g_1 W_\mu^3 + g_2 B_\mu) = \cos\theta_W W_\mu^3 + \sin\theta_W B_\mu, \\ A_\mu &= \frac{1}{|g|}(-g_2 W_\mu^3 + g_1 B_\mu) = -\sin\theta_W W_\mu^3 + \cos\theta_W B_\mu, \end{aligned} \tag{5.35}$$

where θ_W is the Weinberg angle defined as

$$\cos\theta_W = \frac{g_1}{|g|} = \frac{g_1}{\sqrt{g_1^2 + g_2^2}}, \qquad \sin\theta_W = \frac{g_2}{|g|} = \frac{g_2}{\sqrt{g_1^2 + g_2^2}}.$$

2). The masses of $W_\mu^\pm$ and Z are as in (5.32) given by

$$m_{W^+} = m_{W^-} = \frac{a\hbar}{\sqrt{2}c}g_1, \qquad m_Z = \frac{a\hbar}{\sqrt{2}c}|g| \tag{5.36}$$

3). The electric charge in (5.34) is

$$e = g_1 \sin\theta_W. \tag{5.37}$$

4). Both charged currents $J_\mu^\pm$ and the neutral current J_μ^{NC} appearing in (5.34) are derived from (5.28) and (5.33), given by

$$J_\mu^\pm = \frac{1}{2}(J_\mu^1 \pm iJ_\mu^2), \quad J_\mu^{NC} = \frac{1}{2}[\cos\theta_W^2 J_\mu^3 - \sin^2\theta_W J_\mu^L - 2\sin^2\theta_W J_\mu^R], \tag{5.38}$$

where J_μ^a is as in (5.5), and

$$J_\mu^3 = \bar{\nu}_e\gamma_\mu\nu_e - \bar{e}^L\gamma_\mu e^L, \qquad J_\mu^R = \bar{e}^R\gamma_\mu e^R,$$
$$J_\mu^L = \bar{\nu}_e\gamma_\mu\nu_e + \bar{e}^L\gamma_\mu e^L, \qquad J_\mu^{em} = \frac{1}{2}J_\mu^3 + \frac{1}{2}J_\mu^L + J_\mu^R = \bar{\nu}_e\gamma_\mu\nu_e + \bar{e}^R\gamma_\mu e^R.$$

Here J_μ^{em} is the electric current.

5). By (5.24) and (5.25) the mass of an electron is given by

$$m_e = aG_e. \tag{5.39}$$

6). Finally, (5.17) is the Higgs field equation, from which we derive the mass of the Higgs particle as

$$m_H = 2a\sqrt{\lambda}. \tag{5.40}$$

Two remarks are now in order.

Remark 5.1. From (5.39) and (5.40) we see that the electron mass m_e and the Higgs particle mass m_H can not be determined by the electroweak theory, and their values are also helpless for determining the masses of $W_\mu^\pm$ and Z_μ. By the $V-A$ theory, we have

$$\frac{G_F}{\sqrt{2}} = \frac{g_1^2}{8m_W^2} = \frac{1}{4a^2},$$

where G_F is the Fermi constant. Therefore,

$$a^2 = \frac{1}{2\sqrt{2}G_F}. \tag{5.41}$$

Experimentally the value of θ_W is determined by

$$\sin^2\theta_W = 0.2325 \pm 0.008.$$

Then, from (5.36), (5.37) and (5.41) it follows that

$$m_W = 80.22 \pm 0.26\ GeV/c^2,$$
$$m_Z = 91.173 \pm 0.020\ GeV/c^2.$$

Remark 5.2. It is worth mentioning that the transformation (5.31) corresponds to a mixed transformation of $U(1)$ generator $\sigma_0 = 1$ and $SU(2)$ generators σ_a:

$$\begin{pmatrix}\tau_1\\ \tau_2\\ \tau_3\\ \tau_0\end{pmatrix} = U\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_0\end{pmatrix}.$$

Consequently, the GWS electroweak model cannot be decoupled to study individual interactions. In other words, the classical electroweak theory violates the principle of representation invariance (PRI).

5.3. **Electroweak theory obeying PRI.** We have seen that the classical electroweak theory violates PRI. In the following we develop a much simpler electroweak theory based on PID, which not only satisfies the PRI, but also leads to the same outcomes as the classical electroweak theory.

According to the IVB theory of weak interactions, the charged currents and the neutral currents are

$$j_\mu^\pm = \frac{g_1}{\sqrt{2}}J_\mu^\pm, \qquad j_\mu^{NC} = \frac{g_1}{\cos\theta_W}J_\mu^{NC}. \tag{5.42}$$

In particular, we see that the WS theory gives rise to the ratio

$$j_\mu^\pm / j_\mu^{NC} = \frac{1}{\sqrt{2}} \cos\theta_W J_\mu^\pm / J_\mu^{NC},$$

which the new theory should retain as well.

We note that the doublets

$$\widetilde{e}^R = \begin{pmatrix} e^R \\ 0 \end{pmatrix}, \quad \widetilde{e}^L = \begin{pmatrix} 0 \\ e^L \end{pmatrix}, \quad \widetilde{\nu}_e = \begin{pmatrix} \nu_e \\ 0 \end{pmatrix} \tag{5.43}$$

are $SU(2)$ symmetric, i.e. they can not be distinguished by themselves. Hence, the fermionic action $\mathcal{L}_F$ in general is taken in the form

$$\begin{aligned} \mathcal{L}_F =& \bar{L}(i\gamma^\mu D_\mu - m^L)L + \alpha_1 \bar{\widetilde{e}}^L (i\gamma^\mu D_\mu - m^L)\widetilde{e}^L \\ &+ \alpha_2 \bar{\widetilde{e}}^R (i\gamma^\mu D_\mu - m^R)\widetilde{e}^R + \alpha_3 i\bar{\widetilde{\nu}}_e \gamma^\mu D_\mu \widetilde{\nu}_e, \end{aligned}$$

where $\alpha_1, \alpha_2, \alpha_3$ are constants, m_e is the electron mass,

$$m^L = \begin{pmatrix} 0 & 0 \\ 0 & m_e \end{pmatrix}, \qquad m^R = \begin{pmatrix} m_e & 0 \\ 0 & 0 \end{pmatrix}$$

are 2nd-order tensors for the transformation

$$\begin{aligned} L &\to e^{i\frac{1}{2}\theta^a \sigma_a} L, \\ \widetilde{e}^L &\to e^{i\frac{1}{2}\theta^a \sigma_a} \widetilde{e}^L, \\ \widetilde{e}^R &\to e^{i\frac{1}{2}\theta^a \sigma_a} \widetilde{e}^R, \end{aligned}$$

and

$$\begin{aligned} D_\mu \widetilde{e}^L &= (\partial_\mu + i\beta_1 B_\mu - i\frac{g_1}{2} W_\mu^a \sigma_a)\widetilde{e}^L, \\ D_\mu \widetilde{e}^R &= (\partial_\mu + i\beta_2 B_\mu - i\frac{g_1}{2} W_\mu^a \sigma_a)\widetilde{e}^R, \\ D_\mu \widetilde{\nu}_e &= (\partial_\mu + i\beta_3 B_\mu - i\frac{g_1}{2} W_\mu^a \sigma_a)\widetilde{\nu}_e. \end{aligned}$$

By (5.43), $\mathcal{L}_F$ can be equivalently written as

$$\begin{aligned} \mathcal{L}_F \quad = \quad & \bar{L}(i\gamma^\mu D_\mu - m^L)L + i\alpha_3 \bar{\nu}_e \gamma^\mu (\partial_\mu + i\beta_3 B_\mu - i\frac{g_1}{2} W_\mu^3)\nu_e \\ & -\alpha_1 \bar{e}^L \left[i\gamma^\mu (\partial_\mu + i\beta_1 B_\mu - i\frac{g_1}{2} W_\mu^3) - m_e \right] e^L \\ & +\alpha_2 \bar{e}^R \left[i\gamma^\mu (\partial_\mu + i\beta_2 B_\mu - i\frac{g_1}{2} W_\mu^3) - m_e \right] e^R. \end{aligned} \tag{5.44}$$

If we regard W_μ^3 as Z_μ, B_μ as A_μ and g_2 as e, then the currents derived from (5.44) are as follows

$$\begin{aligned} & J_\mu^\pm = \bar{L}\gamma_\mu \sigma^\pm L, \quad \sigma^\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2), \\ & J_\mu^{NC} = \frac{\cos\theta_W}{2}[J_\mu^3 - \alpha_1 \bar{e}^L \gamma_\mu e^L + \alpha_2 \bar{e}^R \gamma_\mu e^R + \alpha_3 \bar{\nu}_e \gamma_\mu \nu_e], \\ & J_\mu^{em} = \frac{1}{2}[\bar{L}\gamma_\mu L - \frac{\alpha_1\beta_1}{e}\bar{e}^L \gamma_\mu e^L + \frac{\alpha_2\beta_2}{e}\bar{e}^R \gamma_\mu e^R + \frac{\alpha_3\beta_3}{e}\bar{\nu}_e \gamma_\mu \nu_e]. \end{aligned} \tag{5.45}$$

The currents in (5.45) will be utterly the same as those from the classical electroweak theory if the parameters α_k and β_k $(1 \le k \le 3)$ are chosen properly.

Now, we take the action

$$L = \int [\mathcal{L}_G + \mathcal{L}_F] dx, \tag{5.46}$$

where $\mathcal{L}_F$ is (5.44) and $\mathcal{L}_G$ as in (5.3) with $B_\mu = A_\mu$ being the electromagnetic potential. The the variational equations of (5.46) with the div_A-free constraint are given by

$$\begin{aligned} &\partial^\nu W^1_{\nu\mu} - \frac{g_1}{2} g^{\nu\nu}(W^2_{\nu\mu} W^3_\nu - W^3_{\nu\mu} W^2_\nu) + \frac{g_1}{2} J^1_\mu \\ &\quad = \left[\eta^1 \partial_\mu + \frac{\eta^1}{4}\left(\frac{m_H c}{\hbar}\right)^2 x_\mu - k^2 W^1_\mu\right]\phi, \end{aligned} \tag{5.47}$$

$$\begin{aligned} &\partial^\nu W^2_{\nu\mu} - \frac{g_1}{2} g^{\nu\nu}(W^3_{\nu\mu} W^1_\nu - W^1_{\nu\mu} W^3_\nu) + \frac{g_1}{2} J^2_\mu \\ &\quad = \left[\eta^2 \partial_\mu + \frac{\eta^2}{4}\left(\frac{m_H c}{\hbar}\right)^2 x_\mu - k^2 W^2_\mu\right]\phi, \end{aligned} \tag{5.48}$$

$$\begin{aligned} &\partial^\nu W^3_{\nu\mu} - \frac{g_1}{2} g^{\nu\nu}(W^1_{\nu\mu} W^2_\nu - W^2_{\nu\mu} W^1_2) + \frac{g_1}{\cos\theta_W} J^{NC}_\mu \\ &\quad = \left[\eta^3 \partial_\mu + \frac{\eta^2}{4}\left(\frac{m_H c}{\hbar}\right)^2 x_\mu - k_0^2 W^3_\mu\right]\phi, \end{aligned} \tag{5.49}$$

$$\partial^\nu(\partial_\nu A_\nu - \partial_\nu A_\nu) - e J^{em}_\mu = 0, \tag{5.50}$$

where $\eta = (\eta^1, \eta^2, \eta^3)$ is the $SU(2)$ gauge tensor, ϕ is a scalar field, $J^\pm_\mu = \frac{1}{2}(J^1_\mu \pm iJ^2_\mu)$, J^{NC}_μ, J^{em}_μ are as in (5.45), and

$$(\varepsilon_{ab}) = \begin{pmatrix} k^2 & 0 & 0 \\ 0 & k^2 & 0 \\ 0 & 0 & k_0^2 \end{pmatrix}$$

is a 2nd-order $SU(2)$ tensor with

$$k = \frac{m_W c}{\hbar}, \qquad k_0 = \frac{m_Z c}{\hbar}. \tag{5.51}$$

It is clear that the equations (5.47)-(5.50) are covariant under transformations of representations of $U(1) \times SU(2)$. Namely, PRI holds true for this model.

Then by taking the divergence on both sides of (5.47)-(5.49) and making the inner product with $\eta = (\eta^1, \eta^2, \eta^3)$, we obtain the field equation for ϕ:

$$\begin{aligned} \partial^\mu \partial_\mu \phi + \left(\frac{m_H c}{\hbar}\right)^2 \phi =& \eta^b \varepsilon_{ab} W^a_\mu \partial^\mu \phi + \frac{g_1}{2} \eta^a \partial^\mu J^a_\mu \\ &+ \frac{g_1}{2} \eta^a \varepsilon^{abc} g^{\nu\alpha} \partial^\mu (W^b_{\nu\mu} W^c_\alpha), \end{aligned} \tag{5.52}$$

which is the field equation describing the Higgs particle.

From (5.47)-(5.49) we see that when ϕ possesses a nonzero ground state, i.e.

$$\phi = 1 + \phi_0,$$

then the intermediate vector bosons $W^\pm$ and Z with masses m_W and m_Z are generated. Furthermore we can easily derive all six conclusions 1)-6) in the last subsection based on the classical electroweak theory.

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