Homework Spin Operators!

1) Write down the matrix representation of
   (a) $S_z$
   (b) $S_x$
   (c) $S_y$
   in the $|S_z^e\rangle$ and $|S_z^s\rangle$ basis.
   [Hint: Use the resolution of the identity for the eigenstates of each operator.]

2) After you obtain the matrices above, construct the commutators,
   $[S_x, S_y]$, $[S_y, S_z] + [S_z, S_x]$. Simplify and comment on your result.
   [Note: $[A, B] = AB - BA$.]
D  A little bit of help on the spin operators homework:

1. Consider the ket vectors $|+\rangle$ and $|−\rangle$. Let these ket vectors represent the up-spin and down-spin states of an electron along the z-orientation. (i.e., $|S^+_z\rangle$ and $|S^-_z\rangle$) A state with spin = +1/2 and is represented by the vector $|+\rangle$. What is meant by this statement is that $S_z |+\rangle = +\hbar(1/2) |+\rangle$. The state with spin = -1/2 is represented by the vector $|−\rangle$. Again, this statement implies $S_z |−\rangle = −\hbar(1/2) |−\rangle$) That is, these are eigenstates of the $S_z$ operator. These two vectors form a 2-dimensional vector space that is complete and orthonormal. In matrix notation, these ket vectors may be written as

$$|+\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$  \hspace{1cm} (D.9)

and

$$|−\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$  \hspace{1cm} (D.10)

This is based on the isomorphism between $|+\rangle$ and x-polarized light and $|−\rangle$ and y-polarized light.

Since these two vectors form a 2-dimensional vector space that is complete and orthonormal the resolution of the identity in this space can be written as:

$$\{|+\rangle \langle +|\} + \{|−\rangle \langle −|\} = I$$  \hspace{1cm} (D.11)
2. Using these ket vectors the $S_z$ operator can be represented as follows:

\[
S_z \equiv S_z \left[ \{|+\rangle \langle +|\} + \{|-\rangle \langle -|\} \right] \\
= \left[ \frac{\hbar}{2} |+\rangle \langle +| - \frac{\hbar}{2} |-\rangle \langle -| \right] \\
= \frac{\hbar}{2} [|+\rangle \langle +| - |-\rangle \langle -|] \tag{D.12}
\]

(Note that the quantity in square brackets [...] on the left side in Eq. (D.12) is just the identity as per Eq. (D.11). Also note that we have used $S_z |+\rangle = +\hbar(1/2) |+\rangle$ and $S_z |-\rangle = -\hbar(1/2) |-\rangle$ to obtain Eq. (D.12). Obtain similar expressions for $S_x$ and $S_y$.

3. $S_z$ can then be written in matrix form using the basis-ket vectors $|+\rangle$ and $|-\rangle$ as:

\[
S_z \equiv \begin{pmatrix}
\langle + | S_z | + \rangle & \langle + | S_z |-\rangle \\
\langle - | S_z | + \rangle & \langle - | S_z |-\rangle
\end{pmatrix} \\
= \frac{\hbar}{2} \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix} \tag{D.13}
\]

Write down similar matrix forms for the expressions you obtained for $S_x$ and $S_y$ in the previous problems. These three matrices are called the Pauli-spin matrices.

4. Using the three matrices you have for $S_x$, $S_y$, and $S_z$, confirm that these matrices do not commute.

5. Pauli-spin matrices are $2\times2$ matrices. Which means they will act on $2\times1$ vectors. As noted earlier

\[
|+\rangle \equiv \begin{pmatrix}
1 \\
0
\end{pmatrix} \tag{D.14}
\]
and

\[ |\rightarrow \rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]  

(D.15)

And the Pauli-spin matrices can act on either these vectors or linear combinations of these vectors. Such vectors obtained from arbitrary linear combinations of \(|+\rangle\) and \(|-\rangle\) are called “spinors” (which comes from \textbf{spin}-\textbf{vectors}. And in general the coefficients in front of each vector \(|+\rangle\) and \(|-\rangle\) can be complex.