5 A brief note on change of basis

1. We spoke about the definition of a wavefunction. It is the projection of the abstract “ket” |ψ⟩ onto the coordinate representation “ket”, i.e., ⟨x|ψ⟩ ≡ ψ(x), the wavefunction.

2. But then the abstract “ket” |ψ⟩ can be represented using any complete set of kets, and we have chosen above to use the coordinate representation, which we know to be complete from the identity:

\[ \int dx \ |x\rangle \langle x| = 1 \] (5.0.1)

But we could have chosen to use some other set, for example the momentum representation, since:

\[ |ψ⟩ \equiv \int dx \ |x⟩ \langle x|ψ⟩ = \int dk \ |k⟩ \langle k|ψ⟩ = \int dk \ \tilde{ψ}(k) \]
\[ = \sum_i |i⟩ \langle i|ψ⟩ = \sum_i |i⟩ c_i. \] (5.0.2)

where ψ(x), \( \tilde{ψ}(k) \) and \( c_i \) are just projects of the abstract “ket” |ψ⟩ onto the “axes” of the chosen representation.

3. The statement along with our discussion on measurement as a “dot” product may be viewed to imply that, in fact, it is the projection of |ψ⟩ in some basis that is measured (or calculated) and from such a projection we (sometimes) struggle to recreate the entire “ket” |ψ⟩.

4. That begs the question, how do these projections relate to each other. That is how is ψ(x) related to \( \tilde{ψ}(k) \) and \( c_i \) and vice versa.

5. This is the question we address here as part of discussion on “change of basis”.

6. Consider the following sequence of arguments:

\[ ψ(x) = \langle x|ψ⟩ = \langle x | I | ψ⟩ = \left\langle x \left| \int dk \ |k⟩ \langle k|ψ⟩ \right. \right\rangle \] (5.0.3)

where we have inserted the identity in terms of the momentum representation. Now, since \( ⟨x| \) is a vector and \( \int dk \) is essentially a continuous summation, we can take the vector inside the summation (to be discussed in class), leading to:

\[ ψ(x) = \int dk \langle x|k⟩ \langle k|ψ⟩ \] (5.0.4)

Now, \( ⟨x|k⟩ \) is the projection (or representation) of the momentum eigenstate in the coordinate representation. That is, \( ⟨x|k⟩ \) is a function of x, that is also an eigenstate of the momentum operator.
7. So, what is $\langle x|k \rangle$? Well, $\langle x|k \rangle = \exp(ikx)$, because $\exp(ikx)$ is an eigenstate of the momentum operator and is a function of $x$. Read the previous point again to make sure you understand this.

8. That leads to:

$$\psi(x) = \int dk \exp(ikx) \langle k || \psi \rangle = \int dk \exp(ikx) \tilde{\psi}(k)$$

(5.0.5)

9. Some of you might recognize that the above equation is a Fourier transform. Indeed, one needs to construct a Fourier transform to convert a wavefunction into its momentum space analogue and vice versa. But this is now an example of a change of basis. That is if we have $\tilde{\psi}(k)$, we can always use the above equation to get $\psi(x)$, that is the projection of $|\psi\rangle$ on to a different basis.

10. **Homework:** Work out the transformation of $\psi(x)$ onto some discrete basis $\{ |i \rangle \}$.