Topics in Dynamic Macroeconomics

Meeting III: Stochastic Growth

Teaching Notes. Indiana University 11/6/06-11/10/06

(Beware of the typos - do not quote!)

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Fluctuations and Growth

• Reference: Jones and Manuelli (Handbook, 2005)

• How does one interpret the evidence between instability and growth?

• What does theory predict about the relationship between the variance of productivity shocks and growth?

• Comment: Why is it that the RBC literature does not (usually) discuss this?
Empirical Evidence

• Ramey and Ramey (AER, (1995)) estimated

\[ \gamma_{it} = \beta X_{it} + \lambda \sigma_i + u_{it} \]

\[ u_{it} = \sigma_i \epsilon_{it}, \quad \epsilon_{it} \sim N(0, 1). \]

where \( X_{it} \) includes:

- Average \( I/Y \)
- Average \( \gamma_{Pop} \)
- \( H_0 \) (secondary enrollment)
- \( Y_0 \) (initial output)

• Barlevy (AER, (2004)) adds \( \sigma_{\ln(I/Y)} \)
<table>
<thead>
<tr>
<th>Variables</th>
<th>(1) 92–Country</th>
<th>(2) 92–Country</th>
<th>(3) OECD</th>
<th>(4) OECD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>0.07 (3.72)</td>
<td>0.08 (3.73)</td>
<td>0.16 (5.73)</td>
<td>0.16 (4.48)</td>
</tr>
<tr>
<td>$\sigma_i$</td>
<td>$-0.21 (-2.61)$</td>
<td>$-0.109 (-1.22)$</td>
<td>$-0.39 (-1.92)$</td>
<td>$-0.401 (-1.93)$</td>
</tr>
<tr>
<td>Average $I/Y$</td>
<td>0.13 (7.63)</td>
<td>0.12 (6.99)</td>
<td>0.07 (2.76)</td>
<td>0.071 (2.67)</td>
</tr>
<tr>
<td>Average $\gamma_{Pop}$</td>
<td>$-0.06 (-0.38)$</td>
<td>$-0.115 (-0.755)$</td>
<td>0.21 (0.70)</td>
<td>0.230 (0.748)</td>
</tr>
<tr>
<td>$H_0$</td>
<td>0.0008 (1.18)</td>
<td>0.0007 (1.03)</td>
<td>0.0001 (2.00)</td>
<td>0.0001 (1.954)</td>
</tr>
<tr>
<td>$Y_0$</td>
<td>$-0.009 (-3.61)$</td>
<td>$-0.009 (-3.53)$</td>
<td>$-0.017 (-5.70)$</td>
<td>$-0.017 (-4.7445)$</td>
</tr>
<tr>
<td>$\sigma_{\ln(I/Y)}$</td>
<td>-</td>
<td>$-0.023 (1.81)$</td>
<td></td>
<td>$0.007 (0.22)$</td>
</tr>
</tbody>
</table>

Note: t-statistics in parentheses
Source: Columns (1) and (3) Ramey and Ramey (1995)
Columns (2) and (4), Barlevy (2002)
• Kroft and Lloyd-Ellis (2002) estimate

\[
\gamma_{ist} = \beta X_{it} + \lambda w \sigma_{iw} + \lambda_b \sigma_{ib} + \nu_{ist},
\]

\[
\nu_{ist} = \sigma_{iw} \varepsilon_{it} + \mu_{is}, \quad \varepsilon_{it} \sim N(0, 1),
\]

\[
\mu_{is} = \begin{cases} 
\mu_{ie} \quad \text{with probability } p_i = \frac{T_{ie}}{T} \\
\mu_{ir} \quad \text{with probability } 1 - p_i
\end{cases}
\]

• Regime switching

• $\sigma_{ib}$ is the standard deviation of $\mu_{is}$
<table>
<thead>
<tr>
<th>Independent Variable</th>
<th>92-Country Sample (2,208 observations)</th>
<th>OECD Sample (888 observations)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>0.00132 (0.022)</td>
<td>0.095 (1.89)</td>
</tr>
<tr>
<td>Within-phase volatility ($\sigma_{iw}$)</td>
<td>2.63 (4.69)</td>
<td>0.90 (1.44)</td>
</tr>
<tr>
<td>Between-phase volatility ($\sigma_{ib}$)</td>
<td>-2.65 (-6.35)</td>
<td>-1.11 (-2.33)</td>
</tr>
<tr>
<td>Average investment share of GDP</td>
<td>-0.01 (-0.26)</td>
<td>-0.004 (-0.073)</td>
</tr>
<tr>
<td>Average population growth rate</td>
<td>0.58 (1.24)</td>
<td>0.28 (0.62)</td>
</tr>
<tr>
<td>Initial human capital</td>
<td>0.001 (0.66)</td>
<td>-0.000001 (-0.096)</td>
</tr>
<tr>
<td>Initial per capita GDP</td>
<td>0.002 (0.25)</td>
<td>-0.00008 (-1.30)</td>
</tr>
</tbody>
</table>

Note: t-statistics in parentheses.
Source: Kroft and Lloyd-Ellis (2002).
<table>
<thead>
<tr>
<th>Independent Variable</th>
<th>(1)</th>
<th>(2)</th>
<th>(3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volatility ( (\sigma_i) )</td>
<td>-2.772</td>
<td>-1.700</td>
<td>-0.270</td>
</tr>
<tr>
<td></td>
<td>(0.282)</td>
<td>(0.645)</td>
<td>(0.091)</td>
</tr>
<tr>
<td>GDP per capita (1960)</td>
<td>-2.229</td>
<td>-1.856</td>
<td>-0.953</td>
</tr>
<tr>
<td></td>
<td>(0.235)</td>
<td>(0.422)</td>
<td>(0.220)</td>
</tr>
<tr>
<td>Human capital (1960)</td>
<td>0.037</td>
<td>0.040</td>
<td>0.026</td>
</tr>
<tr>
<td></td>
<td>(0.015)</td>
<td>(0.018)</td>
<td>(0.017)</td>
</tr>
<tr>
<td>Average investment share of GDP</td>
<td>0.083</td>
<td>0.143</td>
<td>0.120</td>
</tr>
<tr>
<td></td>
<td>(0.013)</td>
<td>(0.021)</td>
<td>(0.024)</td>
</tr>
<tr>
<td>Average population growth rate</td>
<td>-0.624</td>
<td>-0.562</td>
<td>-0.465</td>
</tr>
<tr>
<td></td>
<td>(0.153)</td>
<td>(0.205)</td>
<td>(0.465)</td>
</tr>
<tr>
<td>Volatility * GDP</td>
<td>0.340</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>(0.036)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Volatility * GDP (1960)</td>
<td>-</td>
<td>0.212</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(0.082)</td>
<td></td>
</tr>
<tr>
<td>Volatility * M3/Y</td>
<td>-</td>
<td>-</td>
<td>0.004</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>(0.001)</td>
</tr>
<tr>
<td>( R^2 )</td>
<td>0.77</td>
<td>0.58</td>
<td>0.57</td>
</tr>
</tbody>
</table>

Note: Sample 1950-1998. Robust standard errors in parentheses
Source: Fatás (2001)
Growth: Model

- Planner’s problem.

- Representative agent has preferences given by

\[ E\left\{ \int_0^\infty e^{-\rho t} \frac{c_t^{1-\theta}}{1-\theta} dt \right\} \]

- Technology: Two types of technologies to produce consumption (two sector model)

\[ dk_t = \left( (A - \delta_k)k_t - c_{1t} \right) dt + \sigma_k k_t dW_t + \eta_k k_t dZ^k_t, \]

\[ db_t = \left( (r - \delta_b)k_t - c_{2t} \right) dt + \sigma_b b_t dW_t + \eta_b b_t dZ^b_t, \]

- \((W_t, Z^k_t, Z^b_t)\) is a vector of three independent standard Brownian motion processes.
Let \( x_t = k_t + b_t \) and assume that capital is malleable (i.e. can be shifted across sectors instantaneously)

- Planner’s problem:

\[
\max E \left\{ \int_0^\infty e^{-\rho t} \frac{c_t^{1-\theta}}{1-\theta} dt \right\}
\]

subject to

\[
dx_t = \left[ (\alpha_t (A - \delta_k) + (1 - \alpha_t)(r - \delta_b))x_t - c_t \right] dt + \left[ (\alpha_t \sigma_k + (1 - \alpha_t) \sigma_b) dW_t + \alpha_t \eta_k dZ_t^k + (1 - \alpha_t) \eta_b dZ_t^b \right] x_t.
\]

\(- \alpha_t \equiv k_t/x_t\)

- The HJB equation is

\[
\rho V(x) = \max_{c,\alpha} \left[ \frac{c^{1-\theta}}{1-\theta} + V'(x)(\mu(\alpha) x - c) + \frac{V''(x)x^2}{2} \sigma^2(\alpha) \right],
\]
where

\[ \mu(\alpha) = r + \alpha(A - r) - (\alpha \delta_k + (1 - \alpha)\delta_b), \]
\[ \sigma^2(\alpha) = (\alpha \sigma_k + (1 - \alpha)\sigma_b)^2 + \alpha^2 \eta_k^2 + (1 - \alpha)^2 \eta_b^2. \]

- Solution is

\[ V(x) = v \frac{x^{1-\theta}}{1 - \theta} \]

where

\[ v = \left[ \frac{\rho - (1 - \theta)[\mu(\alpha^*) - \delta(\alpha^*) - \theta \sigma^2(\alpha^*)]}{\theta} \right]^{-\theta} \]
\[ \delta(\alpha) = \alpha \delta_k + (1 - \alpha)\delta_b \]
• Optimal policies

\[ \alpha^* = \frac{A - \delta_k - (r - \delta_b) - \sigma_b(\sigma_k - \sigma_b) + \eta^2_b}{(\sigma_k - \sigma_b)^2 + \eta^2_b + \eta^2_k}, \]

\[ c = \frac{\rho - (1 - \theta)[\mu(\alpha^*) - \delta(\alpha^*) - \theta\frac{\sigma^2(\alpha^*)}{2}]}{\theta} x \]

- Existence requires \( \rho - (1 - \theta)[\mu(\alpha^*) - \delta(\alpha^*) - \theta\frac{\sigma^2(\alpha^*)}{2}] > 0 \)

\[ dx_t = \left[ \frac{\mu(\alpha^*) - (\delta(\alpha^*) + \rho)}{\theta} - (1 - \theta)\frac{\sigma^2(\alpha^*)}{2} \right] x_t dt + 
\left[ (\alpha^* (\sigma_k - \sigma_b) + \sigma_b) dW_t + \alpha_k^* \eta_k dZ^k_t + (1 - \alpha^*) \eta_b dZ^b_t \right] x_t, \]

- The instantaneous growth rate of the economy, \( \gamma \), and its variance,
\( \sigma_{\gamma}^2 \), satisfy

\[
\gamma = \frac{\mu(\alpha^*) - (\delta(\alpha^*) + \rho)}{\theta} - (1 - \theta)\frac{\sigma^2(\alpha^*)}{2},
\]

\[
\sigma_{\gamma}^2 = (\alpha^*(\sigma_k - \sigma_b) + \sigma_b)^2 + \alpha^2 \eta_k^2 + (1 - \alpha^*)^2 \eta_b^2.
\]

- In general the same factors that influence \( \gamma \) also affect \( \sigma_{\gamma}^2 \)!
One Sector $Ak$ Model

- The technology is given by

$$dk_t = [(\hat{A} - \delta_k)k_t - c_t]dt + \sigma_y\hat{A}k_t dW_t.$$ 

where we view the law of motion for capital as

$$dk_t = -\delta_k k_t dt + dX_{kt},$$

- This is a special case defined by

$$\eta_b = \sigma_b = \eta_k = \sigma_k = 0$$

$$A = \hat{A} - \delta_k$$

$$\sigma_k = \sigma_y A.$$ 

and pick $r$ so that $\alpha^* = 1$
The pair \((\gamma, \sigma_{\gamma})\) is
\[
\gamma = \frac{\hat{A} - (\rho + \delta_k)}{\theta} - (1 - \theta)\frac{\sigma_{\gamma}^2}{2},
\]
\[
\sigma_{\gamma}^2 = \sigma_{y}^2 \hat{A}^2.
\]

The relationship between \(\gamma\) and \(\sigma_{\gamma}\) (as \(\sigma_{y}^2\) varies) depends on the value of \(\theta\).

- If \(0 < \theta < 1\), substitution effect dominates (negative).
- If \(1 < \theta\), income effect dominates (positive).

The equilibrium interest rate is
\[
r^* = \hat{A} - \delta_k - \theta \sigma_{y}^2 \hat{A}^2
\]
Two Sector $A_k$: Model

- The technology is given by
  
  \[ dk_t = (A_k t - c_{1t})dt + \sigma_y A_k t dW_t. \]

  \[ db_t = (r b_t - c_{2t})dt \]

  with

  \[ c_t = c_{1t} + c_{2t} \]

- This is a special case defined by

  \[ \eta_b = \sigma_b = \eta_k = 0 \]

  \[ A - \theta \sigma_k^2 < r < A \]

  \[ A = \hat{A} - \delta_k \]

  \[ \sigma_k = \sigma_y A. \]
• The pair \((\gamma, \sigma_\gamma)\) is

\[
\gamma = \frac{r - \rho}{\theta} + \left( \frac{A - r}{\theta \sigma_k} \right)^2 \frac{1 + \theta}{2},
\]

\[
\sigma_{\gamma}^2 = \left( \frac{A - r}{\theta \sigma_k} \right)^2.
\]

• The relationship between \(\gamma\) and \(\sigma_\gamma\) (as \(\sigma_k^2\) varies) is independent of \(\theta\).

• Let

\[
\hat{r} = A - \frac{\theta \sigma_k^2}{1 + \theta}.
\]

then

\[
\frac{\partial \gamma}{\partial r} \geq 0 \iff r \geq \hat{r},
\]
and

\[ \frac{\partial \sigma_\gamma^2}{\partial r} < 0 \]

- The relationship between \( \gamma \) and \( \sigma_\gamma \) (as \( r \) varies) has a U-shape.

- Can this explain nonlinearities?

- In low \( \sigma_\gamma \) countries (high \( r \)), fluctuations are “detrimental” to growth.

- In high \( \sigma_\gamma \) countries (low \( r \)), fluctuations are “beneficial” for growth.
Two Sector $A_k$ Model with Correlated Shocks

- Similar technology as before

\[ dk_t = (A_k - c_{1t})dt + \sigma k_t dW_t + \eta k_t dZ^k_t. \]

\[ db_t = (r b_t - c_{2t})dt + \sigma b_t dW_t \]

- The interpretation is that there is an aggregate shock ($W_t$) that affects both sectors, while the $A$ sector displays, in addition, a sector specific shock.
The solution is
\[ \alpha^* = \frac{A - r}{\theta \eta^2}, \]
\[ \gamma = \frac{r - \rho}{\theta} - (1 - \theta)\frac{\sigma^2}{2} + \left( \frac{A - r}{\theta \eta} \right)^2 \frac{1 + \theta}{2}, \]
\[ \sigma_{\gamma}^2 = \sigma^2 + \left( \frac{A - r}{\theta \eta} \right)^2. \]

• An increase in \( \sigma \).
  
  – Hits both sectors
  
  – No sectoral reallocation
  
  – Increases \( \sigma_{\gamma} \)
  
  – Effect depends on \( \theta \).
• A decrease in $r$.
  
  – Increases in $\sigma_\gamma$.
  
  – Nonmonotonic impact on $\gamma$. [For high $r$, negative]
  
  – Relationship between $\sigma_\gamma$ and $\gamma$ is an inverted U-shape.

• An increase in $\eta$.
  
  – Lowers share of $k$.
  
  – Lowers $\sigma_\gamma$. This change increases the ‘riskiness’ of the $A$ technology and results in a portfolio reallocation as the representative agent decreases the share of capital in the high return sector (technology). The change implies that $\sigma_\gamma$ and $\gamma$ decrease. Thus, differ
• The correlation between the two sectoral shocks is

\[ \nu = \frac{\sigma}{(\sigma^2 + \eta^2)^{1/2}} \]
• Experiment: Change \((\sigma, \eta)\) in such a way that the variance of the growth rate is unchanged.

\[
\sigma^2_\gamma = \sigma^2 + \left( \frac{A - r}{\theta \eta} \right)^2
\]

• If different economies have different \(\sigma^2\) with the same \(\sigma^2_\gamma\) (of course, \(\eta\) adjusts) their growth rates are

\[
\nu = \left( 1 + \left( \frac{A - r}{\theta} \right)^2 \frac{1}{\sigma^2 (\sigma^2_\gamma - \sigma^2)} \right)^{-1},
\]

\[
\gamma = \frac{r - \rho}{\theta} - \sigma^2 + \frac{1 + \theta}{2} \sigma^2_\gamma.
\]

• Mapping from \(\sigma^2_\gamma\) to \(\gamma\) is a correspondence!

• Higher \(\sigma^2\) \(\rightarrow\) lower \(\nu\) and lower \(\gamma\)
Physical and Human Capital

- Technology

\[
\begin{align*}
dk_t &= \left( [F(k_t, h_t) - \delta_k k_t - x_t - c_t] - \delta_k k_t - x_t - c_t \right) dt + \sigma_y F(k_t, h_t) dW_t, \\
 dh_t &= -\delta_h h_t + x_t dt + \sigma_h h_t dW_t + \eta h_t dZ_t,
\end{align*}
\]

- Let \( x_t = k_t + h_t \), and \( \alpha_t = k_t/x_t \).

- Feasibility is

\[
\begin{align*}
 dx_t &= \left( [F(\alpha_t, 1 - \alpha_t) - (\delta_k \alpha_t + \delta_h (1 - \alpha_t))] x_t - c_t \right) dt + \\
&\quad \quad \sigma_y F(\alpha_t, 1 - \alpha_t) x_t dW_t + \sigma_h (1 - \alpha_t) x_t dW_t \\
&\quad \quad + \eta (1 - \alpha_t) x_t dZ_t.
\end{align*}
\]
• The HJB equation is

\[ \rho V(x) = \max_{c,\alpha} \left[ \frac{c^{1-\theta}}{1-\theta} + V'(x)[(F(\alpha, 1-\alpha) - \delta(\alpha))x_t - c_t] + \frac{V''(x)x^2}{2}\sigma^2(\alpha) \right], \]

where

\[ \delta(\alpha) = \delta_k\alpha + \delta_h(1-\alpha), \]
\[ \sigma^2(\alpha) = \sigma_y^2 F(\alpha, 1-\alpha)^2 + \sigma_h^2 (1-\alpha)^2 + \eta^2 (1-\alpha)^2 + \sigma_y \sigma_h F(\alpha, 1-\alpha)(1-\alpha). \]

• The solution to the HJB equation is

\[ V(x) = v \frac{x^{1-\theta}}{1-\theta} \]

\[ \rho = \theta v^{-1/\theta} + (1-\theta)\{F(\alpha, 1-\alpha) - \delta(\alpha) - \frac{\theta}{2}\sigma^2(\alpha)\}, \]

\[ \alpha = \arg \max (1-\theta)\{F(\alpha, 1-\alpha) - \delta(\alpha) - \frac{\theta}{2}\sigma^2(\alpha)\}. \]
• Implications for growth

\[ \gamma = F(\alpha, 1 - \alpha) - \delta(\alpha) - \nu^{-1/\theta}, \]
\[ \sigma^2_{\gamma} = \sigma^2_y F(\alpha, 1 - \alpha)^2 + \sigma^2_h (1 - \alpha)^2 + \eta^2 (1 - \alpha)^2 + \sigma_y \sigma_h F(\alpha, 1 - \alpha)(1 - \alpha) \]

• Example:

\[ F(x, y) = Ax^\omega y^{1-\omega}, \quad 0 < \omega < 1. \]

\[ \sigma_h = \eta = 0 \]

\[ \delta_k = \delta_h \]

• Solution. First order condition (\alpha)

\[ \phi(\alpha) \hat{F}(\alpha)[1 - \theta \sigma^2_y \hat{F}(\alpha)] = 0, \]
where

\[
\hat{F}(\alpha) \equiv A\alpha^\omega (1 - \alpha)^{1-\omega}, \\
\phi(\alpha) = \frac{\omega}{\alpha} - \frac{1 - \omega}{1 - \alpha}.
\]

- The second order condition

\[-\omega(1 - \omega)[\alpha^{-2} + (1 - \alpha)^{-2}]\hat{F}(\alpha)[1 - \theta\sigma_y^2\hat{F}(\alpha)] - \theta\sigma_y^2\hat{F}(\alpha)^2\phi(\alpha) < 0.\]

- Since \(\hat{F}(\alpha) > 0\), the solution is either \(\phi(\alpha) = 0\) [\(\alpha^* = \omega\)], or \(\hat{F}(\alpha^*) = 1/\theta\sigma_y^2\)

- **Case A**: \(\sigma_y^2 \leq \frac{1}{\theta\hat{F}(\omega)}\).

  - This implies \(1 - \theta\sigma_y^2\hat{F}(\alpha) > 0\), for all \(\alpha\).
- Solution is \( \alpha^* = \omega \).

- **Case B:** \( \sigma_y^2 > \frac{1}{\theta \hat{F}(\omega)} \)

- Two solutions to the equation \( 1 = \theta \sigma_y^2 \hat{F}(\alpha). \) \([\alpha^- < \omega < \alpha^+]\).

- The second order condition holds

- The optimal \((\gamma, \sigma_\gamma)\) pairs satisfy (in Case A)

\[
\gamma_A = \frac{\hat{F}(\omega) - (\rho + \delta)}{\theta} - \frac{1 - \theta}{2} \sigma_y^2 \hat{F}(\omega)^2, \\
\sigma_{\gamma A} = \sigma_y \hat{F}(\omega),
\]

- \( \Delta \sigma_y > 0 \implies \Delta \sigma_{\gamma A} > 0 \implies \Delta \gamma_A < 0 \) (if \( 0 < \theta < 1 \)) [negative in \((\gamma, \sigma_\gamma)\)]
• The optimal \((\gamma, \sigma_\gamma)\) pairs satisfy (in Case B)

\[
\gamma_B = \frac{1}{\theta} \left[ \frac{1 + \theta}{2} \frac{1}{\theta \sigma_y^2} - (\rho + \delta) \right],
\]

\[
\sigma_{\gamma B} = \frac{1}{\theta \sigma_y},
\]

- \(\Delta \sigma_y > 0 \implies \Delta \sigma_{\gamma B} < 0 \implies \Delta \gamma_A < 0\). [positive in \((\gamma, \sigma_\gamma)\)]

• If \(0 < \theta < 1\) then

- Case A implies a negative relationship between \(\sigma_\gamma\) and \(\gamma\) (as \(\sigma_y\) varies).

- Case B implies a positive relationship between \(\sigma_\gamma\) and \(\gamma\) (as \(\sigma_y\) varies)

• For any \(\sigma_\gamma\) there are two values of \(\sigma_y\) (one corresponding to Case A, and the other to Case B) that result in the same variance of the growth rate, but different average growth rates.
Figure 1: The mapping between $\sigma_\gamma$ and $\gamma$. $[0 < \theta < 1]$
• The case $\theta > 1$ is similar
Figure 2: Figure 2: The mapping between $\sigma_\gamma$ and $\gamma$. [$\theta > 1$]
Recessions as Opportunities to Invest in Human Capital

- Are bad times (for the economy) good times for graduate programs?

- Are unemployed enjoying leisure or producing human capital?
Opportunity Cost: Model

• Preferences

\[ U = E \left[ \int_0^\infty e^{-\rho t} \frac{c_t^{1-\theta}}{1-\theta} dt \mid F_0 \right] \]

• Technology

\[ c_t + x_t \leq z_t A k_t^\alpha (n_t h_t)^{1-\alpha}, \]
\[ dk_t \leq [z_t A k_t^\alpha (n_t h_t)^{1-\alpha} - k_t - c_t]dt. \]
\[ dh_t = [1 - \delta + B(1 - n_t)]h_t dt + \sigma_t [1 - \delta + B(1 - n_t)]h_t dW_t, \]
Opportunity Cost: Solution

• Given, $n_t h_t$ (and full depreciation), $k$ is chosen to maximize output.

\[ c_t = A^* \hat{z}_t n_t h_t, \]

where $A^* = (A\alpha)^{1/(1-\alpha)}(\alpha^{-1} - 1)$ and $\hat{z}_t = z_t^{1/(1-\alpha)}$.

• Conjecture:

\[ V(\hat{z}_t, h_t) = v(\hat{z}_t h_t)^{1-\theta} \]

• The HJB equation is

\[
\rho v(\hat{z}h)^{1-\theta} \frac{1}{1-\theta} = \max_x \left\{ \frac{[A^*(\mu - x)\hat{z}h]^{1-\theta}}{1-\theta} + v(\hat{z}h)^{1-\theta} x - v(\hat{z}h)^{1-\theta} \theta \frac{\sigma^2 h x^2}{2} \right\}
\]

where $\mu \equiv 1 - \delta + B$, and $x = 1 - \delta + B(1 - n)$. 
• Choosing $x$ is equivalent to choosing $n$.

• Solution is

$$x^2 = \frac{2(1 + \mu \sigma^2_h)}{(1 + \theta)\sigma^2_h} x + \frac{2(\rho - \mu)}{\theta(1 + \theta)\sigma^2_h}.$$

• Bounded utility requires (even in the case $\sigma_h = 0$), $\rho - \mu > 0$.

• Positive root implies $\Delta \sigma_h > 0 \implies \Delta x < 0$.

• The stochastic process for $h_t$ is given by

$$dh_t = xh_t dt + \sigma_h h_t dW_t$$
• Assume

\[ dz_t = z_t(\sigma_w dW_t + \sigma_m dM_t), \]

then

\[ d\hat{z}_t = \frac{\alpha}{(1 - \alpha)^2} \frac{\sigma_w^2 + \sigma_m^2}{2} \hat{z}_t dt + \frac{\alpha}{(1 - \alpha)} \hat{z}_t (\sigma_w dW_t + \sigma_m dM_t). \]

• Equilibrium:

\[ \frac{dc_t}{c_t} = \frac{dh_t}{h_t} + \frac{d\hat{z}_t}{\hat{z}_t} + \frac{\alpha x}{(1 - \alpha)} \sigma_h \sigma_w dt, \]

\[ \gamma = x (1 + \alpha \frac{\sigma_h \sigma_w}{(1 - \alpha)}) + \alpha \frac{\sigma_w^2 + \sigma_m^2}{(1 - \alpha)^2} + \frac{\alpha}{(1 - \alpha)} \sigma_w^2 + \sigma_m^2 + \frac{\alpha}{(1 - \alpha)} \sigma_w^2 + \sigma_m^2 (1 - \alpha) \sigma_m^2 \]
• The share of the time allocated to human capital formation—the engine of growth in this economy—is independent of the variability of the technology shock in the goods sector, as measured by \((\sigma_w, \sigma_m)\).

• High \((\sigma_w, \sigma_m)\) economies are also high growth economies. Thus, if cross-country differences in \(\sigma_\gamma\) are mostly due to differences in \((\sigma_w, \sigma_m)\), the model implies a positive correlation between the standard deviation of the growth rate and mean growth.

• It can be shown that increases in \(\sigma_h\) result in decreases in \(\sigma_h x\). Thus, if countries differ in this dimension the model also implies a positive relationship between \(\sigma_\gamma\) and \(\gamma\).

• In the model, investment in physical capital (as a fraction of output) is \(\alpha\), independently of the distribution of the shocks. Thus, there is no sense that a regression that shows that variability does not affect the rate of investment provides evidence against the role of shocks in development.
• This lack of (measured) effect on both physical and human capital investment should not be interpreted as evidence against the proposition that incentives for human or physical capital accumulation matter for growth. It is easy enough to include a tax/subsidy to the production of human capital —consider a policy that affects $B$— and it follows that this policy affects growth.
Fiscal Policy and Growth

- Reference: Eaton (ReStud, (1981))
- Interplay between policy and real shocks
Fiscal Policy and Growth: Model

- Preferences:

\[ U = E \left[ \int_0^\infty e^{-\rho t} \frac{c_t^{1-\theta}}{1-\theta} dt \mid F_0 \right]. \]

- Technology

\[ dY_t = (A k_t - c_t) dt + \sigma A k_t dW_t \]
\[ dG_t = g A k_t dt + g' \sigma A k_t dW_t \]
\[ dT_t = \tau A k_t dt + \tau' \sigma A k_t dW_t. \]

Given that

\[ dk_t = dY_t - dG_t - c_t dt \]

aggregate feasibility is

\[ dk_t = ((1 - g) A k_t - c_t) dt + (1 - g') \sigma A k_t dW_t \quad (7) \]
• The government finances its deficit by issuing bonds.

\[ B_t + dG_t - dT_t = p_t dB_t \]

• Let \( h \) be the ratio of the value of government bonds to the capital stock. Thus,

\[ h_t = \frac{p_t B_t}{k_t} \]

• Rates of return:

  - Capital (it is the after tax marginal product)

\[ \tilde{r}_{kt} = (1 - \tau)Adt + (1 - \tau')\sigma AdW_t \]

  or

\[ \tilde{r}_{kt} = r_k dt + \sigma_k dW_t \]
- Bonds

\[ \tilde{r}_{bt} = r_b dt + \sigma_b dW_t \]

where \((r_b, \sigma_b)\) will be determined in equilibrium.

- The evolution of the budget constraint. Let

\[ b_t = \frac{h_t}{1 + h_t} = \frac{p_t B_t}{p_t B_t + k_t}. \]

Conjecture that \(b_t = b\). Then, the budget constraint is

\[
\begin{align*}
\frac{dx_t}{dt} &= \left[ ((1 - b)(1 - \tau)A + b r_b)x_t - c_t \right] dt + \\
&\quad \left[ ((1 - b)(1 - \tau')\sigma A + b \sigma_b)x_t \right] dW_t.
\end{align*}
\]

In equilibrium, it must be the case that

\[ k_t = (1 - b)x_t. \]

From the budget constraint, it follows that

\[
\begin{align*}
\frac{dk_t}{dt} &= \left[ ((1 - b)(1 - \tau)A + b r_b) - (1 - b)c_t/k_t \right] k_t dt + \\
&\quad \left[ ((1 - b)(1 - \tau')\sigma A + b \sigma_b)k_t \right] dW_t.
\end{align*}
\]
It follows that the stochastic and the deterministic components of equations (7) and (8) must be equal. This requires

\[(1 - b)(1 - \tau)A + br_b - (1 - b)c_t/k_t = (1 - g)A - c_t/k_t\]

\[(1 - b)(1 - \tau')\sigma A + b\sigma_b = (1 - g')\sigma A\]

or,

\[r_b = (1 - \tau)A - c/k - A(g - \tau)/b\]

\[\sigma_b = [(1 - \tau' - (g' - \tau')/b)]\sigma A\]

Note that if \(g' = \tau'\) then \(\sigma_b = \sigma_k = (1 - \tau')\sigma A\) and capital and bonds are equally risky.

- The optimal consumption-saving problem. The consumer solves

\[\max U = E \left[ \int_0^\infty e^{-\rho t} \frac{c_t^{1-\theta}}{1 - \theta} dt \mid F_0 \right]\]
subject to

\[ dx_t = \left[ ((1 - b)r_k + br_b)x_t - c_t \right] dt + \left[ (1 - b)\sigma_k + b\sigma_b \right] x_t dW_t. \]

Instead of solving this problem directly (which is not hard), let’s first transform the problem into a “Merton” problem with one safe and one risky asset. Let a fraction \( \omega_i \) denote the fraction of wealth held in the risky asset. The remaining fraction \( (1-\omega_i) \) is allocated in proportions \( \hat{b} \) and \( (1-\hat{b}) \) to debt and capital respectively. Let

\[ \hat{b} = \frac{b(1 - \tau')}{(g' - \tau')}. \]

Note that this portfolio has mean return

\[ r_s = \hat{b}r_b + (1 - \hat{b})r_k \]

or

\[ r_s = (1 - \tau)A - \frac{(1 - \tau')}{(g' - \tau')} \frac{bc}{k} + A(g - \tau) \]
and standard deviation

\[ \sigma_s = \hat{b}\sigma_b + (1 - \hat{b})\sigma_k \]

or,

\[ \sigma_s = \sigma_k - (1 - \tau')\sigma A = 0. \]

Thus, this portfolio is safe.

- The consumer problem is equivalent to

\[
\max U = E \left[ \int_0^\infty e^{-\rho t} \frac{c_t^{1-\theta}}{1-\theta} dt \mid F_0 \right]
\]

subject to

\[
dx_t = \left[ (\omega_i r_k + (1 - \omega_i) r_s) x_t - c_t \right] dt + \\
\omega_i \sigma_k x_t dW_t.
\]
• Since this is similar to the portfolio problem that was discussed before, we simply state the solution

\[ \omega_i = \frac{r_k - r_s}{\theta \sigma_k^2} \]

and

\[ c_t = cx_t \]

where

\[ c = \frac{\rho - (1 - \theta)\left(\frac{r_k - r_s)^2}{2\theta \sigma_k^2} + r_s\right)}{\theta} \]

and

\[ b_i = \hat{b}(1 - \omega_i) \]

• Next we need to impose that, in equilibrium,

\[ b_i = b \]
and
\[
\frac{c_t}{k_t} = \frac{c}{(1 - b)}.
\]
These are two equations and two unknowns, \(b\) and \(c/k\).

- The solution is such that
\[
h = \frac{A(\tau - g) + \theta(1 - g')(g' - \tau')\sigma^2}{c}
\]
and, if the government issues debt, then it must be the case that \(h > 0\).

- If utility is log, then \(\sigma\) has no effect on \(c\). The impact of changes in \(\sigma\) on the demand for bonds depend on the sign of \((g' - \tau')\). [Intuition: uncertainty increases the demand for the safe asset. When \(g' > \tau'\), \(\sigma_b < \sigma_k\).]
• Solution. Let $\alpha$ be the share of capital. Then

$$\alpha = \frac{r_k - r_b - \sigma_b(\sigma_k - \sigma_b)}{(\sigma_k - \sigma_b)^2} \quad (9a)$$

$$c_t = \frac{\rho - (1 - \theta)[\alpha r_k + (1 - \alpha) r_b - \theta(\alpha \sigma_k + (1 - \alpha) \sigma_b)^2]}{} \quad (9b)$$

• The stock of capital evolves according to

$$dk_t = ((1 - g)A - \frac{ct}{kt})kt dt + \sigma(1 - g')Ak_t dW_t.$$ 

• Note that

$$\frac{ct}{kt} = c\frac{x_t}{kt} = c(1 + \frac{1 - \alpha}{\alpha}) = \frac{c}{\alpha}.$$
In equilibrium, the growth rate of private wealth and the growth rate of the capital stock are the same

\[ \alpha r_k + (1 - \alpha) r_b - c = (1 - g) A - \frac{c}{\alpha}, \]  

(10a)

\[ \alpha \sigma_k + (1 - \alpha) \sigma_b = \sigma (1 - g') A. \]  

(10b)

The system formed by the four equations described in (9) and (10)
Solution:

\[
\Delta_r = r_k - r_b, \\
\Delta_\sigma = \sigma_k - \sigma_b.
\]

then

\[
\alpha = \frac{\sigma_\gamma - \sigma_k + \Delta_\sigma}{\Delta_\sigma}, \\
\Delta_r = \theta \sigma_\gamma \Delta_\sigma
\]

Given that

\[
\gamma = \alpha r_k + (1 - \alpha) r_b - c, \\
\sigma_\gamma = \sigma (1 - g') A,
\]
• Equilibrium

\[
\gamma = \left( 1 - \tau \right) A - \rho \frac{1 - \theta - \tau' + \theta g'}{1 - g'} \sigma_{\gamma}^2 \tag{11a}
\]

\[
\sigma_{\gamma} = \sigma (1 - g') A. \tag{11b}
\]

• Properties of the solution:

- If \( \sigma \) varies (cross country), effect on \( \gamma \) depends on \( 1 - \theta - \tau' + \theta g' \).

- High \( \tau \) \( \implies \) low \( \gamma \). [Standard cost of capital effect.]

- High \( \tau' \) \( \implies \) high \( \gamma \). [Government shares risk. Lower effective \( \sigma_k \)]

- Average size of government \( (g) \), no impact on growth.

- Differences in the random component of gov’t consumption, \( g' \), induces positive correlation between \( \gamma \) and \( \sigma_{\gamma} \).
Summary:

\[ \alpha = \frac{r_k - r_b - \sigma_b(\sigma_k - \sigma_b)}{(\sigma_k - \sigma_b)^2}, \]

\[ c = \frac{\rho - (1 - \theta)[\alpha r_k + (1 - \alpha) r_b - \theta(\alpha \sigma_k + (1 - \alpha) \sigma_b)^2]}{\theta} \]

\[ \alpha r_k + (1 - \alpha) r_b - c = (1 - g) A - \frac{c}{\alpha} \]

\[ \alpha \sigma_k + (1 - \alpha) \sigma_b = \sigma (1 - g') A \]
Conclusion

- Uncertainty affects investment and consumption decisions, as well as equilibrium growth.

- Ex-ante (without a model) it is not easy to determine if higher instability increases or decreases investment and growth.