Topics in Dynamic Macroeconomics

Meeting I: Background and Investment

Teaching Notes. Indiana University 11/6/06-11/10/06

(Beware of the typos - do not quote!)

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Questions

- Does uncertainty (e.g. about demand) accelerates or delays investment?
- Can uncertainty affect asset prices even if all investors are risk neutral?
- Is the rate of technological progress faster in more stable economies?
- Is faster adoption of technologies associated with a better productivity (income) distribution?
- Are more flexible technologies more valuable in unstable economies?
- How does income variability affect consumption choices?
• Does inequality (in incomes) result in higher interest rates?

• Fluctuations and growth:
  – Do more unstable economies grow faster?
  – Evidence and tests?
  – How do government policies affect mean growth (and its variability)?

• Theme: Uncertainty and Economic Performance.
Methodology

• Study dynamic decision problems

• Basic tool: Stochastic Calculus

• It is a little different from regular calculus.

• Review:
  – Stochastic processes (continuous time)
  – Dynamic programming (continuous time)
Stochastic Processes: Discrete Time

• In discrete time

\[ X_t = \mu + X_{t-1} + \varepsilon_t \]

where

• \( \mu = \text{drift} \)

• \( \varepsilon_t \) are i.i.d.

• Another example: autoregressive process

\[ X_t = (1 - a) \mu + a X_{t-1} + \varepsilon_t \]
• These are Markov processes

\[ P[X_t \in A \mid X_{t-1}, X_{t-2}, \ldots] = P[X_t \in A \mid X_{t-1}] \]

• Martingale: is a stochastic process satisfying

\[
\begin{align*}
E[ & \mid X_t ] < \infty, \\
E[X_t \mid X_{t-1}] &= X_{t-1}
\end{align*}
\]

• Sometimes, it is useful to have a notation for the information set (filtration). Let \( \{ \mathcal{F}_t \} \) be a collection of (increasing) information sets. Then, the second condition for a martingale can be written as

\[
E[X_t \mid \mathcal{F}_{t-1}] = X_{t-1}
\]
Standard Wiener Process

• Let’s imagine the time between observations to become arbitrarily small, and consider a process that has no drift term and independent, normally distributed increments.

\[ W_{t+\Delta} = W_t + \varepsilon_{t+\Delta}, \quad \varepsilon_{t+\Delta} \sim N(0, \Delta) \]

• Let’s divide a time interval, \( T \), into \( n \) subintervals of size \( \Delta \). Thus, \( T = \Delta n \). Then,

\[ W_{t+T} - W_t = \sum_{j=1}^{n} \varepsilon_{t+j\Delta} \]

is also a normal random variable; i.e. \( W_{t+T} - W_t \sim N(0, n\Delta) \). This is the idea of a Brownian motion!
• More formally, a Standard Brownian Motion (SBM) is a stochastic process satisfying:

- \( W_t(\omega) \) is a continuous function of \( t \), with \( W_0(\omega) = 0 \).
- \( W_t - W_s \) is independent of information available at time \( s \) (i.e. \( \mathcal{F}_s \))
- \( W_t - W_s \sim N(0, t - s) \)

• Back to the discrete time. If we define a small interval of time as \( dt \), then

\[
dW_t = \lim_{dt \to 0} W_{t+dt} - W_t = \lim_{dt \to 0} \varepsilon_{t+dt}, \; \varepsilon_{t+dt} \sim N(0, dt)
\]

• Properties (if we treat \( dW_t \) as a well defined stochastic process...which is not)

1. \( E[dW_t] = 0 \),
2. $E[dW_t^2] = dt$,

3. $E[dW_t dt] = 0$ (it is of order $dt^{3/2}$)

4. $dW_t^2 = dt$ (this looks fishy, but it is true!)

5. $dW_t dt = 0$ (this also looks fishy, but it is (also) true!)

6. $W_t(\omega)$ is nowhere differentiable.

7. $W_t(\omega)$ is a process of unbounded variation.

8. $W_t(\omega)$ is a process of bounded quadratic variation.

- What is $dW_t$? Typically it stands for

$$\int_0^t dW_s = W_t$$
It turns out that defining stochastic integrals takes a bit of work. For our purposes, we will pretend that expressions like

\[ \int_0^t X_s dW_s \]

are well defined for suitable integrands \( X_s \). The key properties are:

- \( Y_t \equiv \int_0^t X_s dW_s \) is a martingale.

- \( E[\int_0^t X_s dW_s] = 0 \)
Common Processes

• A \((\mu, \sigma)\) Arithmetic Brownian motion \([\((\mu, \sigma)\) ABM]\)

\[
dX_t = \mu dt + \sigma dW_t, \quad \text{(stochastic differential equation)}
\]

or,

\[
X_t = X_0 + \int_0^t \mu ds + \int_0^t \sigma dW_s
\]

which implies that

\[
X_t = X_0 + \mu t + \sigma dW_t.
\]

It follows that

\[
X_t - X_0 \sim N(\mu t, \sigma^2 t)
\]

– Positive or negative.

– Variance of forecast goes to \(\infty\).
A \((\mu, \sigma)\) Geometric Brownian motion \([(\mu, \sigma)\ GBM]\)

\[
dX_t = \mu X_t dt + \sigma X_t dW_t,
\]
or,

\[
X_t = X_0 + \int_0^t \mu X_s ds + \int_0^t \sigma X_s dW_s
\]

which implies that (given some technical conditions)

\[
E[X_t] = X_0 + \int_0^t \mu E[X_s] ds.
\]

If \(E[X_t] = \phi(t)\) is a differentiable function of time, then

\[
\phi(t) = X_0 + \int_0^t \mu \phi(s) ds,
\]

\[
\dot{\phi}(t) = \mu \phi(t)
\]

The solution is

\[
E[X_t] = X_0 e^{\mu t}
\]

- \(X_t\) is nonnegative (it has an absorbing state at 0)
The instantaneous growth rate is $\mu \left( \frac{E[dX_t/X_t]}{dt} = \mu \right)$

The instantaneous standard deviation is $\sigma \left( E[(dX_t/X_t - \mu dt)^2/dt] = \sigma^2 \right)$
Ito’s Lemma

- Consider a (twice differentiable) function $f$, of time and a random variable $X_t$ that satisfies

\[ dX_t = \mu_t X_t dt + \sigma_t X_t dW_t \]

- How do we compute $df(t, X_t)$? Use Taylor’s theorem

\[
 f(t+dt, X+dX) = f(t, X) + f_t dt + f_x dX + \frac{1}{2}[f_{tt}(dt)^2 + f_{tX} dX dt + f_{XX}(dX)^2 + ..] 
\]

However, let’s consider dividing by $dt$ (dropping all terms in $(dt)^n$ for $n > 1$). This implies that:

- $(dt)^2 = 0$

- $dX dt = \mu_t X_t (dt)^2 + \sigma_t X_t dW_t dt = 0$
\[-(dX)^2 = \mu_t^2 X_t^2 (dt)^2 + 2\mu_t X_t \sigma_t X_t dW_t dt + \sigma_t^2 X_t^2 dt = \sigma_t^2 X_t^2 dt.\]

Then, we get that

\[f(t + dt, X + dX) = f(t, X) + f_t dt + f_x(\mu_t X_t dt + \sigma_t X_t dW_t) + \frac{1}{2} f_{XX} \sigma_t^2 X_t^2 dt,\]

or

\[df(t, X) = \left[ f_t + f_x \mu_t X_t + \frac{1}{2} f_{XX} \sigma_t^2 X_t^2 \right] dt + f_x \sigma_t X_t dW_t,\]

which is just another stochastic differential equation (SDE) which can be integrated to get

\[f(t, X_t) = f(0, X_0) + \int_0^t \left[ f_s + f_x \mu_s X_s + \frac{1}{2} f_{XX} \sigma_s^2 X_s^2 \right] ds + \int_0^t f_x \sigma_s X_s dW_s.\]
• The danger of not believing in Ito’s Lemma. Consider the SDE

\[ dX_t = W_t dW_t \]

Standard calculus would suggest that

\[ X_t = \frac{W_t^2}{2} \]

Is this right? Let’s use Ito’s Lemma and set \( f(t, W) = \frac{W^2}{2} \). Then, Ito’s Lemma implies

\[ df = [W_t \times 0] dt + \frac{1}{2} dt + W_t dW_t \neq W_t dW_t. \]

Let’s try

\[ X_t = f(t, W_t) = \frac{W_t^2}{2} - \frac{t}{2}. \]

In this case, Ito’s Lemma implies

\[ df = [W_t \times 0] dt - \frac{1}{2} dt + \frac{1}{2} dt + W_t dW_t = W_t dW_t. \]
• More than one variable. Let

\[ dX_t = \mu_t X_t dt + \sigma_t X_t dW_t, \]
\[ dY_t = \eta_t Y_t dt + \nu_t Y_t dZ_t, \]
\[ dW_t dZ_t = \rho dt \]

Let \( f(t, X, Y) \) be \( C^{1,2} \) then

\[
df = [f_t + f_x \mu_t X_t + f_Y \eta_t Y_t] dt + \left[ \frac{1}{2} (f_{XX} \sigma_t^2 X_t^2 + f_{YY} \nu_t^2 Y_t^2) + f_{XY} \rho \sigma_t X_t \nu_t Y_t \right] dt + f_X \sigma_t X_t dW_t + f_Y \nu_t Y_t dZ_t
\]
Dynamic Programming

- Consider a standard consumption-saving problem:

\[
\max E[\int_0^\infty e^{-\rho t} u(c_t) dt]
\]

subject to

\[
da_t = [ra_t + y - c_t] dt + \sigma a_t dW_t
\]

This is similar to the following discrete time problem

\[
a_{t+1} = a_t [1 + r + \sigma \varepsilon_{t+1}] + y - c_t.
\]

If we take a discrete time approximation to the continuous time problem, and consider intervals of time of length \( dt \), we have that the relevant version of Bellman’s equation is:

\[
V(a_t) = \max \{ \int_0^{dt} e^{-\rho s} ds u(c) + e^{-\rho dt} E[(V(a_t + da_t))] \}
\]
or,

\[ V(a_t)(1 - e^{-\rho dt}) = \max\left\{ \frac{1 - e^{-\rho dt}}{\rho} u(c) + e^{-\rho dt} E[(V(a_t + da_t) - V(a_t))] \right\} \]

or

\[ V(a_t)(1 - e^{-\rho dt}) = \max\left\{ \frac{1 - e^{-\rho dt}}{\rho} u(c) + e^{-\rho dt} E[dV(a_t)] \right\} \]

- Next, use Ito’s Lemma to compute \( E[dV(a_t)] \). Recall that, given the process

\[ da_t = \left[ ra_t + y - c_t \right] dt + \sigma a_t dW_t \]

and if \( V \) is \( C^2 \) (showing that this is the case is tricky!), we get

\[ dV(a) = \left[ V'(a)\mu a + \frac{1}{2} V''(a)\sigma^2 a^2 \right] dt + V'(a)\sigma a dW. \]
Taking expectations, we obtain

\[ E[dV(a)] = [V'(a)\mu a + \frac{1}{2}V''(a)\sigma^2 a^2]dt \]

- Back to the Bellman equation.

\[ V(a_t)(1-e^{-\rho dt}) = \max\left\{ \frac{1-e^{-\rho dt}}{\rho}u(c) + e^{-\rho dt}[V'(a)\mu a + \frac{1}{2}V''(a)\sigma^2 a^2]dt \right\} \]

- Divide both sides by \( dt \) and take the limit as \( dt \to 0 \) to get

\[ \rho V(a) = \max_c \{ u(c) + V'(a)(ra + y - c) + \frac{1}{2}V''(a)\sigma^2 a^2 \} \]
and this is the Hamilton-Jacobi-Bellman equation.
Investment: Irreversible and Indivisible

• Basic reference: McDonald and Siegel (QJE, (1986))

• Problem: A firm has to decide when to invest in a plant.
  – Demand (and profits) fluctuate. [Stochastic environment]
  – Once the investment has been made, the plant cannot be sold. [Irreversible investment]
  – Plants come in one size. [Indivisible investment.]

• Do firms in more unstable industries (countries) delay investment?

• What is the efficient investment criteria? (e.g. is it “invest as soon as the expected present discounted value of the project exceeds the cost”?)
- Tobin’s q.
- Firm valuation.
Investment: Economic Environment

• Profits (if the firm has invested) are given by $\pi_t$ where

$$d\pi_t = \mu \pi_t dt + \sigma \pi_t dW_t$$

• The interest rate is constant, $r$, and the cost of the investment project is $I > 0$.

• Let $X_t$ be the expected present discounted value of a firm that has invested in a project at time $t$. Thus,

$$X_t = E_t[\int_0^\infty e^{-r(s-t)}\pi_s ds] = \int_0^\infty e^{-r(s-t)}E_t[\pi_s] ds$$

$$= \int_0^\infty e^{-r(s-t)}\pi_t e^{\mu(s-t)} ds = \frac{\pi_t}{r - \mu}, \quad r - \mu > 0.$$
• Using Ito’s Lemma it follows that

\[ dX_t = \mu X_t dt + \sigma X_t dW_t. \]
Investment: Analysis

- **Optimal decision rule**: invest the first time that the value of the firm (after investment), \( X_t \), reaches some value, \( X^* \).

- The value of such a policy is given by
  \[
  V(X_0) = E[e^{-rT}(X_T - I)],
  \]
  where \( T \) is a stopping time defined by
  \[
  T = \inf\{t : X_t \geq X^*\}
  \]

- One approach: Since \( X_T = X^* \), then
  \[
  V(X_0) = (X^* - I)E[e^{-rT} \mid X_0]
  \]
  and the expectation of such a stopping time is known. Then, the optimal \( X^* \) is chosen.
• Alternative approach: Compute the value \textbf{before} the firm invests.

In this case the HJB equation is

$$rV(X) = \left\{ 0 + V'(X)\mu X + \frac{1}{2}V''(X)\sigma^2 X^2 \right\}$$

• Indifference

$$V(X^*) = X^* - I$$

• Guess

$$V(X) = \alpha_1 X^{\lambda_1} + \alpha_2 X^{\lambda_2}$$

• Under the guess, HJB is

$$r\alpha X^\lambda = \alpha \lambda \mu X^\lambda + \frac{1}{2} \alpha \lambda (\lambda - 1) \sigma^2 X^\lambda$$
or

\[ r - \mu \lambda = \frac{\sigma^2}{2} \lambda (\lambda - 1) \]

- Two roots: \( \lambda_1 < 0 < 1 < \lambda_2 \).

- If \( X_t = 0 \), then the project will never be built. Thus, \( V(0) = 0 \). This requires \( \alpha_1 = 0 \).

- For a fixed \( X^* \), value matching requires

\[ \alpha_2(X^*)^\lambda_2 = X^* - I \]

\[ V(X; X^*) = (X^* - I) \left( \frac{X}{X^*} \right)^\lambda_2 \]
• Optimal choice of $X^*$ is such that

$$\frac{\partial V(X; X^*)}{\partial X^*} \bigg|_{X=X^*} = 0$$

• The solution is

$$X^* = \frac{\lambda_2}{\lambda_2 - 1} I,$$

$$\pi^* = \frac{\lambda_2}{\lambda_2 - 1} (r - \mu) I$$

• The value of a firm that “owns” this investment option is

$$V(X) = \begin{cases} 
\frac{I}{\lambda_2 - 1} \left( \frac{X}{X^*} \right)^{\lambda_2} & \text{for } X \leq X^* \\
\frac{X - I}{X^*} & \text{for } X \geq X^* 
\end{cases}$$

• Properties:
\(-\Delta \sigma > 0 \implies \Delta \lambda_2 < 0 \implies \Delta \pi^* > 0. [\text{Higher uncertainty increases the profitability threshold.}]
\)

\(-\text{It can be shown (if } \mu - \sigma^2/2 > 0 \text{) that}\)

\[
E[T \mid X_0] = \frac{\ln(X^*/X)}{\mu - \sigma^2/2}
\]

Thus, more uncertainty, on average, results in delays in investment.

\(-\text{Consider two firms that have the same } X \text{ (assume } X \leq X^* \text{ in both cases) but belong to two different industries. Differentiating the } V(X) \text{ with respect to } \lambda_2 \text{ implies}\)

\[
\frac{dV(X; \lambda_2)}{d\lambda_2} = V(X; \lambda_2) \ln \left( \frac{X}{X^*} \right) < 0.
\]

The “inactive” firm in the more unstable industry (country?) is more valuable. Uncertainty affects asset prices even if investors are risk neutral.
• How do “standard” investment rules perform?

• “Invest the first time that the expected present discounted value of profits exceeds the cost of the project”.
  
  – This corresponds to: invest when \( X = I \).
  
  – We showed that the optimal rule is \( X^* > I \).
  
  – Is the rule approximately correct? No. \( \lim_{\sigma \to 0} \lambda_2 = 1 \implies \frac{X^*}{I} \to \infty \),

• “Invest whenever Tobin’s \( q \) is greater than 1.”
  
  – Average \( q = \frac{\text{Value of the firm}}{\text{Value of capital}} \implies q = \frac{X^*}{I} \).
  
  – Thus, for \( I < X < X^* \), this version of Tobin’s \( q \) would suggest that the firm is underinvesting.
- Marginal $q \equiv \text{“Change in the value of the firm/Value of capital”} = (X^* - V(X^*)) / I = 1$!
A Digression on Optimally Choosing the Boundary

For any fixed \( X^* \), let

\[
\alpha_2(X^*) = (X^* - I)(X^*)^{-\lambda_2}
\]

then, the value function can be written as

\[
V(X; X^*) = \alpha_2(X^*)X^{\lambda_2}.
\]

The value matching condition is

\[
V(X^*; X^*) = X^* - I.
\]

Totally differentiating

\[
\frac{\partial V(X; X^*)}{\partial X} \bigg|_{X=X^*} + \frac{\partial V(X; X^*)}{\partial X^*} \bigg|_{X=X^*} = 1.
\]

Since

\[
\frac{\partial V(X; X^*)}{\partial X^*} \bigg|_{X=X^*} = 0
\]
at an optimum, then optimality requires (in this case)

\[ \frac{\partial V(X; X^*)}{\partial X} \bigg|_{X=X^*} = 1. \]

In general, in optimal stopping problems the condition that determine the optimal choice of the boundary are labeled smooth pasting conditions. If the payoff after stopping is given by a function \( F(X) \), then the smooth pasting condition is

\[ \frac{\partial V(X)}{\partial X} \bigg|_{X=X^*} = \frac{\partial F(X)}{\partial X} \bigg|_{X=X^*} \]

The value matching condition is

\[ V(X^*) = F(X^*) \]
Investing in Technology Upgrades

- How often should a new technology be adopted?

- How does instability affect the cross sectional dispersion of productivity?
Investing in Technology Upgrades: Environment

- Current technology yields profits given by $x$. The best available technology results in profits given by $X_t$, where

$$dX_t = \mu X_t dt + \sigma X_t dW_t.$$ 

- If a new technology is acquired, the old technology is worthless (irreversibility)

- Cost of acquiring new technology: $cX$.

- Discount rate, $r$, satisfies: $1/c > r > \mu$. 
Investing in Technology Upgrades: Analysis

• **Conjecture**: Acquire the best available technology when $X_t = X^* (x)$, for some threshold $X^* (x)$.

• Let the expected present discounted value of profits of a firm with current technology $x$ when best technology is given by $X$ be $V (x, X)$. The HJB equation is

$$V (x, X) = \max \{ [ \int_0^t e^{-rs} ds ] x + e^{-rdt} E [(V (x, X + dX)) \}$$

or,

$$V (x, X) (1 - e^{-rdt}) = \max \{ \frac{1 - e^{-rdt}}{\rho} x + e^{-rdt} E [(V (x, X + dX) - V (x, X)) \}$$

or

$$V (x, X) (1 - e^{-rdt}) = \max \{ \frac{1 - e^{-rdt}}{\rho} x + e^{-rdt} E [(dV (x, X)) \}$$

where the latter is understood to be for a fixed $x$. 
• Next, use Ito’s Lemma to compute $E[(dV(x, X))]$. Recall that, given the process

$$dX_t = \mu X_t dt + \sigma X_t dW_t$$

and if $V$ is $C^2$, we get

$$dV(x, X) = \left[ \frac{\partial V(x, X)}{\partial X} \mu X + \frac{1}{2} \frac{\partial^2 V(x, X)}{\partial X^2} \sigma^2 X^2 \right] dt + \frac{\partial V(x, X)}{\partial X} \sigma X dW.$$  

Taking expectations, we obtain

$$E[dV(x, X)] = \left[ \frac{\partial V(x, X)}{\partial X} \mu X + \frac{1}{2} \frac{\partial^2 V(x, X)}{\partial X^2} \sigma^2 X^2 \right] dt$$

• Substitute this into the HJB equation to get

$$V(x, X)(1 - e^{-rdt}) = \max \left\{ \frac{1 - e^{-rdt}}{\rho} x + e^{-rdt} \left[ \frac{\partial V(x, X)}{\partial X} \mu X + \frac{1}{2} \frac{\partial^2 V(x, X)}{\partial X^2} \sigma^2 X^2 \right] dt \right\}$$
Dividing by $dt$ and taking limits as $dt \to 0$,

$$rV(x, X) = x + \frac{\partial V(x, X)}{\partial X} \mu X + \frac{1}{2} \frac{\partial^2 V(x, X)}{\partial X^2} \sigma^2 X^2 \quad (1)$$

- The value of the firm during the no investment period is the solution to the PDE (1).

- Boundary conditions:
  
  - Value matching (VM)
    
    $$V(x, X^*(x)) = V(X^*(x), X^*(x)) - cX^*(x) \quad (2)$$

  - Smooth pasting (SP)
    
    $$\left. \frac{\partial V(x, X)}{\partial X} \right|_{X=X^*} = \left. \frac{\partial V(x, X^*)}{\partial x} \right|_{X=X^*} + \left. \frac{\partial V(X^*, X)}{\partial X} \right|_{X=X^*} - c$$
or

\[ V_x(x, X^*(x)) = V_x(X^*(x), X^*(x)) + V_X(X^*(x), X^*(x)) - c. \] (3)

- **Conjecture**: The (homogeneous part of the) HJB is of the form

\[
rV(x, X) = \frac{\partial V(x, X)}{\partial X} \mu X + \frac{1}{2} \frac{\partial^2 V(x, X)}{\partial X^2} \sigma^2 X^2
\]

and the conjectured (form) of the solution is

\[ V^H(x, X) = \alpha(x) X^\lambda. \]

Under the guess the HJB is

\[
r \alpha(x) X^\lambda = \lambda \alpha(x) X^{\lambda - 1} \mu X + \frac{1}{2} \lambda (\lambda - 1) \alpha(x) X^{\lambda - 2} \sigma^2 X^2
\]

or,

\[
r - \lambda \mu = \frac{\sigma^2}{2} \lambda (\lambda - 1).
\]
The roots of this quadratic are

\[ \lambda_1 < 0 < 1 < \lambda_2. \]

What is the value of the firm if \( X_t = 0 \)? In that case it is optimal never to upgrade \((X_s = 0, \text{ for all } s > t)\), and

\[ V(x, 0) = \frac{x}{r}. \]

For this to be the case, it must be that the coefficient of \( \lambda_1 \) is 0.

• **Conjecture**: The particular solution to the HJB equation is:

\[ V^P(x, X) = \frac{x}{r}. \]

• **Conjecture**: The solution to the HJB is

\[ V(x, X) = \frac{x}{r} + \alpha_2(x)X^{\lambda_2}. \]
• Boundary Conditions:
  
  – Issue: The “continuation value” is endogenous.

  – Seek a fixed point in the space of functions $\alpha_2(x)$.

  – Economic intuition: Value should be homogeneous of degree one. This implies the following conjecture
    \[ V(x, X) = \frac{x}{r} + \alpha_2 x^{1-\lambda_2} X^{\lambda_2} \]

  – Given this (more refined) guess, the boundary conditions are:
    \[ \frac{x}{r} + \alpha_2 x^{1-\lambda_2} (X^*)^{\lambda_2} = \frac{X^*}{r} + \alpha_2 X^* - cX^* \]
    (4)

    \[ \alpha_2 \lambda_2 x^{1-\lambda_2} (X^*)^{\lambda_2-1} = \frac{1}{r} + \alpha_2 - c \]
    (5)

• Solution: Equations (4) and (5) are two equations in two unknowns
\((X^*, \alpha_2)\).

- The value of \(X^*\) should be proportional to \(x\). Let \(z^*x = X^*\). Then the system is

\[
\frac{1}{r} + \alpha_2(z^*)^{\lambda_2} = \frac{\dot{z}^*}{r} + \alpha_2z^* - cz^* ,
\]

\[
\alpha_2\lambda_2(z^*)^{\lambda_2 - 1} = \frac{1}{r} + \alpha_2 - c.
\]

- The solution is (solve for \(\alpha_2\) in (SP) and substitute into VM)

\[
\alpha_2 = \frac{1/r - c}{\lambda_2(z^*)^{\lambda_2 - 1} - 1}
\]

and

\[
\frac{1}{r} + \frac{1/r - c}{\lambda_2(z^*)^{\lambda_2 - 1} - 1}(z^*)^{\lambda_2} = \left(\frac{1}{r} - c\right)z^* + \frac{1/r - c}{\lambda_2(z^*)^{\lambda_2 - 1} - 1}\dot{z}^*
\]
which is equivalent to
\[
\frac{1}{1 - rc} = \frac{(\lambda_2 - 1)(z^*)^{\lambda_2}}{\lambda_2(z^*)^{\lambda_2 - 1} - 1} \equiv M(z^*, \lambda_2)
\]

\[
M(1, \lambda_2) = 1,
\lim_{z \to \infty} M(z, \lambda_2) = \infty,
\frac{\partial M(z, \lambda_2)}{\partial z} > 0.
\]

- Given \(1 - rc > 0\), there is a unique solution \(z^* > 1\).

- **Effect of multiple options.** If the firm could upgrade just once, the continuation payoff would be

\[
\frac{X^*}{r} - cX^*.
\]
In this case, the optimal $z^*$, denoted $\hat{z}$ solves

$$\frac{1}{1 - rc} = \frac{(\lambda_2 - 1)\hat{z}}{\lambda_2} \equiv \hat{M}(\hat{z}, \lambda_2).$$

Since for all $z \geq 1$,

$$M(z, \lambda_2) \geq \hat{M}(z, \lambda_2)$$

then

$$z^* < \hat{z}$$

*Result:* multiple investment options (the current investment competes with future investments) result in more frequent upgrades.

*• Effect of economic instability.* Standard (but painful!) algebra shows that

$$\frac{\partial M(z, \lambda_2)}{\partial \lambda_2} = \frac{(z)^{\lambda_2}}{\lambda_2(z)^{\lambda_2-1} - 1}[(z)^{\lambda_2-1} - 1 - (\lambda_2 - 1) \ln(z)] > 0.$$
Since $\Delta \sigma > 0 \implies \Delta \lambda_2 < 0$, if follows that

$$\Delta \sigma > 0 \implies \Delta z^* > 0.$$ 

Increased instability delays investment. The expected time between technology upgrades (investment) is

$$E[T(x)] = \frac{\ln(z^*)}{\mu - \sigma^2/2}$$ 

where

$$T(x) = \inf\{t : X_t = z^*x$$

- **Effect of technology (economic) growth.**
  
  $$\Delta \mu > 0 \implies \Delta \lambda_2 < 0 \implies \Delta z^* > 0$$

  Intuition: option value of waiting increases.

- **Cross sectional dispersion:** High $(\mu, \sigma)$ industries (countries) show more dispersion in productivity per worker (if one worker operates one machine).
• Average productivity: It can be shown (need to compute the stationary distribution to actually prove this!) that average productivity when the least productive plant has productivity $x_t$, is

$$\bar{X}_t = \int_0^{z^*} z x_t \frac{d z}{z^*} = \frac{1}{2} x_t z^*.$$

Since the lowest productivity increases at the $\mu$, variance does not affect mean productivity growth.

• Productivity (income) distribution: Lots of overtaking as all the firms (workers) upgrade to the best technology.

  - Alternative: change the cost of upgrading so that $X^*(x) \neq z x$, for any $z$. 

Investment: The Role of Flexibility

• Question: Are more flexible investment options more valuable in more unstable environments?

• Reference: Kort, Murto and Pawlina (working paper 2005)
Flexibility: Model

• An investment project can be completed in one or two stages.

• If completed in one stage ("Lumpy", $L$) it costs $I$. The productivity of the project is denoted by $R_2$.

• If completed in two stages ("Sequential", $S$) it has the following cost-return structure:
  
  – Stage 1: Costs $\alpha \kappa I$, and has productivity $R_1$.
  
  – Stage 2: Costs $(1 - \alpha) \kappa I$, and increases productivity by $R_2 - R_1$.

• Assumptions:
– The additional cost (cost of flexibility) is $\kappa > 1$.

– $R_1 > \alpha R_2$. [Sequential option is sufficiently productive.]

– Firms maximize the expected present discounted value of profits at the interest rate $r$.

– Demand shocks satisfy

$$dX_t = \mu X_t dt + \sigma X_t dW_t$$

with $r > \mu$. [Existence.]
• What is the value of a firm that has the option to invest in a project that costs $I$ and that, once implemented, yields an infinite stream of earnings given by

$$\pi_t = X_t R_2?$$

• The value of the firm, after the project has been implemented is:

$$E_t\{\int_t^\infty e^{-r(u-t)} X_u R_2 \} = \int_t^\infty e^{-r(u-t)} E_t\{X_u\} R_2$$

• Since for a geometric Brownian motion $E_t\{X_u\} = X_t e^{\mu (u-t)}$, we have that the value of the ongoing project is

$$V(X_t) = \frac{X_t}{r - \mu}.$$
• What is the value of the firm before it invests?

• Let $F_L(X)$ be the value of a firm that has an option to use the $L$ technology to invest in a project before the project is implemented. Then, in equilibrium, it must be that investing the value of the firm at the safe interest rate, $rF_L(X)$, must equal the value of holding on to the option which, in this case, it is given by the expected capital gain. Thus, no arbitrage, implies

$$rF_L(X_t) = E_t\{dF_L(X_t)\}.$$

• Using Ito’s Lemma, it follows that

$$E_t\{dF_L(X_t)\} = F'_L(X_t)\mu X_t + \frac{1}{2} F''_L(X_t)\sigma^2 X_t^2$$
• Thus, no arbitrage implies

$$rF_L(X_t) = F'_L(X_t)\mu X_t + \frac{1}{2}F''_L(X_t)\sigma^2 X_t^2$$

which, of course is the HJB equation that we have seen before.

• To solve this nonlinear ODE we conjecture that the solution is of the form

$$F_L(X_t) = \beta X_t^\lambda$$

for some $\lambda$.

• By substituting the conjecture into the no arbitrage (or HJB) equation we get that the conjecture solution satisfies the ODE if and only is $\lambda$ is a root of

$$r - \mu \lambda = \frac{\sigma^2}{2}\lambda(\lambda - 1)$$
• Two roots: \( \lambda_1 < 0 < 1 < \lambda_2 \).

• If \( X_t = 0 \), then the project will never be built. Thus, \( F_L(0) = 0 \). This requires \( \alpha_1 = 0 \).

• For a fixed \( X^* \), value matching requires

\[
F_L(X_L^*) = \frac{R_2X_L^*}{r - \mu} - I
\]

or, in this case,

\[
\beta_2(X_L^*)^{\lambda_2} = \frac{R_2X_L^*}{r - \mu} - I
\]

while smooth pasting implies

\[
F'_L(X_L^*) = \frac{R_2}{r - \mu}
\]
or, this case,

\[ \lambda_2 \beta_2 (X_L^*)^2 = \frac{X_L^*}{r - \mu}. \]

- It follows that the solution is

\[ X_L^* = \frac{\lambda_2 (r - \mu)I}{\lambda_2 - 1 \frac{R_2}{R_2}} \]

and,

\[ F_L(X) = \left[ \frac{R_2X_L^*}{r - \mu} - I \right] \left( \frac{X}{X_L^*} \right)^{\lambda_2} = \left[ \frac{I}{\lambda_2 - 1} \right] \left( \frac{X}{X_L^*} \right)^{\lambda_2}, \quad \text{for } X < X_L^* \]

- The solution of this system of two equations in two unknowns gives the value of the firm committed to using the \( L \) technology.
\[ F_L(X) = \begin{cases} 
\frac{I}{\lambda_2 - 1} \left( \frac{X}{X_L^*} \right)^{\lambda_2} & \text{for } X \leq X_L^* \\
\frac{R_2 X}{r - \mu} - I & \text{for } X \geq X_L^* 
\end{cases} \]
Only “S” Available

- Let’s solve backwards. If the first stage has already been built, then the decision to build the second stage is just as in the Lumpy case.

- The benefit is
  \[(R_2 - R_1)X_t\]

- and the cost is
  \[(1 - \alpha)\kappa I\]

- The optimal decision rule is to invest in the expansion when \(X_t\) first hits \(X_2^*\), where

\[
X_2^* = \frac{\lambda_2}{\lambda_2 - 1} \frac{(r - \mu)(1 - \alpha)\kappa I}{(R_2 - R_1)}
\]
• It also follows that the incremental value of this firm (after the first stage has been built) is

\[
F_2(X) = \begin{cases} 
\left[ \frac{(R_2-R_1)X^*}{r-\mu} - (1 - \alpha) \kappa I \right] \left( \frac{X}{X^*} \right)^{\lambda_2} & \text{for } X \leq X^*_2 \\
\frac{(R_2-R_1)X}{r-\mu} - (1 - \alpha) \kappa I & \text{for } X \geq X^*_2 
\end{cases}
\]

• Note: Why “incremental”? Given that the firm has already built the first stage, it has a locked market value equal to

\[
\frac{R_1X}{r-\mu}.
\]

The option to build the second stage simply adds to this value. Thus, after both stages have been built, the market value (ignoring investment costs) is the same as the value of a firm that only had access to the Lumpy technology.
• The decision to build the first stage:
  
  – “Project” pays 0 until it is built.
  
  – At the time it is built the payoff is
    
    \[
    \frac{R_1X}{r-\mu} - \alpha\kappa I + \text{Exp. Present value of Profits} + F_2(X) \quad \text{Option Value of Expansion}
    \]
  
• Then, either an arbitrage condition, or the HJB implies that the value of the firm before it invests in the first stage satisfies

\[
rF_1(X_t) = F_1'(X_t)\mu X_t + \frac{1}{2}F_1''(X_t)\sigma^2 X_t^2
\]

with boundary conditions

\[
F_1(X_1^*) = \frac{R_1X_1^*}{r-\mu} - \alpha\kappa I + F_2(X_1^*)
\]

(VM)
and

\[ F'_1(X^*_1) = \frac{R_1}{r - \mu} + F'_2(X^*_1) \]  \hspace{1cm} \text{(SP)}

- As before, we need a conjecture to solve the ODE. The conjecture that works is

\[ F_1(X_t) = \beta X_t^\lambda. \]

For this to be a solution it is necessary that the root \( \lambda \) satisfy the quadratic equation

\[ r - \mu \lambda = \frac{\sigma^2}{2} \lambda (\lambda - 1). \]

Since the value of the option is 0 when \( X = 0 \), we need to discard the negative root (i.e. set the constant equal to 0) and restrict the solution to the class

\[ F_1(X_t) = \beta_1 X_t^{\lambda_2} \]

where \( \lambda_2 > 1 \).
• Imposing the Value Matching (VM) and Smooth Pasting (SP) conditions we get

\[ \beta_1(X_1^*)^{\lambda_2} = \frac{R_1 X_1^*}{r - \mu} - \alpha \kappa I + \left[ \frac{(R_2 - R_1) X_2^*}{r - \mu} - (1 - \alpha) \kappa I \right] \left( \frac{X_1^*}{X_2^*} \right)^{\lambda_2} \] (VM)

and

\[ \lambda_2 \beta_1(X_1^*)^{\lambda_2} = \frac{R_1 X_1^*}{r - \mu} + \lambda_2 \left[ \frac{(R_2 - R_1) X_2^*}{r - \mu} - (1 - \alpha) \kappa I \right] \left( \frac{X_1^*}{X_2^*} \right)^{\lambda_2} \] (SP)

This is a system of two nonlinear equations in two unknowns \((\beta_1, X_1^*)\). The solution is

\[ X_2^* = \frac{\lambda_2}{\lambda_2 - 1} \frac{(r - \mu) \alpha \kappa I}{R_1} \] (6)

and

\[ \beta_1 = \left[ \frac{R_1 X_1^*}{r - \mu} - \alpha \kappa I \right] (X_1^*)^{-\lambda_2} + \frac{(1 - \alpha) \kappa I}{\lambda_2 - 1} (X_2^*)^{-\lambda_2}. \]
Thus, the value of the firm is given by,

\[
F_1(X) = \begin{cases} 
\left[ \frac{R_1 X^*_1}{r-\mu} - \alpha \kappa I \right] \left( \frac{X}{X^*_1} \right)^{\lambda_2} + \frac{(1-\alpha)\kappa I}{\lambda_2-1} \left( \frac{X}{X^*_1} \right)^{\lambda_2} & \text{for } X \leq X^*_1 \\
\frac{R_1 X}{r-\mu} - \alpha \kappa I + \left[ \frac{(R_2-R_1)X^*_2}{r-\mu} - (1-\alpha)\kappa I \right] \left( \frac{X}{X^*_2} \right)^{\lambda_2} & \text{for } X^*_1 \geq X \geq X^*_2 
\end{cases}
\]

- Proposition: Let \( X \in (0, X^*_1) \) then there exists a \( \hat{\kappa} \) such that

1. If \( \kappa < \hat{\kappa} \implies F_1(X) > F_L(X) \);

2. If \( \kappa > \hat{\kappa} \implies F_1(X) < F_L(X) \).

3. \[
\hat{\kappa} = \left[ \left( \frac{R_1}{R_2} \right)^{\lambda_2} \frac{1}{\alpha^{\lambda_2-1}} + \left( \frac{R_2-R_1}{R_2} \right)^{\lambda_2} \frac{1}{(1-\alpha)^{\lambda_2-1}} \right]^{1/(\lambda_2-1)}
\]
• **Proof:** Let $X \leq X_1^*$ (After some algebraic manipulations)

\[
\frac{F_L(X)}{F_1(X)} = \frac{\left(\frac{X_1^*}{X_L^*}\right)^{\lambda_2}}{\lambda_2 \alpha \kappa + (1 - \alpha) \kappa \left(\frac{X_1^*}{X_2^*}\right)^{\lambda_2}}
\]

is independent of $X!$. Given that

\[
\frac{X_1^*}{X_L^*} = \alpha \kappa \frac{R_2}{R_1}
\]

and

\[
\frac{X_1^*}{X_2^*} = \alpha \frac{R_2 - R_1}{1 - \alpha R_1},
\]

it follows that

\[
\frac{F_L(X)}{F_1(X)} = 1
\]
if and only if

\[ \hat{\kappa} = \left[ \left( \frac{R_1}{R_2} \right)^{\lambda_2} \frac{1}{\alpha^{\lambda_2-1}} + \left( \frac{R_2 - R_1}{R_2} \right)^{\lambda_2} \frac{1}{(1 - \alpha)^{\lambda_2-1}} \right]^{1/(\lambda_2-1)}. \]

If \( X \geq X_1^* \) a similar result obtains. ■

- Define

\[ \rho \equiv \frac{R_1}{R_2} > \alpha \]

then we can express the cutoff value of \( \kappa \) as

\[ \hat{\kappa}(\rho) = \left[ \rho \left( \frac{\rho}{\alpha} \right)^{\lambda_2-1} + (1 - \rho) \left( \frac{1 - \rho}{1 - \alpha} \right)^{\lambda_2-1} \right]^{1/(\lambda_2-1)}. \]

- What is the effect of an increase in \( \sigma \) on \( \hat{\kappa}(\rho) \)?
  - [Painful algebra shows that] \( \Delta \lambda_2 > 0 \implies \Delta \hat{\kappa}(\rho) > 0. \)
– Since $\Delta \sigma > 0 \implies \Delta \lambda_2 < 0$,

– We get $\Delta \sigma > 0 \implies \Delta \hat{\kappa}(\rho) < 0$.

• What does this imply? Consider an investment project indexed by its cost of flexibility, $\kappa$. To be precise, let’s assume that $\kappa < \hat{\kappa}(\rho; \sigma)$ and, hence that the sequential (flexible) strategy is preferred. Now, let $\sigma' > \sigma$, and assume that, now, $\kappa > \hat{\kappa}(\rho; \sigma')$. In the new, more unstable environment, the lumpy (inflexible) strategy dominates!

• Intuition: As $\sigma$ increases, then the fraction of the time spent in the intermediate region —where flexibility pays— is reduced!

• A more general case. It can be shown that:

  – $F_S(X) > F_L(X)$ if $X < X_1^*$. 
- $F_S(X) < F_L(X)$ if $X > X_2^*$.

- What happens if, instead of choosing one investment strategy, the firm waits? The candidate solution, for $X_1^* < X < X_2^*$ is: Wait. If $X$ drops to some $X_{P-}$ (with $X_{P-} > X_1^*$), choose the sequential strategy. If $X$ increases to some $X_{P+}$ (with $X_{P+} < X_2^*$), choose the lumpy strategy.

- The solution is then:

  - “Go sequential” if $X < X_{P-}$. ($X_{P-} > X_1^*$)
  
  - “Wait” if $X_{P-} \leq X \leq X_{P+}$. ($X_{P+} < X_2^*$)
  
  - “Go lumpy” if $X > X_{P+}$.

- What is the value of fine tuning?
– Intuition: In more uncertain environments this is a valuable option.

– Result: No. In more uncertain environments things change so much that the value of fine tuning is not high!