The Analytics of the Sign Restriction Approach to Shock Identification: with an Application to the Tax Multiplier Debate

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Motivation

- A given reduced-form VAR model is consistent with (infinitely) many structural models.
  - All just identified structural models have the same likelihood.
- Parametric SVAR models (including recursive/Cholesky identification approach) restrict attention to just one particular structural model.
- Sign restrictions instead do not pin down a unique structural model.
  - This is sometimes viewed as a weakness of the sign restriction approach.
  - However, this non-uniqueness property can help us to understand the scope of the identification problem, e.g. under which conditions the empirical macro puzzles can arise.
Motivation

- Sign Restrictions have become increasingly popular:
  - Monetary Policy (e.g. Uhlig 2005)
    - Exchange Rate (Ferrant et al 2006, Scholl et al 2008)
    - House Prices (Jarocinski et al 2008), Bank Loans (Eickmeier et al 2000)
  - We know little about the theoretical properties of the SR approach.
    - Is the set of SR solutions non-empty?
    - Does the set of sign restrictions pin down the sign of the object of interest, i.e. of the impulse responses left unrestricted?
    - What does the penalty function approach do?
Motivation

- Prominent empirical macro puzzles:
  - The price puzzle: prices *increase* in response to a monetary policy tightening.
  - The output puzzle: output *increases* in response to a monetary policy tightening.
  - Various exchange rate puzzles.
  - The tax multiplier puzzle: for “standard” assumptions about the size of tax elasticities, discretionary tax cuts *depress* output.

- Common feature of these examples:
  - A non-policy variable shows unexpected response to policy shock.

- We take the tax multiplier debate as an example to illustrate our methodological findings.
Contributions

- **Contribution 1:** Analytical characterization of the identification problem for contemporaneous sign restrictions.
  - In a bivariate model, we can characterize analytically the full set of sign restriction solutions.
  - We establish a simple condition for identification (non-emptiness of the set of sign restriction solutions).
  - We show under which conditions the sign of the impulse response of interest is not determined by sign restrictions alone.
  - The penalty function approach has a closed-form solution. The penalty function is an additional identifying restriction.
  - Many results derived in the bivariate context carry over to multivariate models with multiple sign restrictions.

- **Contribution 2:** We show for a prominent example that the full set of sign restriction solutions is sufficiently large to challenge received wisdom.
  - We find two non-empty sets of positive and negative tax multipliers for plausible assumptions on the size of tax elasticities.
The Identification Problem

- The reduced-form VAR model:
  \[ X_t = \mu + B(L)X_{t-1} + u_t, \text{ with } E(u_t u_t') = \Sigma \]

- The structural VAR model:
  \[ u_t = Fe_t \text{ with } \Sigma = FF' \]

- Identification problem (\(F\) is not unique):
  \[ \Sigma = \tilde{A}Q \underbrace{Q'\tilde{A}'}_{FF'} \]
  where \(QQ' = I_K\) and \(\tilde{A}\) is the Cholesky factor.

- Sign restrictions identification: Find rotation/reflection matrices \(Q\) such that \(F\) satisfies sign restrictions.
A Simple Bivariate Model

- We start from a simple two-equation VAR where:
  - variable 1 is a non-policy variable (output);
  - variable 2 is a policy variable (tax revenue);

- We impose the following **contemporaneous** sign restrictions:

  \[
  \begin{bmatrix}
  F_{11} & F_{12} \\
  F_{21} & F_{22}
  \end{bmatrix}
  =
  \begin{bmatrix}
  + & ? \\
  + & +
  \end{bmatrix}
  \]

- \(F_{12}\) is the object of interest. We expect \(F_{12} < 0\).
- Shock 1 controls for co-movements that originate from non-policy shocks. Here, shock 1 provides the cyclical adjustment of tax revenue.
Analytical Approach

- **KEY:** Cholesky Factor \( \tilde{A} \) has an analytical solution:

\[
\begin{bmatrix}
    F_{11} & F_{12} \\
    F_{21} & F_{22}
\end{bmatrix} =
\begin{bmatrix}
    \sigma_1 & 0 \\
    \sigma_2 \rho_{12} & \sigma_2 \sqrt{1 - \rho_{12}^2}
\end{bmatrix}
\begin{bmatrix}
    \cos(\theta) & -\sin(\theta) \\
    \sin(\theta) & \cos(\theta)
\end{bmatrix}
\]

- **Sign restrictions:**

\[
\begin{align*}
F_{11} &= \sigma_1 \cos(\theta) \geq 0, \\
F_{21} &= \sigma_2 \cos(\theta - \varphi) \geq 0 \\
F_{22} &= -\sigma_2 \sin(\theta - \varphi) \geq 0 \\
F_{12} &= -\sigma_1 \sin(\theta)
\end{align*}
\]

where \( \varphi = \arccos(\rho_{12}) \).
The Set(s) of Sign Restriction Solutions

Proposition 1. Let $S$ be the set of all solutions satisfying the sign restrictions. Then, the set $S$, for given $\phi \in (0, \pi)$, is

$$S \equiv \{ \theta \in [-\pi, \pi] : -\frac{\pi}{2} + \phi \leq \theta \leq \frac{\pi}{2} \}.$$

$S$ is non-empty for all $\phi \in (0, \pi)$, i.e. for less than perfect error correlation, $\rho_{12} \in (-1, 1)$.

- $F_{12}$ is smaller than zero for $0 \leq \theta \leq \phi$ (subset of “no-puzzle” solutions).
- $F_{12}$ is larger than zero for $-\frac{\pi}{2} + \phi \leq \theta \leq 0$ and $\phi \leq \theta \leq \frac{\pi}{2}$ (subset of “puzzle” solutions).
- The subsets have equal size for $\phi = \frac{\pi}{3}$, i.e. $\rho_{12} = 0.5!$
- The “puzzle” subset is empty for $\phi > \frac{\pi}{2}$, i.e. $\rho_{12} < 0$ (negative error correlation).
The Special Role of the Cholesky Factor

- Recall: The “puzzle” subset is empty for $\varphi > \frac{\pi}{2}$, i.e. $\rho_{12} < 0$ (negative error correlation).
- Recall also that $F$ is equal to the Cholesky factor $\tilde{A}$ for $\theta = 0$, i.e. $Q = I$.
- Now: the Cholesky factor is a sign restriction solution iff $0 \leq \rho_{12} < 1$ (or $\varphi \leq \frac{\pi}{2}$). The same condition guarantees the non-emptiness of the “puzzle” subset.
- Note: the Cholesky factor implies that $F_{12} = 0$. If $\tilde{A}$ is a sign restriction solution, the sign of $F_{12}$ is not determined!
- Implication: there is a simple test for “sign determinacy” (also in multivariate context) — check whether Cholesky factor is a sign restriction solution.
The Penalty Function Approach

- **Scope**: select one solution from the set $S$ based on a criterion function (e.g. Uhlig JME 2005).
- We show that the penalty function is an **additional identification restriction** over and beyond sign restrictions:

$$
\max_{\theta} \Omega_1 (\theta) \equiv \frac{F_{11}}{\sigma_1} + b \frac{F_{21}}{\sigma_2} = \cos(\theta) + b \cos(\theta - \varphi)
$$

- Standard penalty function approach assumes $b = 1$.
- Our optimisation problem satisfies the conditions of the Weierstrass theorem. We derive the unique global maximum $\theta^* (b, \varphi)$.
- For the standard penalty function ($b = 1$) the maximum obtains for $\theta^* = \frac{\varphi}{2}$. Varying $b$ over $(0, \infty)$ explores the set of “no-puzzle” sign restriction solutions.
Multivariate extensions (work in progress)

- Bivariate case intuitive: only one degree of freedom (rotation angle). Multivariate case more complicated: \( n(n-1)/2 \) degrees of freedom

- However, several key results continue to hold:
  - if sign restrictions only on the first two variables all SR solutions derived for the bivariate case remain intact (now a subset of all SR solutions). Only difference: error correlation between the two variables of interest might be affected (e.g. adding commodity prices to monetary VAR model might resolve price puzzle).
  - even if we add sign restrictions on other variables the SR solutions presented for the bivariate case can remain SR solutions ("seemingly unrestricted variables").
  - easy test for existence of "puzzle" solutions: check whether Cholesky factor satisfies all sign restrictions.
  - The standard penalty function still has a closed-form solution: the optimization problem is recursive \( \theta_{1i}^* = f(\theta_{12}^*, \ldots, \theta_{1i-1}^*) \).
Application: Tax Multiplier Debate

- Our benchmark: Mountford and Uhlig (JAE 2009)
- U.S. data, sample period 1955 I - 2000 IV
- Data: logarithms of real per-capita GDP, tax revenue, spending + 7 control variables
- Correlation coefficient between tax and output residuals: $\rho_{ty} = 0.53$.
- Three shocks: business cycle shock (cyclical adjustment, restrictions on $Y$, $T$, $C$ and $I$), tax shock (restrictions on $T$), spending shock (restrictions on $G$)
- MU report results exclusively for the penalty function approach!
- Main finding: the tax cut multiplier is positive and large, larger than spending multiplier
The Output Elasticity of Tax Revenue

- Rotation angles $\theta$ do not have an economic interpretation.
- The business cycle shock controls for co-movements in taxes and output due to the business cycle, i.e. provides a cyclical adjustment of tax revenue.
- This idea also motivates the traditional SVAR literature (e.g. Blanchard and Perotti QJE 2002 based on earlier OECD work). In that literature a parametric restriction is imposed on the output elasticity of tax revenue ($a_1$ in BP’s notation).
- We can construct the elasticity implied by the sign restriction solutions for a given rotation angle:

$$a_1^{SR} (\theta) = \frac{F_{21} (\theta)}{F_{11} (\theta)}$$
Output elasticity of taxes ($a_{1}^{SR}$)

- Cholesky (Y, T)
- Cholesky (T, Y)
- MU

Percentage change vs. $\cos \theta$
Output Elasticity of Tax Revenue

- What is a plausible value for $a_1$ according to existing empirical literature?
- Blanchard et al (2010): “... an elasticity of tax revenue with respect to output of approximately 1”
  - Consistent with OECD estimates (Giorno et al. 1995).
Impact tax cut multiplier

Dollar change in output for a one dollar tax cut

- Cholesky (Y,T)
- MU
- $a_{1}^{SR} = 1$
- $a_{1}^{SR} = 3$
- Cholesky (T,Y)
Dynamic tax cut multiplier

Dollar change in output for a one dollar tax cut

- \( a_{1}^{SR} = 3 \)
- \( a_{1}^{SR} = 1 \)

Quarter
Conclusions

- For a given reduced-form model, we can characterize analytically the full set of sign restriction solutions.
- We characterize key properties of this set.
- This framework can be applied to any question analyzed in the sign restriction literature.
- Work in progress:
  - Multivariate extensions
  - Priors over the distribution of rotation angles (here: aggregate tax elasticity)
  - Seemingly-unrestricted-variables property affects the size of the spending multiplier
- In a companion paper we develop the analytics of the traditional approach to SVAR identification.