Optimization Problem

- Find the values of n variables $x_1, x_2, \ldots x_n$ that minimize or maximize an objective function of these variables $f(x_1, x_2, \ldots x_n)$.
- E.g.
  - An objective function: the volume of a container (e.g. a circular cylinder).
  - Parameters: r, h
  - a way to search
  - constraints: the size of the surface area

- Assume the function f has first and second partial derivatives,

$$\begin{cases} \dfrac{\partial f(x_1, x_2, \ldots, x_n)}{\partial x_1} = 0 \\ \dfrac{\partial f(x_1, x_2, \ldots, x_n)}{\partial x_2} = 0 \\ \ldots \\ \dfrac{\partial f(x_1, x_2, \ldots, x_n)}{\partial x_n} = 0 \end{cases} \Rightarrow \begin{cases} x_1 = ? \\ x_2 = ? \\ \ldots \\ x_n = ? \end{cases}$$

DEFINITION:

A function f of two variables:

1. has a **local maximum** at (a, b) if
$$f(x, y) \le f(a, b)$$
for all points in a rectangular region containing (a, b).

2. has a **local minimum** at (a, b) if
$$f(x, y) \ge f(a, b)$$
for all points in a rectangular region containing (a, b).

To find the relative maximum and minimum values of f:

1. Find f_x, f_y, f_{xx}, f_{yy}, and f_{xy}.

2. Solve the system of equations $f_x = 0$, $f_y = 0$. Let (a, b) represent a solution.

3. Evaluate D, where $D = f_{xx}(a, b) \cdot f_{yy}(a, b) - [f_{xy}(a, b)]^2$.

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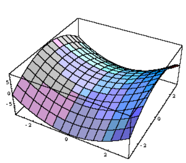
4. Then:

a) f has a maximum at (a, b) if $D > 0$ and $f_{xx}(a, b) < 0$.

b) f has a minimum at (a, b) if $D > 0$ and $f_{xx}(a, b) > 0$.

c) f has neither a maximum nor a minimum at (a, b) if $D < 0$. The function has a saddle point at (a, b).

d) This test is not applicable if $D = 0$.



Example 1: Find the local maximum or minimum values of $f(x, y) = x^2 + xy + y^2 - 3x$.

1. Find f_x, f_y, f_{xx}, f_{yy}, and f_{xy}.

$$f_x = 2x + y - 3 \qquad f_y = x + 2y$$
$$f_{xx} = 2 \qquad f_{yy} = 2$$
$$f_{xy} = 1$$

Example 1 (continued):

2. Solve the system of equations $f_x = 0$ and $f_y = 0$.

$$2x + y - 3 = 0 \qquad x + 2y = 0$$
$$x = -2y$$

Using substitution,

$$\begin{aligned} 2(-2y) + y - 3 &= 0 \\ -4y + y - 3 &= 0 \\ -3y &= 3 \\ y &= -1. \end{aligned}$$

Example 1 (continued):

Then, substituting back,

$$\begin{aligned} x &= -2(-1) \\ x &= 2. \end{aligned}$$

Thus, (2, –1) is the only critical point.

3. Find D.

$$\begin{aligned} D &= f_{xx}(2,-1) \cdot f_{yy}(2,-1) - [f_{xy}(2,-1)]^2 \\ &= 2 \cdot 2 - [1]^2 \\ &= 3 \end{aligned}$$

Considering the problem of finding the maximum of a function $f(x_1,x_2)$ subject to a constraint relating x_1 and x_2, which we write in the form

$$g(x_1,x_2) = 0$$

One approach would be to solve the constraint equation an thus express x_2 as a function of x_1 in the form x_2 $h(x_1)$. This can then be substituted into $f(x_1,x_2)$ to give a function of x_1 alone of the form $f(x_1,h(x_1))$.

The Method of Lagrange Multipliers

To find a maximum or minimum value of a function f (x, y) subject to the constraint g(x, y) = 0:

1. Form a new function:

$$F(x, y, \lambda) = \underbrace{f(x, y)}_{\text{original function}} - \underbrace{\lambda g(x, y).}_{\lambda(\text{constraint})}$$

The variable λ (lambda) is called a **Lagrange multiplier**.

The Method of Lagrange Multipliers (continued)

2. Find the first partial derivatives F_x, F_y, and F_λ.
3. Solve the system

$$F_x = 0, \quad F_y = 0, \quad \text{and} \quad F_\lambda = 0,$$

Let (a, b, λ) represent a solution of this system. We normally must determine whether (a, b, λ) yields a maximum or minimum of the function f.

The method of Lagrange multipliers can be extended to functions of three (or more) variables.

Example 2: Find the maximum value of

$$A(x, y) = xy$$

subject to the constraint x + y = 20.

First note that x + y = 20 is equivalent to x + y – 20 = 0.

1. We form the new function, F, given by

$$F(x, y, \lambda) = xy - \lambda \cdot (x + y - 20).$$

Example 2 (continued):

2. We find the first partial derivatives:
   $F_x = y - \lambda$
   $F_y = x - \lambda$
   $F_\lambda = -(x + y - 20)$
3. We set each derivative equal to 0 and solve the resulting system:

$$\begin{aligned} y-\lambda &= 0 \\ x-\lambda &= 0 \\ -(x+y-20) &= 0 \end{aligned}$$

Example 2 (concluded):

From the first two equations, we can see that $x = \lambda = y$. Substituting x for y in the last equation, we get

$$\begin{aligned} x+x-20 &= 0 \\ 2x &= 20 \\ x &= 10 \end{aligned}$$

Thus, $y = x = 10$. The maximum value of A subject to the constraint occurs at (10, 10) and is

$$\begin{aligned} A(10,10) &= 10\cdot 10 \\ &= 100 \end{aligned}$$

Example 3: The standard beverage can has a volume A. What dimensions yield the minimum surface area? Find the minimum surface area.
(Assume the shape of the can is a right circular cylinder.)

We want to minimize the function s, given by

$$s(h, r) = 2\pi rh + 2\pi r^2$$

subject to the volume constraint $\pi r^2 h = A$ or $\pi r^2 h - A = 0$.

Example 3 (continued):

1. We form the new function, S, given by
   $$S(h, r, \lambda) = 2\pi rh + 2\pi r^2 - \lambda\cdot(\pi r^2 h - A).$$
2. We find the first partial derivatives:
   $S_h = 2\pi r - \lambda\pi r^2$
   $S_r = 2\pi h - 4\pi r - 2\lambda\pi rh$
   $S_\lambda = -(\pi r^2 h - A)$

Example 3 (continued):

3. We set each derivative equal to 0 and solve the resulting system:

$$\begin{aligned} 2\pi r - \lambda\pi r^2 &= 0 \\ 2\pi h - 4\pi r - 2\lambda\pi rh &= 0 \\ -(\pi r^2 h - A) &= 0 \end{aligned}$$

Since π is a constant, solve the first equation for r.

$$\pi r(2-\lambda r) = 0$$

$$\pi r = 0 \quad \text{or} \quad 2-\lambda r = 0$$

$$r = 0 \quad \text{or} \quad r = \frac{2}{\lambda}$$

Example 3 (continued):

Since $r = 0$ cannot be a solution to the problem, we will continue by substituting $2/\lambda$ into the second equation.

$$\begin{aligned} 2\pi h + 4\pi\cdot\frac{2}{\lambda} - 2\lambda\pi\cdot\frac{2}{\lambda}\cdot h &= 0 \\ 2\pi h + \frac{8\pi}{\lambda} - 4\pi h &= 0 \\ \frac{8\pi}{\lambda} &= 2\pi h \\ \frac{4}{\lambda} &= h \end{aligned}$$

Example 3 (continued):
Since $r = 2/\lambda$ and $h = 4/\lambda$, it follows that $h = 2r$.
Substituting 2r for h in the third equation yields

$$-(\pi r^2 (2r) - A) \qquad = 0$$

$$r^3 = A/2\pi$$