Strategic Behavior and Endogenous Risk of Contagion in a Financial Network: A Network Formation Game

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ABSTRACT

This paper attempts to analyze the endogenous risk of contagion within an interbank network resulting from an external shock hitting the system. It uses a network formation game theoretic model to investigate the strategic interactions that may be observed once the shock has entered the network and thus endogenized itself. The primary results of this model indicate that the marginal utility that banks derive out of being in the network or alternatively, initiating connections to neighboring banks is negatively related to their degree of centrality as well as exposure levels. As such, their optimal strategy seems to be one that balances the level of their centrality and exposure, which in turn is a measure of their risk of contagion. Moreover, it is found that if a subnetwork of banks is highly connected then, with a very high probability, it is likely that it will entirely collapse. It is shown that there exists a stationary probability distribution that ensures this.

Key Words: Network Formation Games, Interbank Network, Degree of Centrality, Transition Probabilities, Dynamic Programming

JEL codes: A14; C71; C72; C73; D85

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1 Introduction

The recent recession has provided a strong impetus to the (already popular) topic of interbank contagion as a subject of research. A fair amount of empirical work done in this area try to model the phenomenon of shock propagation across banks using advanced econometric tools. Although, a complete theoretical model that can explain such a contagion is still missing, there have been attempts to address issues that constitute a part of the larger problem (e.g., Babus (2005)) with the aid of theory. When we refer to theory, we must specify the particular strand of theory we are borrowing our tools from. For instance, there are papers on frenzies, manias, panics and crashes that employ the conventional game theoretic style of modeling. On the other hand, there are other tools used from information economics, that model risk sharing as a factor driving contagion (Allen and Gale (1998)). Although one might argue that both these approaches are off-shoots of the same strand of theory, the fact that there are subtle differences in the techniques, leading to relative advantages of using one method over the other (based on the question under investigation), can hardly be denied. This comparison is important as it can help us understand how the choice of modeling techniques can play a crucial role in the simplification (or complication) of the model. This provides a strong justification for introducing a new set of tools borrowed from the field of Network Theory.

The use of networks as a modeling tool is not a new one. In fact, a quick literature review of the fields of computer science (information networks), electrical engineering (circuits), mechanical engineering (transport networks), political science (social and political networks) among others, will provide numerous applications of networks. In the past decade, advanced economic theory has joined this list as well. Due to its nice descriptive properties and the power to explain complex (simultaneous) strategic interactions between economic agents, the use of networks to model economic questions has gained increasing popularity.

We will abstract from the formal definition of a network and its properties (which are based on the types of networks) here, as it is beyond the scope of this proposal. However, in simple terms, a network can be viewed as a set of nodes (which could be individuals, firms, etc.) and the links between these nodes (for example, in a friendship network, the link between any two nodes could be that of a friend and so on). It is important to note that there are broadly, two types of networks: directed and undirected (also known as linking networks). In a directed network, links are directed rays from one node to the other implying that between two nodes $i$ and $j$, the relationship $i$ shares with $j$ may not be the same as the one $j$ shares with $i$. A quick example would be to assume that $i$ is the father and $j$ is the son. So, the link from $i$ to $j$ would be that of a father and from $j$ to $i$ would be that of a son. Due such a structure of these links, these are known as arcs in the context of directed networks.

In this context, another distinction that is vital to mention is that of a network formation game and a game on the network. A network formation game refers to a situation where a game starts on an exogenously given network, but as the game proceeds with changing strategies of the player, the network itself changes. Thus, the equilibrium (if one exists) is reached in a network which has endogenously evolved through these strategic interactions and is not the exogenous network that we started with. On the other
hand, a game on the network is constrained by the network given to us. It is closer to the conventional extensive form games, where the network itself does not change and players try to find their best strategies given the network.

In this paper, we propose to use directed networks to model contagion as a network formation game. And the rules of network formation are given by the set of rules introduced by Jackson-Wollinsky (1996). The interbank network consists of an intricate set of arcs between its nodes, which are banks. These arcs are not necessarily the same between banks. Bank $i$ might be a lender, whereas bank $j$ would be a borrower. Thus, it makes sense to use directed networks to model this. Moreover, since we are interested in looking at how an exogenous shock hits and propagates through the system, we have to model it as a network formation game. This is because, once the shock hits, it is the change in the strategies between banks that decides the course of shock diffusion. Thus, once the shock has diffused, the network we have may not be the same one as we started with. Thus, this game can be perceived as a two stage game, where in the first stage a network is formed based on exogenously given information that include the ‘strength’ of a bank (credit rating), its measure of centrality (how well connected it is) and whether it is a borrower or lender in that stage. The shock hits at the end of the first stage. Thus, in the second stage, the shock as well as the evolution of a new network (if any) becomes an endogenous process.

The rest of the paper has been divided into four sections. Section 2 outlines the setup and details of the model with subsections that lay out the primitives, related details and the results. Section 3 concludes.

2 Setup

This section provides a timeline of the game played within an interbank network. But before we move forward, it might be useful to put together some other details that explain the network. The interbank network that we will be looking at may be viewed as a game. The set of players include banks, customers, regulator and a central bank (or the Fed). The set of strategies is different for each player and may be defined as follows:

**Strategies:**

1. Banks - \{borrow, lend, withdraw\}
2. Customers - \{deposit, withdraw\}
3. Regulator - \{broadcast, don’t broadcast\}
4. Central Bank - \{bailout, don’t bailout\}

The payoffs for these players will be defined a little later, after the introduction of some important and relevant concepts. We assume an environment of incomplete information where the *type* of a given player
(post-shock) may not be known to most of the other players and vice-versa. A more elaborate description of this statement will be provided in the following sections.

Mathematically, we introduce the following notations. We assume that there are \( n \) banks denoted by
\[
B = \{b_1, \ldots, b_n\}_{1 \times n}
\]
There are \( k \) customer nodes denoted by
\[
C = \{c_1, \ldots, c_k\}_{1 \times k}
\]
There is one regulator denoted by
\[
R = \{r\}_{1 \times 1}
\]
and one central bank (Federal Reserve) denoted by
\[
Fe = \{f\}_{1 \times 1}
\]
We have three markets, namely, the bank-customer market, the interbank market and the fed-bank market. The bank-customer nodes (alternatively, the market) are given by
\[
N_m = B \times C
\]
The inter-bank nodes are given by
\[
N_B = B \times B
\]
The fed-bank nodes are given by
\[
N_f = Fe \times B
\]
The strategy sets relevant for each market is dependent on the players in the respective markets. We shall refer to these strategies as arcs. The set of arcs within the bank-customer market is given by
\[
A_m = \{ \; a_{1m}, \; a_{2m}, \; a_{3m} \; \}
\]
with deposit withdraw not connected
The set of arcs within the inter-bank market is given by
\[
A_B = \{ \; a_{1b}, \; a_{2b}, \; a_{3b} \; \}
\]
with borrow lend not connected
The set of arcs in the fed-bank market is given by
\[
A_f = \{ \; a_{1f}, \; a_{2f}, \; a_{3f} \; \}
\]
with bailout don’t bailout inactive
So, a typical element of the bank-customer market may be denoted as
\[
M_{\text{network}} = \{a_{im}, (b_j \times c_k)\} \in A_m \times B \times C
\]
A typical element of the interbank market may be denoted as

\[ B_{\text{network}} = \{a_{ib}, (b_j \times b_k)\} \in A_B \times B \times B \]

A typical element of the fed-bank market may be denoted as

\[ F_{\text{network}} = \{a_{if}, (f, b_j)\} \in A_f \times Fe \times B \]

Thus, the entire network with all the players and their strategies constitutes the financial market, which we call the 'mega-network' and define as

\[ \text{Mega network} = \left[ f, r, \{a_{im}, (b_j \times c_k)\}, \{a_{ib}, (b_j \times b_k)\} \right] \]

At this point, it might be useful to stop and think about what is so attractive about a network form of model. One of the most important factors is the ease in which a network model captures simultaneous interactions across agents. Secondly, it is very sensitive to the fact that the exact position of a node within the network does have an effect on its choice of strategies and hence, the equilibrium of the game. This, in more technical terms, may be defined as the centrality of a node. So we go ahead and introduce the concept of degree of centrality. The degree of centrality of the \(i\)th node is denoted by \(x_i\) where \(x_i : A_B \times \{i\} \times N_b \rightarrow [0, 1]\).

This also enables us to rank the banks based on their degree of centrality which refers to the number of links incident upon a node (i.e. number of ties that a node has). Although there are various forms of centrality measures that may be used, for our purposes the eigen vector centrality measure that is based on the importance of a node in a network is the most relevant. It assigns a relative score to all nodes in the network based on the principle that connections to high scoring nodes contribute more to the score of the node in question than equal connections to low scoring nodes. For example, in our model, a score may be defined as a function \(f(E)\), where \(f : E \rightarrow Z_+\) with \(f' > 0\) and \(E\) is some set the elements which measure the risk of contagion for each bank, so that as \(E\) increases, this score increases. Therefore, the score of the node with high connections with nodes that have high \(E\) (failed banks\(^2\)), will have a high score as well, meaning more susceptible to the shock. Thus, the centrality measure can potentially have a serious impact on the rate as well as path of shock diffusion within an interbank network.

In order to explain the concept of the eigen vector centrality measure, let us use an example. Consider the following network.

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\(^2\)Failed banks may refer to those nodes which have either credit to total asset ratio greater than 1 or have simply been declared insolvent.
Let $x_i$ denote the score of the $i$th node. Let $A_{ij}$ be the adjacency matrix of the network given by.

$$
\begin{array}{cccc}
  i_1 & i_2 & i_3 & i_4 \\
  i_1 & 0 & 1 & 1 & 0 \\
  i_2 & 1 & 0 & 1 & 1 \\
  i_3 & 0 & 1 & 1 & 1 \\
  i_4 & 1 & 0 & 1 & 1 \\
\end{array}
$$

Hence, $A_{ij} = 1$ if $i$ and $j$ are connected and 0 otherwise. For the $i$th node, let the centrality score be proportional to the sum of the score of all nodes, which are connected to it. Hence,

$$x_i = \frac{1}{\lambda} \sum_{j \in M(i)} x_j = \frac{1}{\lambda} \sum_{j=1}^{N} A_{ji}x_j$$

where $M(i)$ is the set of nodes that are connected to the $i$th node, $N$ is the total number of nodes and $\lambda$ is a constant. In matrix form,

$$x = \frac{1}{\lambda} Ax \Rightarrow \lambda x = Ax$$

In general, there will be many different eigenvalues $\lambda$, for which an eigenvector solution exists. However, the additional requirement that all entries in the eigenvector be positive implies (by Perron-Frobenius theorem which asserts that a real square matrix with positive entries has a unique largest real eigenvalue and that the corresponding eigenvector has strictly positive components) that only the greatest eigenvalue results in the desired centrality measure.

For example, in the network above, we have that the eigen vectors of its adjacency matrix can be represented as a matrix where each vector corresponds to an eigenvalue of this adjacency matrix. The eigen vector matrix for this example is given by,
where the first column is the eigen vector corresponding to the first eigen value, the second column is the eigen vector corresponding to the second eigen value, and so on. The eigen values for our adjacency matrix are given by:

$$
\begin{array}{cccc}
-0.4156 + 0.4248i & 0 & 0 & 0 \\
0 & -0.4156 - 0.4248i & 0 & 0 \\
0 & 0 & 2.8312 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
$$

Thus, observe that Perron-Frobenius theorem holds. There is a unique real positive eigen value, which is 2.8312 in this example. And the eigen vector corresponding to this eigen value, given by the third column of the eigen vector matrix above, is the only eigenvector that has all positive elements. This information can be interpreted as follows. Given the adjacency matrix, all links have a connection to \( i_3 \), thus \( i_3 \) has the highest degree of centrality. As such, if the shock hits \( i_3 \), then the likelihood of it diffusing through the network is very close to 1. Therefore, the banks must make their decisions conditional on these links. This is to say that every bank in such a network, must measure its degree of centrality for various combinations of connections and look for the minimum given its constraints. For example, if \( i_3 \) were Bank of America, then the likelihood of \( i_3 \) directing a ‘borrow’ arc to any of the smaller banks (weaker banks) is very low. Thus, most likely, all or most of \( i_3 \)’s connection are that of ‘lend’. Thus, if a shock hits \( i_3 \), it will require extra liquidity and might want to pull out money from the smaller banks it lent to. This, is going to adversely affect the standing of the smaller banks and the shock will diffuse in no time. Whereas, if \( i_3 \) were a small/weak bank, which has borrowed money from \( i_1, i_2 \) and \( i_4 \) and has been hit by a shock, the rate of diffusion might depend on the size of its lenders. Of \( i_1, i_2 \) and \( i_4 \) the stronger banks might be able to sustain the default or postpone repayment by \( i_3 \) till it stabilizes. However, the relatively weaker ones might want to pull the money out of \( i_3 \), thus aggravating the shock. Therefore, the kind of connections and degree of centrality any bank has plays an important role in determining the rate of overall shock diffusion.

Now that we have setup the notations, we can go ahead and outline the structure of the game. We design a three stage game, which is as follows:
3 - Stages:

1. **Stage 0**: The regulator, who appears only in this stage, releases the credit standing of each node in the interbank network. Given this, the connections in the mega-network are formed (we could take this as an exogenously given network). In particular, we have that in this stage, banks form links with one another based on the information exogenously provided to them. As a starting point, we can assume a particular exogenously given network. This is not going to affect the eventual outcomes as stage I is what matters, where the links are formed endogenously. Also note that at this stage, some of these banks are known to be ‘strong’ while others are ‘weak’ based on the announcements made by the regulator about the strength of each bank. So, at stage 0 everyone knows whether a given bank is strong or weak. A bank is strong if it has credit-to-deposit ratio below an exogenously given threshold and is weak otherwise. Customers also have access to the regulator’s information and accordingly decide which banks to use in order to deposit or withdraw money (based on their preference for a strong or weak). Another crucial node in this network is the central bank. Although, the customers cannot build a link to the central bank, the banks can. Also, it is imposed through the rules of network formation that there is a link from the central bank to every other bank that is inactive, and can be activated only when none of the other links could be built. In this stage, therefore, $A_f$ is inactive.

2. **Stage I**: At the beginning of stage I, a shock randomly hits one of the nodes. A shock is an event that changes the credit standing of the bank that it hits, by more than 30%. However, how these credit ratings change is a source of uncertainty as the regulator doesn’t update the information announced at the beginning of the game anymore. The manner in which the shock enters the system may be described as follows. At the beginning of this stage, nature chooses one or a pair of nodes (that are connected directly) for the shock to hit them. That is, the credit standing of this or these node or nodes (respectively) changes by more than 30%. Given this then, they must decide what to do, and based on their choice of strategy the existing network gets modified and thus, this modified network becomes the next step in the path of the shock diffusion. Also, note that the shock can enter the system only through the interbank network.

This stage has an important feature that demands detailed discussion. Although, at a glance this game may seem to be a one shot game with three stages, that is not what it is. In this stage, once the shock has entered the system, it transforms itself into an endogenous shock and it is likely that its nature/impact has undergone a change due to the medium through which it entered the system. In

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3 The regulator is a passive node in the network. That is, it is not a player in the other stages of the game and is active only at the beginning of the game when the announcements are made.

4 Definition of Credit Standing: It is a score that reflects the credit worthiness of a bank. For example, in the US, Fair Isaac Corporation (FICO) provides credit scoring services intended to help financial institutions make reasoned decisions. Here, we assume the regulator to be one such organization or individual that provides credit scores for banks.

5 This is an approximation that we presume based on the discussion in the Basel II convention.

6 A medium here would be the bank that was hit at the beginning of stage I.
other words, the shock gradually diffuses itself through these nodes and every step of diffusion is a stage game by itself, where nodes are trying to figure out their best strategies given that the shock has hit them. So, the flavor of a finitely repeated game is quite prominent and hence, we will model it on those lines.

We assume that the information that the system has been hit by a shock, is known to all. Also, if a bank is insolvent or has more debt than assets, it is known to all as it is likely to activate its connection with the central bank at this stage. Given this and the fact that the regulator doesn’t update the information on credit scores, it is left to the banks to construct a measure of their own in order to estimate the change in credit standing of, at least, the banks they are connected to. Thus, they must try to figure out their own probabilities of getting hit by the shock or alternatively, the risk of contagion. At this point, we introduce a new term called exposure. Exposure is a function of the credit to total asset ratio of a given bank, links it has with failed banks and links it has with existing banks

\[ E_i(CR_i, L_f, L_s) : [0,1] \times A_B \times \{i\} \times N \to [0,1] \]

and

\[ CR_i \in [0,1], \quad (L_f \cup L_s) \in A_B \times \{i\} \times N \]

where \( CR \) is the credit to total asset ratio of the given bank, \( L_f \) is the set of links it has with failed banks and \( L_s \) is the set of links it has with strong banks. \( E_i \) is assumed to be continuous, however, is not differentiable as its domain is a discrete space. We also assume that

\[ \frac{\Delta E_i(CR_i, L_f, L_s)}{\Delta CR_i} > 0; \quad \frac{\Delta E_i(CR_i, L_f, L_s)}{\Delta L_f} > 0; \quad \frac{\Delta E_i(CR_i, L_f, L_s)}{\Delta L_s} < 0 \]

It is important to observe that \( CR \) is equal to \((1 - \frac{\text{equity}}{\text{total assets}})\), and the value of equity could change as an effect of the shock, thereby changing the credit to total asset ratio as well. Thus, for a bank, the closer this ratio is to one, the more exposed it is. It is important for each bank to have such a measure as it provides the banks with an estimate of the probability with which they might be affected by the shock. The manner in which this affects their payoffs and strategies has been discussed in the following section. At this point, the customers also extract signals from the market that they are part of and decide their optimal strategies. Also, banks can activate their connections with the central bank under the rules of network formation. We will leave discussions on these aspects for later.

3. \textit{Stage II}: As a consequence of strategic decisions resulting from the endogenized shock in stage \( I \), a new network evolves from the network we started with.
2.1 The Model

Recall that there are three markets within the mega network that may be operating simultaneously. In order to simplify these sets of complex interactions, we will split them apart and analyze them one after the other. Thereafter, we will build the general model with all parts put together. So, consider the interbank market first. We start with some more necessary definitions and the basic assumptions for the case of the $A_B \times B \times B$ market.

2.1.1 Primitives and Derived Concepts

**Definition 1 (Feasible Coalitions)**

Given a finite player set $D$, a feasible set of coalitions is a nonempty subset $F$ of the collection of all coalitions $P(D)$.

A-1: The feasible set of coalitions in the interbank market

$$F_{\leq 2} = \{S \in P(D) : |S| \leq 2\} \text{ where } D \in N_b$$

**Definition 2 (State)**

The state space is defined as the set $\Omega = G \times F$ of all feasible network-coalition pairs. We denote a typical state as $\omega = (G, S)$.

Thus, each state in $\Omega$ may be interpreted as follows: if $(G, S)$ is the current state, then $G$ is the current status quo network of social interactions and it is coalition $S$’s turn to propose a new network and the next coalition that should move in this new network. The intuition underlying this is the following. If $(G, S)$ has been chosen to make a move, then they decide on their strategies that leads us to the next network. However, their choice of strategies would also imply a change in the likelihood of getting exposed for the nodes directly or indirectly connected to the chosen nodes. In that sense, when $(G, S)$ choose a strategy and therefore, the next network, they are also choosing the next coalition that will move in that network. Of course, whether that coalition ends up being chosen depends on a law of motion or transition probabilities to be defined shortly. Thus, if $(G, S)$ is the current state, then nature will accept proposals by members of coalition $S$ concerning what the new status quo network-coalition pair should be. We will refer to the coalition whose turn it is to move as the status quo coalition.

In other words, in the context of our setup, the entry of the shock may be interpreted as an event where nature chooses a coalition $(|S| \leq 2)$ at random in the beginning of stage $I$ and thereafter, this shock follows a transition path based on a law of motion (to be defined shortly). So, at every step a new coalition is chosen to make a move (i.e., the chosen coalition makes a network proposal for the next state).

Before we can go ahead an formally define a proposal, we need to introduce the following definitions.

**Definition 3 (Network Descriptors)**
For \( G \in P_{f \in F}(A \times N \times N) \),

\[
G(a) = \{(i, i') \in N \times N : (a, (i, i')) \in G\}
\]

and

\[
G(ii') = \{a \in A : (a, (i, i')) \in G\}
\]

Thus in network \( G \), \( G(a) \) is the set of node pairs connected by arc \( a \) and \( G(ii') \) is the set of arcs from node \( i \) and \( i' \).

**Definition 4 (Proposal Constraint)**

Suppose the current status quo is \( \omega = (G, S) \in G \times F_{\leq 2} \) and \( E_d \subseteq N \times N \), then each player’s \( (d) \) proposal constraint set is given by

\[
\Phi_d(G, S) = \begin{cases} 
\{G' \in G : \forall (i, j) \not\in E_d, G'(ij) = G(ij)\} & \text{if } d \in S \\
\{G\} & \text{if } d \not\in S
\end{cases}
\]

The players treat this set as a constraint as it limits their proposal choices.

**A-2: (Continuity of the Constraint Mappings)**

All constraint correspondences, \( \Phi_d(G, S) \) are such that

1. for all states \( \omega = (G, S) \),
   (a) \( G \in \Phi_d(G, S) \)
   (b) \( \{G\} = \Phi_d(G, S) \forall d \not\in S \)

2. \( \Phi_d(\cdot) \) has a closed graph,
   \( Gr\Phi_d(\cdot) = \{(\omega, G) : G \in \Phi_d(\omega)\} \).

Also,

\[\omega \mapsto \Phi(\omega) = \Pi_{d \in D} \Phi_d(\omega)\]

**Definition 5 (Rules of Network Formation)**

The rules of network formation must specify for any more of the game which connections can be changed and which players can change them.

The proposal constraint defines a part of this as it specifies for each player, who is a member of the coalition, whose turn it is to move and what are the proposals that the coalition can pick from. However, this is derived from the basic rules of the game, which are assumed to be the ones proposed by Jackson-Wollinsky (1996).
A-3: (Jackson-Wollinsky Rules of Network Formation)

We assume that there must be mutual consent between a pair of nodes for them to be able to have a connection. We modify the J-W rules a bit in case of the withdrawal of an existing connection. It is assumed that if a node in the interbank network is a lender, then the subtraction requires mutual consent unless the lender becomes insolvent or fails as a bank. However, if a borrower wants to withdraw the connection, it can be done without the agreement of the lender as it can declare itself to be insolvent. In other words, subtraction of a connection may be an individual or a pairwise decision.

Thus, in our model, within the interbank network, a connection is formed between a pair of banks if the lender bank agrees to lend to the borrower bank who has initiated the connection by requesting to borrow. Moreover, the lender bank may decide to subtract the connection once it has lent the money out. However, it would require to demand its money back or be insolvent/failed (in which case the subtraction is one way). On the other hand, if the borrower indicates its inability to repay, which means that it shows willingness to subtract the existing connection, the only option the lender bank has is to agree to it. So, in that sense it is an individual decision made on the part of the borrower. We are missing a crucial point here though - they can initiate their connection with the central bank. However, for now we will assume that they choose not to do so in order to concentrate on one part of the interbank interactions.

Definition 6 (Cardinality of Connections and Arc Feasibility)

Suppose the feasible set of networks is given by

\[ G_{nm} = \{ G \in \mathbf{G} : \forall (i, i') \in N \times N, n(ii') \leq |G(ii')| \leq m(ii') \} \]

where \( n(.) \) and \( m(.) \) are nonnegative integer-valued functions such that for all node pairs \((i, i')\), \( n(ii') \leq m(ii') \) (with \( 0 < m(ii') \) for some node pair), and where \( |G(ii')| \) is the cardinality of the set of arcs from node \( i \) to node \( i' \) in network \( G' \). In our model, since there may or may not exist a connection between two nodes, we have that \( n = 0 \) and \( m = 1 \).

Definition 7 (Hausdorff Metric)

Let the distance between connection \((a, (i_0, i_1)) \in A \times (N \times N)\) and network \( G \in P_{f \in \mathbf{F}}(A \times N \times N)\) be given by

\[ d((a, (i_0, i_1)), G) = \inf_{(a', (i_0', i_1')) \in G} d((a, (i_0, i_1)), (a', (i_0', i_1'))), \]

where

\[ d((a, (i_0, i_1)), (a', (i_0', i_1'))) = d_A(a, a') + d_N(i_0, i_0') + d_N(i_1, i_1') \]

is the metric on \( A \times (N \times N) \). The Hausdorff metric \( h \) is then defined as

\[ \max\{ \sup_{(a, (i_0, i_1)) \in G} d((a, (i_0, i_1)), G'), \sup_{(a', (i_0', i_1')) \in G} d((a', (i_0', i_1')), G) \} \]

We will redefine this general definition of the Hausdorff metric in the following sections once we have discussed a few more concepts. Let us equip the set of coalitions, \( \mathbf{F} \), with the discrete metric \( d_F \) that is,
\[ d_F(S', S) = \begin{cases} 
1 & \text{if } S' \neq S \\
0 & \text{if } S' = S 
\end{cases} \]

**A-4: (Compactness of Metric Space)**

The state space \((G \times F)\) is a compact metric space under the metric \(d_\Omega\) given by

\[ d_\Omega((G', S'), (G, S)) = h(G', G) + d_F(S', S) \] (1)

In other words, our state space is finite.\(^7\)

**Definition 8: (Shock Probability Measure)**

Letting \(B(\Omega) = B(G \times F)\) be the Borel \(\sigma\)-field generated by the metric \(d_\Omega\), we equip our state space \((\Omega, B(\Omega))\) with a probability measure

\[ \zeta = \nu \times \eta \]

where the probability measure \(\eta\) on player coalitions is such that \(\eta(S) > 0\) for all \(S \in F\) and \(\nu\) is the probability measure on networks. In other words, \(\zeta\) may be interpreted as an exogenous probability of nature choosing the first network-coalition pair that gets to make the move. That is, it gets hit by the shock by this probability and then the coalition decides on a strategy. Now we can define our state space, the probability space

\[ (\Omega, B(\Omega), \zeta) = (G \times F, B(G \times F), \nu \times \eta) \]

a compact metric space with metric \(d_\Omega = h + d_F\). Because \(G\) is a compact metric space, \(B(G \times F) = B(G) \times B(F)\) where \(B(F)\) is the set of all subsets of \(F\) (including the empty set).

**A-5: (Shock)**

A shock can hit a coalition or an individual node only once unless the immediate impact of it is the complete collapse of the system, in which case the coalition chosen doesn’t change.

**Definition 9: (Payoffs)**

We define the utility derived from that bank’s degree of centrality as follows:

\[ g_d : x_d \to \mathbb{R} \]

where we assume that a change in \(g_d\) resulting from a change in \(x_d\) moves in the same direction as the latter. This means that bank \(d\), in fact, enjoys having a higher degree of centrality, which is why it accepts the connections targeted at it to begin with. However, this in turn exposes it to the risk of contagion that is

\(^7\)Recall, a discrete metric space is compact if and only if it is finite.
captured by the exposure function. The exposure matrix for the entire system looks as follows:

\[
E = \begin{bmatrix}
  i_1 & \cdots & i_l & \cdots & i_n \\
i_1 & e(1,1) & \cdots & e(1,l) & \cdots & e(1,n) \\
i_2 & e(2,1) & \cdots & e(2,l) & \cdots & e(2,n) \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
i_k & e(k,1) & \cdots & e(k,l) & \cdots & e(k,n) \\
i_n & e(n,1) & \cdots & e(n,l) & \cdots & e(n,n)
\end{bmatrix}
\]

where \( e(i,j) = E_{i,j} \) and \( e(j,i) = E_{j,i} \). Also, \( \sum_j e(i,j) = CR_{i,j} + Assets_i + Assets_{j,i} TA_i = 1. \)

\[\kappa_d(\cdot) : E \rightarrow \mathbb{R}\]

where

\[\frac{\Delta \kappa_d}{\Delta E} < 0\]

Recall, \( \frac{\Delta E}{\Delta CR_i} > 0, \frac{\Delta E}{\Delta L_f} > 0 \) and \( \frac{\Delta E}{\Delta L_s} < 0 \). So, we have

\[\frac{\Delta \kappa_d}{\Delta CR_d} = \frac{\Delta \kappa_d}{\Delta E} \frac{\Delta E}{\Delta CR_d} < 0.\]

\[\frac{\Delta \kappa_d}{\Delta L_f} = \frac{\Delta \kappa_d}{\Delta E} \frac{\Delta E}{\Delta L_f} < 0.\]

\[\frac{\Delta \kappa_d}{\Delta L_s} = \frac{\Delta \kappa_d}{\Delta E} \frac{\Delta E}{\Delta L_s} > 0.\]

and

\[\mu_s(\cdot) : \Omega \times G^m \rightarrow [-M,M], \ M \in \mathbb{R}, \ s \in F, \ |s| \leq 2\]

This means the following: \( \mu_s((\omega_s, (G_s, G_{-s})), (\omega_{s'}, (G_{s'}, G_{-s'}))) \) which is the utility that the coalition \( s \) (involving a given bank \( d \)) derives from being hit by the shock and thereafter, from being a part of the next network when it is \( s' \)'s turn to move.

A part of the payoffs for a given bank includes \( \kappa_d, \mu_d \) as well as \( g_d \). Therefore, we have

\[
u_d(\omega, \omega') = g_d(\cdot) + \kappa_d(\cdot) + \mu_d(\cdot)
\]

\[
g_d(x_d) + \kappa_d(G(E(CR_i, L_f(a, \{d, j\}), L_s(a, \{d, j\})))) + \mu_d(\omega_s, (G_s, G_{-s})) + Prob(\omega_s, \omega_{s'}) \mu_s(\omega_{s'}, (G_{s'}, G_{-s'}))
\]

where \( d \in S \). If \( d \not\in S \), then the payoff remains the same except that the term \( \mu_s(\omega_s, (G_s, G_{-s})) = 0 \) as it is not \( d \)'s turn to move. Therefore, \( d \) doesn’t get any value from the fact that some other coalition has been chosen to make a move. The value \( d \) would get out of the next state, however, is reflected in the
second term of \( \mu_d \), which is the expected payoff from transiting into the next state when it is some other coalition’s turn to move. Note that this is the post-shock utility function. We don’t model the pre-shock utility function as we start with an exogenous network and are interested in what happens once a shock hits this exogenous network. Of course, one may notice that once nature chooses the first coalition according to the measure \( \zeta \), that is the shock hits a coalition, thereafter the shock evolves endogenously and thus, its path gets determined in the process. These endogenous transition probabilities are given by \( P(\Omega) \) which is represented by the following matrix:

\[
P(\Omega) =
\begin{bmatrix}
\omega_1 & \cdots & \omega_j & \cdots & \omega_n \\
\omega_1 & P(\omega_1, \omega_1) & \cdots & P(\omega_1, \omega_j) & \cdots & P(\omega_1, \omega_n) \\
\omega_2 & P(\omega_2, \omega_1) & \cdots & P(\omega_2, \omega_j) & \cdots & P(\omega_2, \omega_n) \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\omega_i & P(\omega_i, \omega_1) & \cdots & P(\omega_i, \omega_j) & \cdots & P(\omega_i, \omega_n) \\
\omega_n & P(\omega_n, \omega_1) & \cdots & P(\omega_n, \omega_j) & \cdots & P(\omega_n, \omega_n)
\end{bmatrix}
\]

where \( \sum_j P(\omega_i, \omega_j) = 1 \) and a typical \( P(\omega_i, \omega_j) \) denotes the probability of moving to \( \omega_j \) given that the current state is \( \omega_i \). Since this notation may be confusing to some, we will use \( P(\omega_j | \omega_i) \) to refer to the same, instead.

The payoffs for a given bank, \( d \), are then given by

\[
V_d(\omega) = u_d(\omega, \omega') + \beta \int_{\Omega} V_d(\omega')dP(\omega'|\omega)
\]

where \( \beta \in [0, 1] \) is the discount rate.

This is a Bellman equation and therefore, we are looking at solving a dynamic programming problem. Each bank maximizes

\[
V_d(\omega) = u_d(\omega, \omega') + \beta \int_{\Omega} V_d(\omega')dP(\omega'|\omega)
\]  \hspace{1cm} (2)

subject to

\[
\omega' \mapsto \Phi_d(\omega') = \Pi_{d \in N_b} \Phi_d(\omega')
\]

A-6 (Continuity of the Utility Functions)

\( g_d, \kappa_d \) and \( \mu_d \) are continuous functions. However, they (in particular, \( \mu \)) need not be differentiable as our state space is discrete. Thus, \( V_d(\cdot) \) is a continuous function but is not necessarily differentiable.

We need to find the value function that solves the bank’s maximization problem\(^8\). Note that the value function is in fact a function of \( x_d, E_d \) and \( \omega \). Each bank maximizes its own utility with respect to the proposal constraint. The proposal constraint may in fact be thought of as a choice of strategy on the part of the banks, such as whether to continue to borrow, or lend or decide to withdraw that link. Each of these choices are associated with a different levels of centrality and exposure. As such, one may think

---

\(^8\)I am working on solving for the class of value function/s that will solve this.
of the proposal constraint as a mapping from the set of strategy choices and therefore, the set of exposure and centrality levels to the set of states. So, instead of subjecting the maximization problem to the proposal constraint, we could subject it to another correspondence that may be defined as

$$\Gamma_d : x_d \times E_d \rightarrow \Omega$$

Alternatively, we could have re-written this as

$$\Gamma_d : [0, 1] \times [0, 1] \rightarrow \Omega$$

$\Gamma_d$ is a continuous and compact-valued correspondence. Thus, each state proposal is a result of the strategy choices or preference centrality and exposure by the player or coalition. We can then state the value function as:

$$V_d : x_d \times E_d \times \Omega \rightarrow \mathbb{R}$$

However, for the ease of notation we will simply denote $V_d(x_d, E_d, \omega)$ as $V_d(\omega)$ as the latter would capture the essence of the former.

**Existence**

We have that by Berge’s Theorem of the maximum,

$$\max_{\omega \in \Gamma_d(x,e)} V_d(\omega)$$

is continuous and the solution correspondence denoted by $S : x \times E \rightarrow \Omega$ with

$$S(\omega) = \arg\max\{\omega \in \gamma_d(x, e); V_d(\omega)\}$$

is nonempty, compact-valued and upper hemi-continuous.

So, we are guaranteed a solution to the maximization problem. From the discussion above, it is clear that the solution, which must tell the bank about the next state to propose, must in fact be in the form of strategies as the next state will be a function of these strategies. For example, if the solution implies that the next proposal must be $\omega' \neq \omega$, then it must be the case that *ceteris paribus*, the proposal in fact, is that the chosen coalition wants to alter the existing status of their connection. Thus, the solution is a strategy or policy that tells the bank about what to do at each state. That is, it should prescribe the state a bank should propose next at any given state. Having said that, one may now observe that (2) resembles a Markov decision process (MDP) which can be solved by linear programming or dynamic programming. Our model, apparently, is one that would try to solve this problem with the latter. The idea underlying this solution method is the following. We start by assuming that we know the state transition function $P(\omega'|\omega)$ and the utility function $u_d(\omega, \omega')$, and we wish to calculate the strategy or policy that maximizes the expected discounted utility. Thus, (2) may be rewritten as
\[ V_d(\omega) = \sum_{\omega'} P(\omega, \omega')[u_d(\omega, \omega') + \beta V_d(\omega')] \]

Note that we switch from integration to summation because here we are summing over discrete states, \( \omega' \). In (2) we were integrating over the whole space, \( \Omega \). The standard class of algorithms to calculate this optimal policy requires storage for two arrays indexed by the state - the first one is the value \( V \), which contains real values, and the second one is the policy or strategy \( \pi \) which contains actions. At the end of the algorithm, \( \pi \) will contain the solution and \( V(\omega) \) will contain the discounted sum of the utility by following that solution from state \( \omega \).

The algorithm has two steps, which are repeated in some order for all the states until no further changes take place. They are

\[ \pi(\omega) = \arg\max_a \left\{ \sum_{\omega'} P_a(\omega, \omega')[u_d(\omega, \omega') + \beta V_d(\omega')] \right\} \]

\[ V_d(\omega) = \sum_{\omega'} P_{\pi(\omega)}(\omega, \omega')[u_d(\omega, \omega') + \beta V_d(\omega')] \]

Their order depends on the variant of the algorithm; one can also do them for all states at once or state by state, and more often to some states than others. As long as no state is permanently excluded from either of the steps, the algorithm will eventually arrive at the correct solution.

Observe that this algorithm requires the knowledge of both \( P(\omega, \omega') \) and \( u_d(\omega, \omega') \). For the transition probability function we propose the following functional form.

**Transition Probabilities**

\[
P(\omega_i, \omega_j) = \begin{cases} 
\frac{d((G, S), (G', S'))}{\sum_{(G', S')} d((G, S), (G', S'))} & \text{if } \omega_i = (G, S) \text{ and } \omega_j = (G', S') \\
0 & \text{otherwise}
\end{cases}
\]  \hspace{1cm} (3)

For example, \( P(\omega_i, \omega_j) = 0 \) if \( \omega_i = (G, S) \) and \( \omega_j = (G', S) \) unless the system has completely collapsed and there is no other coalition to pick from so that the coalition remains unchanged, however the state has changed. The term \( d((G, S), (G', S')) = d(\omega, \omega') \) is defined in a way similar to what we had in definition (7), except now we have some additional terms.

When a shock hits the system a coalition is chosen by nature and it is its turn to move. This coalition then chooses its strategy (or alternatively, proposes a network) and suggests the next coalition that must move. Thereafter, it depends on the transition probabilities as to which state or network-coalition pair really gets picked. Let the original network be denoted as \( G_0 \) and \( G_n \) be one of the networks that may emerge out of the shock. Now suppose \( G_n = G' \) and \( G_0 = G \) then

\[
d_{G}(\omega_i, \omega_j) = h(G', G) + d_F(S', S)
\]  \hspace{1cm} (4)

\(^9\)Recall, that the lebesgue integral may be extended by linearity to non-negative measurable simple functions. It is a similar idea that we have used here.
where,

\[ h(G', G) = d_a(a, a') + d_e(e, e') + d_x(G(x), G'(x')) + d_N(i_0, i_0') + d_N(i_1, i_1') \]

\[ d(a, a') = |\text{arc going out from } i \in G - \text{arc going out from } i \in G'| \]

where \( i \in S \) is the node in the coalition \( S \) that has been hit by the shock.

\[ d_e(e, e') = \begin{cases} 1 & \text{if } |e' - e| > 0 \\ 0 & \text{if } |e' - e| = 0 \end{cases} \]

which means that for a given coalition with \(|S| \leq 2\) or a pair of nodes, if the difference in levels of exposure in \( G' \) and \( G \) is positive, the distance between the networks is 1 and it is 0 otherwise. Also, if two nodes are not connected then this term is 0.

\[ d_x(G(x), G'(x')) = \begin{cases} 1 & \text{if } |x' \in G' - x \in G| > 0 \\ 0 & \text{if } |x' \in G' - x \in G| = 0 \end{cases} \]

which means that for a given coalition with \(|S| \leq 2\) or a pair of nodes, if the difference in the degrees of centrality in \( G' \) and \( G \), the distance between the networks is 1 and it is 0 otherwise.

\[ d(i, i') = |\text{self-loop within } i \in G - \text{self-loop within } i \in G'| \]

\[ d(j, j') = |\text{arc from } i \text{ to } j - \text{arc from } i \text{ to } j'| \]

where \( j \) and \( j' \) are nodes that are connected to \( i \in S \) that is hit by the shock. And

\[ d_F(S', S) = \begin{cases} 1 & \text{if } S' \neq S \\ 0 & \text{if } S' = S \end{cases} \]

Therefore, if for a given coalition, \( d((G', S'), (G, S)) < 1 \), which would require that \( d_F(S', S) = 0 \) then that coalition is no more a feasible coalition in the new state (unless the new state is a state of total collapse). That is, the shock can’t hit a given coalition twice. Thus, \( d_\Omega((G', S'), (G, S)) \geq 1 \) and \( P(\omega_i, \omega_j) \leq 1 \). Also, note that this function is such that along the row of \( P(\Omega) \) we have \( \sum_j P(\omega_i, \omega_j) = 1 \). So, this is a valid probability transition function. Moreover, we can show that it is stationary (refer to the appendix).

Since this is a work in progress, we will postpone the discussion on solving for the optimal policy that yields the maximum payoff to later. Also recall that we are yet to find a value function that solves our problem. These are questions that are being currently investigated and are therefore beyond the scope of this draft. However, there are some interesting results that we derive based on the setup we have.

### 2.1.2 Results

1. **Result 1**: If the shock hits a node with a high degree of the centrality and a high level of exposure in the interbank network, the transition probability measure chooses the state in which the subnetwork (containing this node) collapses almost completely, with the highest probability.
Intuitively, this makes sense because if we have a node that is very highly connected, get hit by the shock, this immediately increases the level of exposure of the nodes that are connected to it, directly or indirectly. As such, they are at a very high risk of contagion. Now note that, the transition probabilities that give us the probability of moving to the next state, which could involve any of these nodes, is a function of these exposures. So, a higher exposure implies a 1 for \( d(e,e') \) and thus, increases the term \( h(\omega,\omega') \) where \( \omega' \) is the next state. So, this has a positive impact on the probability and leads to an increase in the probability of contagion. For a proof, please refer to the appendix.

2. **Result 2:** Marginal utility derived by the banks within an interbank network with respect to a change in the state is negatively related with degree of centrality and level of exposure.

Although, we are yet to find the value function that solves the banks’ maximization problem and are still to find the optimal policy, an informal exercise of looking at the effect of a change in the state on the value function leads to the following expression:

\[
\frac{\Delta \mu(\omega')}{\Delta(\omega')} = -\frac{\beta}{P(\omega' \mid \omega)} \left[ \frac{\Delta g(\omega')}{\Delta(\omega')} + \frac{\Delta \kappa(\omega')}{\Delta(\omega')} \right]
\]

where \( \frac{\Delta \mu(\omega')}{\Delta(\omega')} \) denotes the marginal utility derived from the proposal made and \( \left[ \frac{\Delta g(\omega')}{\Delta(\omega')} + \frac{\Delta \kappa(\omega')}{\Delta(\omega')} \right] \) denote the marginal cost of that proposal in terms of change in degree of centrality \( \frac{\Delta \kappa(\omega')}{\Delta(\omega')} \) and exposure levels \( \frac{\Delta \kappa(\omega')}{\Delta(\omega')} \). Note that the left hand side is negatively related to the right hand side. This has an interesting interpretation - it implies that if the degree of centrality increases and the exposure increases in the new state, the marginal utility that the coalition derives from proposing \( \omega' \) as the next state decreases. Moreover, if exposure decreases more than the increase in centrality, then the marginal utility becomes positive! Similarly, if centrality decreases more than exposure, the effects of exposure are nullified. Also, recall that we assume \( \beta \in [0,1] \) and know that \( P \in [0,1] \). If we had a case where both \( \beta = P = 1 \), then the ratio \( \frac{\beta}{P+\beta} = 0.5 \). So, marginal utility declines by half the magnitude of the joint increase in centrality as well as exposure. This is true when \( \beta = P = 0.5 \), so that the ratio \( \frac{\beta}{P+\beta} = 0.5 \). For a given \( P, \frac{\Delta g(\omega')}{\Delta(\omega')} \) and \( \frac{\Delta \kappa(\omega')}{\Delta(\omega')} \), marginal utility decreases as \( \beta \) increases. This is because a higher \( \beta \) implies ‘more patience’ on the part of the coalition. That is, the cost of an increase in centrality and exposure is higher for them as they would rather wait for a state where they are not exposed as much and accordingly make the proposal. On the other hand, for a given \( \beta, g' \) and \( \kappa' \), marginal utility decreases by a lower amount as compared to the case of \( \beta \) as \( P \) increases. This is because a higher \( P \) implies a higher likelihood of transition to the proposed state, which has a positive effect on the marginal utility as there is, in a way, more weight on the proposal or chances of nature picking the proposal. Please refer to the appendix for a derivation.

Besides these basic results and pending questions, we have a few more propositions that need to be proved. They may be stated as follows.

3. **Proposition 3:** Conditional on Proposition 1 and Proposition 2, there exists a set of best response
strategies that will lead to a stable set consisting of banks that survived the shock.

In other words, this proposition is related to finding the optimal policy or strategy that banks can employ to find out their optimal actions that may minimize the risk of contagion and maximize their payoffs by prescribing the best proposal to make given their state.

4. **Proposition 4**: There is a unique stable set consisting of nodes which can be arrived at from finite number of paths leading to this stable set. In other words, there are only a finite set of adjacency matrices that can lead to such a stable set. Thus, if a given node does not follow its own location specified by one of these paths, it will never be able to reach the stable set and will be hit by the shock.

This is related to Proposition 3. In simple terms, Proposition 4 claims that within the optimal policy set, there exists a particular policy that prescribes the strategies to each bank such that if they followed it all through, they will survive the shock or not even be hit by it. The way it is different than proposition 3 is that it further adds the following - there exists a particular position in the network, that must be chosen at the beginning, say at stage 0, which belongs to a basin of attraction and hence is shock proof. So, if a policy is chosen at the very start in manner that it doesn’t increase the node’s degree of centralitity beyond a prescribed threshold and hence, limits the exposure, the node will definitely remain safe or minimally affected until the stage of complete shock diffusion.

These propositions only pertain to the interbank network. However, this part constitutes only one-third of the paper. We are yet to introduce the interactions within the bank-customer market as well as analyze the cases when the banks decide to initiate the fed-bank connection. These sections are work in progress currently. As such, this draft doesn’t do complete justice to the real flavor of the paper. Having said that, it must be useful to mention that the work presented here has been developed within a short period of time and under a deadline. Therefore, it is likely that there exist loose ends that need to be looked at.

### 3 Conclusion

The model proposed here is not the complete version. It has two of the integral parts of a financial market missing, which will be added to it in the completed version. However, it should be noted that the material presented here are more for the purpose of a first draft and therefore, should be viewed as a starting point. The propositions outlined in the previous section are still being worked on and might have to be rephrased based on the results derived from the model. To the best of our knowledge, this is the first time an attempt to capture the propability of a node being hit by a shock, through a theoretical model, is being made. In fact, it is worth emphasizing that the investigation is not only targeted towards the calculation of this probability, but also aims at modeling strategic interactions (between banks) based on the location of the nodes within an interbank network. Moreover, the use of
network formation game to model interbank contagion is also new. This is definitely not the complete exposition of the model and we hope to work on it more in order to complete the structure. However, given its scope, this draft captures the main ideas of the intended research.
4 References


Appendix

1. To show that

\[
P(\omega_i, \omega_j) = \begin{cases} 
\frac{d(G, S, \omega_i) d(G', S', \omega_j)}{\sum d(G', S', \omega_j) d(G, S, \omega_i)} & \text{if } \omega_i = (G, S) \text{ and } \omega_j = (G', S') \\
0 & \text{otherwise}
\end{cases}
\]

(5)

is stationary.

For stationarity we require to find a stochastic vector \( \Pi \) such that \( \Pi P = \Pi \). Since our \( P(\Omega) \) matrix is an \( n \times n \) matrix, therefore for the sake of simplifying our computation we will assume \( n = 2 \). This will not change our result. The only thing that will change is the elements and the order of the vector \( \Pi \), which will be an \( 1 \times n \) vector in the general case. So, suppose we have

\[
P(\Omega) = \begin{bmatrix} \omega_1 & \omega_2 \\ \omega_1 & P(\omega_1, \omega_1) & P(\omega_1, \omega_2) \\ \omega_2 & P(\omega_2, \omega_1) & P(\omega_2, \omega_2) \end{bmatrix}
\]

That is,

\[
P(\Omega) = \begin{bmatrix} \omega_1 & \omega_2 \\ \omega_1 & \frac{d_{11}}{d} & \frac{d_{12}}{d} \\ \omega_2 & \frac{d_{21}}{d} & \frac{d_{22}}{d} \end{bmatrix}
\]

where \( d_{ij} = d(\omega_i, \omega_j) \) and \( d = \sum d_{ij} \). Now, we set \( \Pi = f(0_{n \times n}) \left[ f(P - I_{n \times n}) \right]^{-1} \) where \( f(A) \) is a function that returns the matrix \( A \) with the last column replaced by ones. For example, we may define \( f \) as follows.

Let \( A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \). And let \( f(A) = A - \begin{bmatrix} 0 & a_{12} - 1 \\ 0 & a_{22} - 1 \end{bmatrix} \).

Applying this to our case,

\[
f(P - I) = \begin{bmatrix} \frac{d_{11}}{d} - 1 & \frac{d_{12}}{d} \\ \frac{d_{21}}{d} & \frac{d_{22}}{d} - 1 \end{bmatrix} - \begin{bmatrix} 0 & \frac{d_{12}}{d} - 1 \\ 0 & \frac{d_{22}}{d} - 1 \end{bmatrix} = \begin{bmatrix} \frac{d_{11}}{d} - 1 & 1 \\ \frac{d_{21}}{d} & 1 \end{bmatrix}
\]

and

\[
f(0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 - 0 \\ 0 & 1 - 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}
\]

Now,

\[
\left[ f(P - I) \right]^{-1} = \frac{1}{\left( \frac{d_{11}}{d} - 1 \right) - \frac{d_{22}}{d}} \begin{bmatrix} 1 & -1 \\ -\frac{d_{22}}{d} & \frac{d_{11}}{d} - 1 \end{bmatrix} = \frac{d}{d_{11} - d - d_{21}} \begin{bmatrix} 1 & -1 \\ -\frac{d_{22}}{d} & \frac{d_{11}}{d} - 1 \end{bmatrix}
\]
Let $D = d_{11} - d - d_{21}$ Therefore,

$$\Pi = f(0), \left[ f(P - I) \right]^{-1} = \left[ -\frac{d_{21}}{D} \frac{(d_{11} - d)}{D} \right]$$

Note that this is a stochastic vector with $-\frac{d_{21}}{D} + \frac{(d_{11} - d)}{D} = \frac{-d_{21} + d_{11} - d}{d_{11} - d - d_{21}} = 1$.

Check conditions for stationarity:

$$\Pi P = \Pi \left[ -\frac{d_{21}}{D} \frac{(d_{11} - d)}{D} \right] \times \left[ \begin{array}{c} \frac{d_{11}}{d} \\ \frac{d_{21}}{d} \\ \frac{d_{22}}{d} \end{array} \right]$$

implies,

$$\left[ -\frac{d_{21} \cdot d_{11}}{D \cdot d} + \frac{d_{21} \cdot (d_{11} - 1)}{D \cdot d} \right] \times \left[ \begin{array}{c} \frac{d_{11}}{d} \\ \frac{d_{21}}{d} \\ \frac{d_{22}}{d} \end{array} \right]$$

Note that,

$$\frac{-d_{21} \cdot d_{11}}{D \cdot d} + \frac{d_{21} \cdot (d_{11} - d)}{D \cdot d} = \frac{-d_{21} \cdot d_{11}}{D \cdot d} + \frac{d_{21} \cdot d_{11}}{D \cdot d} = -\frac{d_{21}}{D}$$

$$\frac{-d_{21} \cdot d_{12}}{D \cdot d} + \frac{d_{22} \cdot (d_{11} - d)}{D \cdot d} = \frac{-d_{21} \cdot d_{12}}{D \cdot d} + \frac{d_{22} \cdot (d_{11} - 1)}{D \cdot d} \frac{-d_{21} \cdot d_{12}}{D \cdot d} + \frac{d_{22} \cdot (d_{11} - d)}{D \cdot d} = \frac{-d_{12} \cdot (d_{21} + d_{22})}{D \cdot d} = -\frac{d_{12} \cdot d}{D \cdot d} = -\frac{d_{12}}{D} = \frac{d_{11} - d}{D}$$

where we have used the fact that $\frac{d_{11}}{d} + \frac{d_{22}}{d} = 1$ and $\frac{d_{12}}{d} + \frac{d_{22}}{d} = 1$. We can similarly find $\Pi$ for the $n \times n$ case.

2. **Proof of Result 1:** If the shock hits a node with a high degree of the centrality and a high level of exposure in the interbank network, the transition probability measure chooses the state in which the subnetwork (containing this node) collapses almost completely, with the highest probability.

Since we are looking at a subnetwork that is highly connected, we may assume that the coalition (or the center of this subnetwork) has direct connections to all other nodes within this sub-network.\(^\text{10}\) Now suppose the state in which the subnetwork collapses completely is denoted by $(G', S')$ and the current state is denoted by $(G, S)$. The distance between the networks $G$ and $G'$ may be calculated as follows:

\[
d_{21}((G', S'), (G, S)) = h(G', G) + d_F(S', S)
\]

\[
= d_a(a, a') + d_e(e, e') + d_x(G(x), G'(x')) + d_N(i_0, i'_0) + d_N(i_1, i'_1) + d_F(S', S)
\]

\[
= 1 + 1 + 1 + 0 + 0 + 0
\]

\[
= 3
\]

\(^\text{10}\)This assumption doesn't alter our findings in any way and can easily be shown to hold even when we relax it. Since this is a draft version, the general proof has not been included here.
Note that we have $d_a(a, a') = |1 - 0| = 1$ because in $G$ the arc exists and in the new state it doesn’t because the network has collapsed, $d_e(e, e') = 1$ because it is given that $e$ is very high i.e. $|e' - e| > 0$, $d_x(G(x), G'(x')) = 1$ as it is given that $x$ is very high, i.e., $|x - x'| > 0$, $d_N(i_0, i'_0) = |1 - 1| = 0$ because the loops in $G$ will continue to be the same in $G'$, $d_N(i_1, i'_1) = |1 - 1| = 0$ because the loops in $G$ will continue to be the same in $G'$ and $d_F(S', S) = 0$ because in $G'$ the entire network has collapsed so the only coalition is $S$, so $S = S'$ and therefore $d_F(S', S) = 0$.

Now suppose there exists a state where the system doesn’t collapse, however a part of the subnetwork (involving $s$) collapses. Denote this state by $(G'', S'')$ and the current state is denoted by $(G, S)$. The distance between the networks $G$ and $G''$ may be calculated as follows:

$$d_{\Omega}((G'', S''), (G, S)) = h(G'', G) + d_F(S'', S)$$
$$= d_a(a, a'') + d_e(e, e'') + d_x(G(x), G'(x'')) + d_N(i_0, i''_0) + d_N(i_1, i''_1) + d_F(S'', S)$$
$$= 1 + 0 + 0 + 0 + 0 + 1$$
$$= 2$$

Note that we have $d_a(a, a'') = |1 - 0| = 1$ because in $G$ the arc exists and in the new state it doesn’t because that part of the subnetwork has collapsed, $d_e(e, e') = 0$ because in the part where the subnetwork has collapsed the exposure levels don’t change, $d_x(G(x), G'(x'')) = 0$ as in the part where the subnetwork has collapsed the degrees of centrality levels don’t change, $d_N(i_0, i''_0) = |1 - 1| = 0$ because the loops in $G$ will continue to be the same in $G''$, $d_N(i_1, i''_1) = |1 - 1| = 0$ because the loops in $G$ will continue to be the same in $G'$ and $d_F(S'', S) = 1$ because in $G$ the part of the subnetwork with $s$ has collapsed; so the next coalition that will be chosen to move $(S'')$ will be such that $S \neq S''$ and therefore $d_F(S'', S) = 1$.

Further, suppose if we had that despite the shock, $d_a(a, a''') = |1 - 1| = 0$ implying that the coalition hit by the shock continues to maintain the link given the high connectivity and high level of exposure. Denote this state by $(G''', S''')$, we have that

$$d_{\Omega}((G''', S'''), (G, S)) = h(G''', G) + d_F(S''', S)$$
$$= d_a(a, a''') + d_e(e, e''') + d_x(G(x), G'''(x''')) + d_N(i_0, i'''_0) + d_N(i_1, i'''_1) + d_F(S''', S)$$
$$= 0 + 0 + 0 + 0 + 0 + 1$$
$$= 1$$

Note that we have $d_a(a, a''') = |1 - 1| = 1$ because in $G'$ the same arc exists, $d_e(e, e''') = 0$ because it is implied that $e''' = e$ as the arc in $G$ remains the same in $G'''$, $d_x(G(x), G'''(x''')) = 1$ as it is given that
$x''' \in G''' = x \in G$ by the same argument, $d_N(i_0, i''') = |1 - 1| = 0$ because the loops in $G$ will continue to be the same in $G'''$, $d_N(i_1, i''') = |1 - 1| = 0$ because the loops in $G$ will continue to be the same in $G'$ and $d_F(S''', S) = 1$ because by assumption unless $s$ is the only coalition that is left or has led to the collapse of the entire network, a shock can’t hit the same coalition more than once. So the next coalition that will be chosen to move ($S''''$) will be such that $S \neq S''''$ and therefore $d_F(S''', S) = 1$.

Using this, we can find the transition probabilities:

$$P((G, S), (G', S')) = \frac{d((G, S), (G', S'))}{\sum_{(G', S')} d((G, S), (G', S'))}$$

$$= \frac{3}{3 + 2 + 1} = \frac{1}{2}$$

$$P((G, S), (G'', S'')) = \frac{d((G, S), (G'', S''))}{\sum_{(G', S')} d((G, S), (G', S'))}$$

$$= \frac{2}{3 + 2 + 1} = \frac{1}{3}$$

$$P((G, S), (G''', S''')) = \frac{d((G, S), (G''', S'''))}{\sum_{(G', S')} d((G, S), (G', S'))}$$

$$= \frac{1}{3 + 2 + 1} = \frac{1}{6}$$

Therefore, $P((G, S), (G', S'))$ is the highest implying that the probability of collapse is the highest in the given situation.

3. **Proof of Result 2**: Marginal utility derived by the banks within an interbank network with a change in the state is negatively related with degree of centrality and level of exposure.

Since we are not guaranteed differentiability of the value function, we will look at discrete changes in order to investigate the impact of a change in state on it. Recall that our value function was given by

$$V_d(\omega) = u_d(\omega', \omega') + \beta \int \Omega V_d(\omega')dP(\omega'|\omega)$$

So,

$$\frac{\Delta V_d}{\Delta \omega'} = \frac{V_d(\omega' + \Delta \omega') - V_d(\omega)}{\Delta \omega'}$$

$$= \frac{1}{\Delta \omega'} \left[ P(\omega'|\omega)\mu(\omega' + \Delta \omega') + \beta[u(\omega', \omega'') - P(\omega'|\omega)\mu(\omega') + \beta[u(\omega', \omega'')]] \right]$$

$$= P(\omega') \frac{\mu(\omega' + \Delta \omega') - \mu(\omega')}{\Delta \omega'} + \beta \left[ \frac{g(\omega' + \Delta \omega') - g(\omega')}{\Delta \omega'} + \frac{\kappa(\omega' + \Delta \omega') - \kappa(\omega')}{\Delta \omega'} + \frac{\mu(\omega' + \Delta \omega') - \mu(\omega')}{\Delta \omega'} \right]$$

$$= (P(\omega') + \beta) \frac{\Delta \mu}{\Delta \omega'} + \beta \left[ \frac{\Delta g}{\Delta \omega'} + \frac{\Delta \kappa}{\Delta \omega'} \right]$$
where we have used the fact that all terms that are functions of the current state cancel out. In order to get to the third line from the second, we have used the notion of an envelope theorem, although it can’t be applied directly here as we don’t have differentiability. But we have made use of the idea to get the relevant terms from the expression for $V(\omega')$. Finally, note that the first term in the last line above is the marginal utility from the proposed network-coalition pair and the second term is the marginal cost. In the optimal proposal these must equal each other. Alternatively, this sum must equal 0. Thus,

$$\left( P(.) + \beta \right) \frac{\Delta \mu}{\Delta \omega'} + \beta \left[ \frac{\Delta g}{\Delta \omega'} + \frac{\Delta \kappa}{\Delta \omega'} \right] = 0$$

yields

$$\frac{\Delta \mu}{\Delta \omega'} = -\frac{\beta}{\left( P(.) + \beta \right)} \left[ \frac{\Delta g}{\Delta \omega'} + \frac{\Delta \kappa}{\Delta \omega'} \right]$$