Cass existence approach of financial equilibria when portfolios are constrained

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Abstract
It is well known that equilibrium asset prices will not offer arbitrage opportunities to individuals. Using an approach that dates back to Cass (1984) [4], we seek to isolate arbitrage free asset prices that are also equilibrium asset prices. However we do this when each agent’s portfolio choice is restricted to a closed, convex set containing zero (as in Siconolfi [23]). In the presence of such portfolio restrictions we need to confine our attention to strong-arbitrage-free asset prices. We also describe a considerably weak condition on the space of income transfers that ensure these asset prices to be part of a financial equilibrium. Staying in the Cass [4] framework, we consider an exchange economy with nominal assets, but allow for multiple (finite) periods and very general preference relations. Moreover we show the existence theorem using the approach of Martins Da-Rocha and Triki [6], and hence do not resort to the Cass trick.

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1 Introduction

Investors facing restrictions on the portfolios that they can trade, is more of a norm than an exception. We consider a model in which investors’ portfolio sets are constrained. As in Balasko, Cass and Siconolfi [2] these constraints are exogenously given (probably arising due to some institutional reasons). Moreover, we consider very general restrictions on portfolio sets as in Siconolfi [23], where each agent’s portfolio set is assumed to be closed, convex and contains zero.

This paper primarily examines the existence of a financial equilibrium in a multiperiod model when investors face such general portfolio restrictions. In two date (one period) models without restrictions on portfolio sets, the existence issue has been extensively studied. Cass ([4]) and Werner ([25], [26]) showed existence with nominal assets. Duffie and Shafer ([9]) showed a generic existence result with real assets. This second approach has been extensively used. Magill and Shafer [15] provide a good survey of financial markets equilibria and contingent markets equilibria. Another approach to prove existence in a differentiable economy is to show existence in a numeraire asset economy and infer the existence in the nominal asset economy (See Villanacci et al. Villanacci et al and Magill and Quinzii [16]).

Multiperiod models are better equipped to capture the evolution of time and uncertainty and are a necessary step before studying infinite horizon models. Following Debreu’s [7] pioneering model we consider an event-tree to represent the evolution of time and uncertainty. Magill and Quinzii ([16]) and Angeloni and Cornet ([1]) are great references for the treatment of multiperiod financial models. Each node in the event tree represents a date event. Given information on asset prices and spot prices at all date events, consumers will choose a consumption and a portfolio of assets (assumed to be constrained here), such that the node specific value of the consumption does not exceed the node specific value of their endowments and the net returns from the portfolio.

In the absence of such portfolio restrictions the notion of absence of arbitrage is clear - if there is no portfolio that yields nonnegative net returns in all nodes and strictly positive returns in some node. However in the case where all agents face restrictions in their asset market participation, the notion of arbitrage and its absence at the individual level may differ from that at the aggregate level. Angeloni and Cornet (2006)[1] make this distinction. Given asset prices, an agent does not have arbitrage opportunities if she cannot find a portfolio within her constrained portfolio set that yields nonnegative net returns in all nodes and strictly positive returns in some node. On the other hand, there are no arbitrage opportunities in the aggregate, if there is no portfolio in the set of pooled
portfolio sets of all agents that yields nonnegative net returns in all nodes and strictly positive returns in some node.

At an equilibrium there must be no arbitrage at the individual level. A natural question then is will any asset price at which there is no arbitrage be an equilibrium asset price. The objective of this paper is to explore this characterization under general portfolio constraints.

In the absence of portfolio constraints, Cass ([4]), Duffie ([8]) and Florenzano and Gourdel ([10]) show this characterization of equilibrium and arbitrage free asset prices. In the presence of such constraints, the approach initiated by Cass ([4]), where one agent has an unconstrained portfolio set, facilitates the existence proof. This approach has been extensively used to show existence ever since, Magill and Shafer ([15]), Florenzano and Gourdel ([10]), Magill and Quinzii ([16]), Angeloni and Cornet ([1]) among others.

This approach of Cass ([4]), breaks the symmetry of the problem and hence it is not possible to give a symmetric existence (symmetric with respect to the agents’ problem). More recently in a working paper, with such general portfolio restrictions, Da-Rocha and Triki ([6]) have been able to show the characterization between equilibrium and arbitrage free asset prices without the use of the Cass approach.

In this paper we explore this characterization issue by showing that any strong-arbitrage-free asset price (where the span of the aggregate payoff space does not intersect the interior of the positive quadrant) can be supported as an equilibrium asset price. The approach here is similar to that in Da-Rocha and Triki ([6]), however the notion of absence of arbitrage and the assumptions on the set of income transfers are weaker that those in Da-Rocha and Triki ([6]). With Cass [4] as a motivation, the condition we require in this paper is that the cone generated by the aggregate income transfer possibilities be contained in the cone generated by the union of individual income transfer possibilities.

In the Cass approach, the unconstrained agent behaves like in an Arrow-Debreu economy and is able to accommodate the equilibrium excess demand for assets. The attainable consumption and asset allocation are then bounded. We observe that if the set of attainable income transfers is bounded then we can guarantee the existence of a weak equilibrium, which differs from an equilibrium only in the requirement that instead of asset market clearing, there is accounts clearing in the asset markets. The notion of weak equilibrium is useful when redundant assets exist.

Section 2 describes the $T$-period model and the notion of a financial equilibrium. Section 3, states the main result and discusses the notion of a weak equilibrium and its existence. Section 5 gives a detailed proof of the central result in this paper.
2 The $T$-period financial exchange economy

2.1 Time and uncertainty in a multiperiod model

We consider a multiperiod exchange economy with $(T + 1)$ dates, $t \in T : = \{0, \ldots, T\}$, and a finite set of agents $I = \{1, \ldots, l\}$. The stochastic structure of the model is described by a finite event-tree $D$ of length $T$ and we shall essentially use the same model as Angeloni and Cornet [1], (we refer to [16] for an equivalent presentation with information partitions). The set $D_t$ denotes the nodes (also called date-events) that could occur at date $t$ and the family $(D_t)_{t \in T}$ defines a partition of the set $D$; for each $\xi \in D$ we denote by $t(\xi)$ the unique $t \in T$ such that $\xi \in D_t$. Also we denote the cardinality of the set $D$ by $|D|$. At each date $t \neq T$, there is an a priori uncertainty about which node will prevail in the next date. There is a unique non-stochastic event occurring at date $t = 0$, which is denoted $\xi_0$, (or simply 0) so $D_0 = \{\xi_0\}$. Finally, every $\xi \neq \xi_0$ in the event-tree $D$ has a unique predecessor denoted $pr(\xi)$ in $D$. The predecessor mapping $pr : D \setminus \{\xi_0\} \rightarrow D$ satisfies $pr(D_t) = D_{t-1}$, for every $t \neq 0$. The element $pr(\xi)$ is called the immediate predecessor of $\xi$ and is also denoted $\xi^-$. For each $\xi \in D$, we let $\xi^+ = \{\xi \in D : \xi = \xi^-\}$ be the set of immediate successors of $\xi$; we notice that the set $\xi^+$ is nonempty if and only if $\xi \in D \setminus D_T$.

Moreover, for $\tau \in T \setminus \{0\}$ and $\xi \in D \setminus \bigcup_{t=1}^{T-1} D_t$ we define, by induction, $pr^\tau(\xi) = pr(pr^{\tau-1}(\xi))$ and we let the set of (not necessarily immediate) successors and the set of predecessors of $\xi$ be respectively defined by

$$D^+(\xi) = \{\xi' \in D : \exists \tau \in T \setminus \{0\} \mid \xi = pr^\tau(\xi')\},$$

$$D^-(\xi) = \{\xi' \in D : \exists \tau \in T \setminus \{0\} \mid \xi' = pr^\tau(\xi)\}.$$

If $\xi' \in D^+(\xi)$ [resp. $\xi' \in D^+(\xi) \cup \{\xi\}$], we shall also use the notation $\xi' > \xi$ [resp. $\xi' \geq \xi$].

We notice that $D^+(\xi)$ is nonempty if and only if $\xi \notin D_T$ and $D^-(\xi)$ is nonempty if and only if $\xi \neq \xi_0$. Moreover, one has $\xi' \in D^+(\xi)$ if and only if $\xi \in D^-((\xi')^\tau)$ and similarly $\xi' \in \xi^+$ if and only if $\xi = (\xi')^-$.  

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4In this paper, we shall use the following notations. A $(\bar{D} \times \bar{J})$–matrix $A$ is an element of $\mathbb{R}^{\bar{D} \times \bar{J}}$, with entries $(a(\xi,j))_{\xi \in D, j \in J}$; we denote by $A(\xi) \in \mathbb{R}^I$ the $\xi$–th row of $A$ and by $A(j) \in \mathbb{R}^D$ the $j$–th column of $A$. We recall that the transpose of $A$ is the unique $(\bar{J} \times \bar{D})$–matrix $A^T$ satisfying $(Ax) \bullet y = x \bullet (A^T y)$, for every $x \in \mathbb{R}^I$, $y \in \mathbb{R}^D$, where $\bullet$ [resp. $\bullet_\circ$] denotes the usual scalar product in $\mathbb{R}^D$ [resp. $\mathbb{R}^I$]. We shall denote by $\text{rank} A$ the rank of the matrix $A$. For every subsets $\bar{D} \subset D$ and $\bar{J} \subset J$, the $(\bar{D} \times \bar{J})$–sub-matrix of $A$ is the $(\bar{D} \times \bar{J})$–matrix $A$ with entries $\bar{a}(\xi,j) = a(\xi,j)$ for every $(\xi,j) \in \bar{D} \times \bar{J}$. Let $x, y$ be in $\mathbb{R}^n$; we shall use the notation $x \geq y$ (resp. $x \succ y$) if $x_h \geq y_h$ (resp. $x_h \succ y_h$) for every $h = 1, \ldots, n$ and we let $\mathbb{R}^+_n = \{x \in \mathbb{R}^n : x \geq 0\}$, $\mathbb{R}^+_n = \{x \in \mathbb{R}^n : x \succ 0\}$. We shall also use the notation $x > y$ if $x \geq y$ and $x \neq y$. We shall denote by $\| \cdot \|$ the Euclidean norm in the different Euclidean spaces used in this paper and the closed ball centered at $x \in \mathbb{R}^d$ of radius $r > 0$ is denoted $B_L(x,r) := \{y \in \mathbb{R}^d : \|y - x\| \leq r\}$.
2.2 The stochastic exchange economy

At each node $\xi \in D$, there is a spot market where a finite set $H = \{1, ..., H\}$ of divisible physical goods is available. We assume that each good does not last for more than one period. In this model, a commodity is a couple $(h, \xi)$ of a physical good $h \in H$ and a node $\xi \in D$ at which it will be available, so the commodity space is $\mathbb{R}^L$, where $L = H \times D$. An element $x$ in $\mathbb{R}^L$ is called a consumption, that is $x = (x(\xi))_{\xi \in D} \in \mathbb{R}^L$, where $x(\xi) = (x(h, \xi))_{h \in H}$, for every $\xi \in D$.

We denote by $p = (p(\xi))_{\xi \in D} \in \mathbb{R}^L$ the vector of spot prices and $p(\xi) = (p(h, \xi))_{h \in H} \in \mathbb{R}^H$ is called the spot price at node $\xi$. The spot price $p(h, \xi)$ is the price paid, at date $t(\xi)$, for the delivery of one unit of the physical good $h$ at node $\xi$. Thus the value of the consumption $x(\xi)$ at node $\xi \in D$ (evaluated in unit of account of node $\xi$) is

$$p(\xi) \cdot_R x(\xi) = \sum_{h \in H} p(h, \xi) x(h, \xi).$$

Each agent $i \in I$ is endowed with a consumption set $X_i \subset \mathbb{R}^L$ which is the set of her possible consumptions. An allocation is an element $x \in \prod_{i \in I} X_i$, and we denote by $x_i$ the consumption of agent $i$, that is the projection of $x$ onto $X_i$.

The tastes of each consumer $i \in I$ are represented by a strict preference correspondence $P_i : \prod_{j \in I} X^j \rightarrow X_i$, where $P_i(x)$ defines the set of consumptions that are strictly preferred by $i$ to $x_i$, that is, given the consumptions $x^j$ for the other consumers $j \neq i$. Thus $P_i$ represents the tastes of consumer $i$ but also her behavior under time and uncertainty, in particular her impatience and her attitude towards risk. If consumers’ preferences are represented by utility functions $u_i : X_i \rightarrow \mathbb{R}$, for every $i \in I$, the strict preference correspondence is defined by $P_i(x) = \{x_i \in X_i \mid u_i(\bar{x}_i) > u_i(x_i)\}$.

Finally, at each node $\xi \in D$, every consumer $i \in I$ has a node-endowment $e_i(\xi) \in \mathbb{R}^H$ (contingent to the fact that $\xi$ prevails) and we denote by $e_i = (e_i(\xi))_{\xi \in D} \in \mathbb{R}^L$ her endowment vector across the different nodes. The exchange economy $\mathcal{E}$ can thus be summarized by

$$\mathcal{E} = [D; H; I; (X_i, P_i, e_i)_{i \in I}].$$

2.3 The financial structure

We consider finitely many financial assets and we denote by $J = \{1, ..., J\}$ the set of assets. An asset $j \in J$ is a contract, which is issued at a given and unique node in $D$, denoted by $\xi(j)$ and called the emission node of $j$. Each asset $j$ is bought (or sold) at its emission node $\xi(j)$ and only yields payoffs at the successor nodes $\xi'$ of $\xi(j)$, that is, for $\xi' > \xi(j)$. We denote by $v(\xi, j)$ the payoff of asset $j$ at node $\xi$. Since we consider only
nominal assets this payoff does not depend on the spot prices. For the sake of convenient notations, we shall in fact consider the payoff of asset $j$ at every node $\xi \in D$ and assume that it is zero if $\xi$ is not a successor of the emission node $\xi(j)$. Formally, we assume that $v(\xi, j) = 0$ if $\xi \notin D^+(\xi(j))$. With the above convention, we notice that every asset has a zero payoff at the initial node, that is $v(\xi_0, j) = 0$ for every $j \in J$. Furthermore, every asset $j$ which is emitted at the terminal date $T$ has a zero payoff, that is, if $\xi(j) \in D_T$, $v(\xi, j) = 0$ for every $\xi \in D$.

For every consumer $i \in I$, if $z_i^j > 0$ [resp. $z_i^j < 0$], then $|z_i^j|$ will denote the quantity of asset $j \in J$ bought [resp. sold] by agent $i$ at the emission node $\xi(j)$. The vector $z_i = (z_i^j)_{j \in J} \in \mathbb{R}^J$ is called the portfolio of agent $i$.

We assume that each consumer $i \in I$ is endowed with a portfolio set $Z_i \subset \mathbb{R}^\mathbb{I}$, which represents the set of portfolios that are admissible for agent $i$.

The price of asset $j$ is denoted by $q_j$ and we recall that it is paid at its emission node $\xi(j)$. We let $q = (q_j)_{j \in J} \in \mathbb{R}^J$ be the asset price (vector).

To summarize, the financial asset structure $\mathcal{F} = (J, (\xi(j), V^j)_{j \in J}, (Z_i)_{i \in I})$ consists of

- a set of assets $J$,

- each asset $j \in J$ is defined by a node of issue $\xi(j) \in D$ and the vector of payoffs across all nodes $V^j = (v(\xi, j))_{\xi \in D} \in \mathbb{R}^D$,

- the portfolio set $Z_i \subset \mathbb{R}^J$ for every agent $i \in I$.

The payoff matrix of $\mathcal{F}$ is the $D \times J$ matrix $V = (v(\xi, j))_{\xi \in D, j \in J}$, and it satisfies the condition $v(\xi, j) = 0$ if $\xi \notin D^+(\xi(j))$ as explained above.

The full payoff matrix $W_\mathcal{F}(q)$ is the $(D \times J)$–matrix with entries

$$w_\mathcal{F}(x)(\xi, j) := v(\xi, j) - \delta_{\xi, \xi(j)}q_j,$$

where $\delta_{\xi, \xi'} = 1$ if $\xi = \xi'$ and $\delta_{\xi, \xi'} = 0$ otherwise.

So, for a given portfolio $z \in \mathbb{R}^J$ (and asset price $q$) the full flow of payoffs is $W_\mathcal{F}(q)z$ and the (full) financial payoff at node $\xi$ is

$$[W_\mathcal{F}(q)z](\xi) := W_\mathcal{F}(q, \xi) \cdot z = \sum_{j \in J} v(\xi, j)z_j - \sum_{j \in J} \delta_{\xi, \xi(j)}q_j z_j$$

$$= \sum_{\{j \in J | \xi(j) < \xi\}} v(\xi, j)z_j - \sum_{\{j \in J | \xi(j) = \xi\}} q_j z_j.$$

We shall extensively use the fact that, for $\lambda \in \mathbb{R}^D$, and $j \in J$, one has:

$$[\lambda W_\mathcal{F}(q)](j) = \sum_{\xi \in D} \lambda(\xi) v(\xi, j) - \sum_{\xi \in D} \lambda(\xi) \delta_{\xi, \xi(j)}$$

$$= \sum_{\xi > \xi(j)} \lambda(\xi) v(\xi, j) - \lambda(\xi(j)) q_j.$$
In the following, when the financial structure \( \mathcal{F} \) remains fixed, while only prices vary, we shall simply denote by \( W(q) \) the full matrix of payoffs. In the case of unconstrained portfolios, namely \( Z_i = \mathbb{R}^j \), for every \( i \in \mathcal{I} \), the financial asset structure will be simply denoted by \( \mathcal{F} = (J, (\xi(j), V^j)_{j \in J}) \).

### 2.4 Financial equilibrium

We now consider a financial exchange economy, which is defined as the couple of an exchange economy \( \mathcal{E} \) and a financial structure \( \mathcal{F} \). It can thus be summarized by

\[
(\mathcal{E}, \mathcal{F}) := \{D, H, \mathcal{I}, (X_i, p, e_i)_{i \in \mathcal{I}}; \mathcal{J}, (\xi(j), V^j)_{j \in J}, (Z_i)_{i \in \mathcal{I}}\}.
\]

Given the price \((p, q) \in \mathbb{R}^L \times \mathbb{R}^J\), the budget set (resp. quasi-budget set) of consumer \( i \in \mathcal{I} \) is\(^5\)

\[
B_i(p, q) = \{(x_i, z_i) \in X_i \times Z_i : \forall \xi \in D, \ p(\xi) \cdot \mathcal{H}[x_i(\xi) - e_i(\xi)] \leq [W_{\mathcal{F}} q]_i(\xi)\} = \{(x_i, z_i) \in X_i \times Z_i : p_\square (x_i - e_i) \leq W_{\mathcal{F}} q z_i\}.
\]

\[
(\text{resp. } \bar{B}_i(p, q) = \{(x_i, z_i) \in X_i \times Z_i : p_\square (x_i - e_i) \ll W_{\mathcal{F}} q z_i\})
\]

When \( \mathcal{F} \) is fixed we can drop the subscript \( \mathcal{F} \) from the budget set. We now introduce the equilibrium notion.

**Definition 2.1** An equilibrium of the financial exchange economy \((\mathcal{E}, \mathcal{F})\) is a list of strategies and prices \((\bar{x}, \bar{z}, \bar{p}, \bar{q}) \in (\mathbb{R}^L)^I \times (\mathbb{R}^J)^I \times \mathbb{R}^L \times \mathbb{R}^J\) such that

(a) for every \( i \in \mathcal{I} \), \((\bar{x}_i, \bar{z}_i)\) maximizes the preferences \( P_i \) in the budget set \( B_i(\bar{p}, \bar{q}) \), in the sense that

\[
(\bar{x}_i, \bar{z}_i) \in B_i(\bar{p}, \bar{q}) \text{ and } [P_i(\bar{x}) \times Z_i] \cap B_i(\bar{p}, \bar{q}) = \emptyset;
\]

(b) \( \sum_{i \in \mathcal{I}} \bar{x}_i = \sum_{i \in \mathcal{I}} e_i \) and \( \sum_{i \in \mathcal{I}} \bar{z}_i = 0.\)

**Definition 2.2** A list of strategies and prices \((\bar{x}, \bar{z}, \bar{p}, \bar{q}) \in (\mathbb{R}^L)^I \times (\mathbb{R}^J)^I \times \mathbb{R}^L \times \mathbb{R}^J\) is a quasi-equilibrium of the financial exchange economy \((\mathcal{E}, \mathcal{F})\) if \( \bar{p} \neq 0 \) and \(^6\)

(a i) for every \( i \in \mathcal{I} \),

\[
(\bar{x}_i, \bar{z}_i) \in B_i(\bar{p}, \bar{q}) \text{ and } [P_i(\bar{x}) \times Z_i] \cap \bar{B}_i(\bar{p}, \bar{q}) = \emptyset;
\]

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\(^5\)For \( x = (x(\xi))_{\xi \in \mathcal{D}}, p = (p(\xi))_{\xi \in \mathcal{D}} \in \mathbb{R}^L \times \mathbb{R}^J \) (with \( x(\xi), p(\xi) \in \mathbb{R}^n \)) we let \( p_\square x = (p(\xi) \cdot x(\xi))_{\xi \in \mathcal{D}} \in \mathbb{R}^n.\)

\(^6\)For \( x = (x(\xi))_{\xi \in \mathcal{D}} \in \mathbb{R}^n \), we denote \( x(-\xi) = (x(\xi'))_{\xi \in \mathcal{D} \setminus \xi}.\)
\( (a ii) \) for every \( i \in I \), \( \bar{p} \square (\bar{x}_i - e_i) = W(\bar{q})z_i; \)

\( (a iii) \) for every \( \xi \in \mathcal{D} \), \( (x_i(-s), x_i(s)) \in P_i(\bar{x}) \) implies \( \bar{p}(s) \bullet_H (x_i(s)) \geq \bar{p}(s) \bullet_H (\bar{x}_i(s)); \)

\( (b) \) \( \sum_{i \in I} \bar{x}_i = \sum_{i \in I} e_i \) and \( \sum_{i \in I} \bar{z}_i = 0. \)

### 2.5 Arbitrage and equilibrium

In the case where portfolio sets are constrained the absence of arbitrage opportunities at the individual level will differ from that at the aggregate level. As outlined in Angeloni and Cornet [1] we have the following definition.

**Definition 2.3** Given the financial structure \( \mathcal{F} = (J, (\xi(j), V_j))_{j \in J}, (Z_i)_{i \in I} \),

(i) A portfolio \( \bar{z}_i \in Z_i \) is said to have no arbitrage opportunities or to be arbitrage-free for agent \( i \in I \) at the price \( q \in \mathbb{R}^J \) if there is no portfolio \( z_i \in Z_i \) such that \( W_F(q)z_i \geq W_F(q)\bar{z}_i \), that is, \( [W_F(q)z_i](\xi) \geq [W_F(q)\bar{z}_i](\xi) \), for every \( \xi \in \mathcal{D} \), with at least one strict inequality, or, equivalently, if

\[
W_F(q)(Z_i - \bar{z}_i) \cap \mathbb{R}^D_+ = \{0\}.
\]

(ii) We say \( q \) is an arbitrage free asset price or the financial structure \( \mathcal{F} \) is said to be arbitrage-free at \( q \) if there exists no portfolios \( z_i \in Z_i \) \( (i \in I) \) such that \( W_F(q)(\sum_{i \in I} z_i) > 0 \), or, equivalently, if:

\[
W_F(q)(\sum_{i \in I} Z_i) \cap \mathbb{R}^D_+ = \{0\}.
\]

When asset market participation is restricted, as we will show in the course of this paper, there are fewer asset prices that can be supported as equilibrium asset prices than those that do not offer arbitrage. We will thus consider a subset of the no arbitrage asset prices that are given by the following definition.

**Definition 2.4** We say that the asset price \( q \) is aggregate arbitrage-free if one of the following equivalent conditions hold\(^7\)

\( (i) \) \( \left( W_F(q)(\sum_{i \in I} Z_i) \right) \cap \mathbb{R}^D_+ = \{0\}. \)

\( (ii) \) There exists \( \lambda \in \mathbb{R}^D_+ \) such that \( \lambda \bullet_D w = 0 \) for all \( w \in \left( W_F(q)(\sum_{i \in I} Z_i) \right) \).

If \( q \) is an aggregate arbitrage-free asset price then the financial structure is arbitrage free at \( q \). Let the financial structure \( \mathcal{F} \) be arbitrage-free at \( q \), and let \( \bar{z}_i \in Z_i \) \( (i \in I) \) such that \( \sum_{i \in I} W_F(q)\bar{z}_i = 0 \), then it is easy to see that, for every \( i \in I \), \( \bar{z}_i \) is arbitrage-free at \( q \). The converse is true, when \( \sum_{i \in I} W_F(q)Z_i \subset \text{cone } [\bigcup_{i \in I} W(q)(Z_i - \bar{z}_i)] \). The later is true

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\(^7\)Given a subset \( A \subset \mathbb{R}^n \) we denote \( (A) := \text{Span } A \).
in particular when some agent’s portfolio set is unconstrained, that is, \( Z_i = \mathbb{R}^J \) for some \( i \in I \).

Consider the following non-satiation assumption:

**Assumption NS (i)** For every \( \bar{x} \in \prod_{i \in I} X_i \) such that \( \sum_{i \in I} x_i = \sum_{i \in I} e_i \),

(Non-Satiation at Every Node) for every \( \xi \in \mathcal{D} \), there exists \( x \in \prod_{i \in I} X_i \) such that, for each \( \xi \neq \xi_i, \), \( x_i(\xi) = \bar{x}_i(\xi) \) and \( x_i \in P_i(\bar{x}) \);

(ii) if \( x_i \in P_i(\bar{x}) \), then \([x_i, \bar{x}_i] \subset P_i(\bar{x})\).

It is well known that if preferences are non-satiated then there is no arbitrage at the individual level. In particular, under (NS), if \((\bar{x}, \bar{z}, \bar{p}, \bar{q})\) is an equilibrium of the economy \((\mathcal{E}, \mathcal{F})\), then \( \bar{z}_i \) is arbitrage-free at \( \bar{q} \) for every \( i \in I \) (see Angeloni and Cornet [1]).

### 3 Existence of equilibrium

#### 3.1 The main existence result

We will prove that when agents’ portfolio sets are constrained, any aggregate strong-no-arbitrage asset price can be characterized as an equilibrium asset price. Our approach however does not cover the general case of real assets which needs a different treatment.

Let us consider, the financial economy

\((\mathcal{E}, \mathcal{F}) = [\mathcal{D}, \mathcal{H}, \mathcal{I}, (X_i, P_i, e_i)_{i \in I}; \mathcal{J}, (\xi(j), V^j)_{j \in \mathcal{J}}, (Z_i)_{i \in I}]\).

Define the set of attainable consumptions by

\[ \hat{X} = \{ x \in \prod_{i \in I} X_i \mid \sum_{i \in I} x_i = \sum_{i \in I} e_i \} \]

and for each \( i \in I \), let \( \hat{X}_i \) be the projection of \( \hat{X} \) on \( X_i \).

We introduce the following assumptions.

**Assumption (C) (Consumption Side)** For all \( i \in I \) and all \( \bar{x} \in \prod_{i \in I} X_i \),

(i) \( X_i \) is a closed and convex subset of \( \mathbb{R}^L \) and \( \hat{X}_i \) is compact\(^8\) in \( \mathbb{R}^L \);

(ii) [Continuity] the preference correspondence \( P_i : \prod_{i \in I} X_i \to X_i \), is lower semicontinuous\(^9\) and \( P_i(\bar{x}) \) is open in \( X_i \) (for its relative topology);

---

\(^8\)Note: \( \hat{X}_i \) is compact if \( X_i \) is bounded below.

\(^9\)A correspondence \( \varphi : X \to Y \) is said to be lower semicontinuous at \( x_0 \in X \) if, for every open set \( V \subset Y \) such that \( V \cap \varphi(x_0) \) is not empty, there exists a neighborhood \( U \) of \( x_0 \) in \( X \) such that, for all \( x \in U \), \( V \cap \varphi(x) \) is nonempty. The correspondence \( \varphi \) is said to be lower semicontinuous if it is lower semicontinuous at each point of \( X \).
(iii) [Convexity] $P_i(\bar{x})$ is convex;

(iv) (Irreflexivity) $\bar{x} \notin P_i(\bar{x})$;

(v) (Non-Satiation of Preferences at Every Node) if $\sum_{i \in I} \bar{x}_i = \sum_{i \in I} e_i$ for every $\xi \in D$ there exists $x \in \prod_{i \in I} X_i$ such that, for each $\xi' \neq \xi$, $x_i(\xi') = \bar{x}_i(\xi')$ and $x_i \in P_i(\bar{x})$;

(vi) (Strong Survival Assumption) $e_i \in \text{int} X_i$.

Note that these assumptions on $P_i$ are satisfied in particular when agents preferences are given by a utility function that is continuous, monotonic increasing, and quasi-concave.

**Assumption (F) (Financial Side)**

1. For every $i \in I$, $Z_i$ is closed, convex and contains zero;
2. For every asset price $q \in \mathbb{R}^J$, for every $i \in I$, $W(q)Z_i$ is a closed subset of $\mathbb{R}^D$;

We can now state the main theorem characterizing equilibrium prices with arbitrage free prices under the appropriate compatibility condition.

**Theorem 3.1** (a) Suppose the financial exchange economy $(\mathcal{E}, \mathcal{F})$ satisfies C and F. Let $\bar{q} \in \mathbb{R}^J$ be an aggregate arbitrage-free asset price such that the following conditions hold:

1. Closed Cone $\left( \bigcup_{i \in I} W(\bar{q})(Z_i) \right)$ is convex; and
2. (WEQ) $-\sum_{i \in I} Z_i \cap \text{Ker} W(\bar{q}) \subset \sum_{i \in I} \left( A(Z_i) \cap \text{Ker} W(\bar{q}) \right)$.

Then there exists $(\bar{x}, \bar{z}, \bar{p})$ with $\bar{p} \neq 0$, such that $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is a quasi-equilibrium.

(b) If we additionally assume that

1. Closed Cone $\left( \bigcup_{i \in I} W(\bar{q})(Z_i) \right)$ is linear,

then there exists $(\bar{x}, \bar{z}, \bar{p})$ with $\bar{p}(\xi) \neq 0$, for all $\xi \in D$ such that $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is an equilibrium.

Assumption WEQ is taken from Martins - Triki [6] and allows us to show that existence of a weak equilibrium implies the existence of an equilibrium. It is worth noticing that the two assumptions in the theorem are satisfied in the following examples.

**Example 3.1** For every $i \in I$, $Z_i$ is closed, convex and contains zero and $\bigcup_{i \in I} Z_i$ is a vector space.

**Example 3.2** (Cass Condition) For every $i \in I$, $Z_i$ is closed, convex and contains zero and for some $i_0 \in I$, $Z_i \subset Z_{i_0}$ and $Z_{i_0}$ is a vector space. Then $\bigcup_{i \in I} Z_i = Z_{i_0}$ is a vector space and we are in the structure of Example 3.3.

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Example 3.3 Let $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$ and

$$\forall i \in \mathcal{I}_1, \quad Z_i = \{ z \mid z \cdot e_1 \geq 0, e_1 = (1, 0) \}$$

$$\forall i \in \mathcal{I}_2, \quad Z_i = \{ z \mid z \cdot e_2 \geq 0, e_2 = (0, 1) \}.$$

That is every agent in $\mathcal{I}_1$ can buy the first asset and every agent in $\mathcal{I}_2$ can sell the first asset.

It is also worth noticing that in Condition (i) in Theorem 3.1, taking $\bigcup_{i \in \mathcal{I}} W(\bar{q})Z_i$ to be ‘closed cone’ instead of a linear space allows us to consider the following example. Let $\mathcal{I} = 1, 2$ and

$$Z_1 = \{ z \in \mathbb{R}^2 \mid z_2 \geq (z_1)^2 \}$$

$$Z_1 = \{ z \in \mathbb{R}^2 \mid z_2 \leq -(z_1)^2 \}.$$

Example 3.4 Give counterexample here.

3.2 Quasi-equilibrium

Gottardi and Hens consider a two date incomplete markets model without consumption in the first date. Their definition of a quasi-equilibrium was suitably modified by Seghir et al. to include consumption in first date. They define a quasi-equilibrium as in the following definition. The lemma that follows shows that in a two date model, a quasi-weak-equilibrium in our model implies a quasi-weak-equilibrium in the sense of Seghir et al.

Definition 3.1 (Seghir et al.) A list of strategies and prices $(\bar{x}, \bar{z}, \bar{p}, \bar{q}) \in (L^L)^I \times (L^J)^I \times \mathbb{R}^L \times \mathbb{R}^J$ is a quasi-equilibrium of the financial exchange economy $(E, F)$, if $\bar{p} \neq 0$,

(a i) for every $i \in \mathcal{I}, x_i \in P_i(\bar{x})$ and $\bar{p} \cdot (x_i - e_i) \leq Vz_i \Rightarrow \bar{p}(0) \cdot H_i(x_i(0) - e_i(0)) + q \cdot z_i \geq 0$;

(a ii) for every $i \in \mathcal{I}, \bar{p} \cdot (x_i - e_i) = W(\bar{q})z_i$;

(a iii) for every $\xi \in D_i, (x_i(-s), x_i(s)) \in P_i(\bar{x})$ implies $\bar{p}(s) \cdot H_i(x_i(s)) \geq \bar{p}(s) \cdot H_i(\bar{x}_i(s))$;

(b) $\sum_{i \in I} \bar{x}_i = \sum_{i \in I} e_i$ and $\sum_{i \in I} \bar{z}_i = 0$.

Let

$$\tilde{B}_i(p, q) = \{ (x_i, z_i) \in X_i \times Z_i \mid p(0) \cdot H_i(x_i(0) - e_i(0)) < -q \cdot z_i; p \cdot (x_i - e_i) \leq Vz_i \}$$

Notice that (a i) of this definition is true if an only if $(P_i(\bar{x}) \times Z_i) \cap \tilde{B}_i(\bar{p}, \bar{q}) = \emptyset$.

Comparing Definitions 2.2 and 3.1 we see that the quasi-equilibrium conditions a( ii), (a iii) and (b) are the same. The following lemma shows how (a i) in Definition 2.2 is related to (a i) of Definition 3.1.
Lemma 3.1 Under the assumption\(^{10}\) (F0): \(\exists \hat{z}_i \in A(Z_i)\) such that \(V \hat{z}_i >> 0\),

\[ (P_i(\bar{x}) \times Z_i) \cap \tilde{B}_i(\bar{p}, \bar{q}) = \emptyset \Rightarrow (P_i(\bar{x}) \times Z_i) \cap \tilde{B}_i(\bar{p}, \bar{q}) = \emptyset. \]

Proof of Lemma 3.1. Suppose on the contrary there is some \((x_i, z_i) \in (P_i(\bar{x}) \times Z_i) \cap \tilde{B}_i(\bar{p}, \bar{q}) \neq \emptyset\). Then

\[ \bar{p}(0) \cdot \mathbf{H}(x_i(0) - e_i(0)) < -\bar{q} \cdot z_i \]

\[ p \cdot 1(x_i - e_i) \leq V z_i. \]

Take \(\hat{z}_i \in AZ_i\) with \(V \hat{z}_i >> 0\). Then for all \(t > 0\) small enough

\[ \bar{p}(0) \cdot \mathbf{H}(x_i(0) - e_i(0)) < -\bar{q} \cdot z_i - \bar{q} \cdot z_i(t \hat{z}_i) = -\bar{q}(z + t \hat{z}_i) \]

\[ p \cdot 1(x_i - e_i) << V (z_i + t \hat{z}_i) \]

and clearly \(z_i + t \hat{z}_i \in Z_i\) since \(\hat{z}_i \in AZ_i\). This contradicts \((P_i(\bar{x}) \times Z_i) \cap \tilde{B}_i(\bar{p}, \bar{q}) = \emptyset\). □

Example 3.5 Counterexample. In general, if closed cone \(\bigcup_{i \in I} W(\bar{q})Z_i\) is convex and not linear then we may not have a weak-equilibrium and hence no equilibrium.

4 Existence of a weak equilibrium

The main result of our paper will be proved as a consequence of the following more general result, which is also interesting by itself.

Definition 4.1 A weak-equilibrium (resp. quasi-weak-equilibrium) in the economy \((\mathcal{E}, \mathcal{F})\) is a list \((\bar{x}, \bar{z}, \bar{p}, \bar{q})\) satisfying all the condition in Definition 2.1 (resp. 2.2) except that asset market clearing, \(\sum_{i \in I} z_i\) is replaced by accounts clearing in asset markets \(\sum_{i \in I} W(\bar{q})Z_i = 0\).

Theorem 4.1 Suppose the financial exchange economy \((\mathcal{E}, \mathcal{F})\) satisfies C and F. Let \(\bar{q} \in \mathbb{R}^J\) be an aggregate arbitrage-free asset price such that

\[ \text{Closed Cone} \left( \bigcup_{i \in I} W(\bar{q})(Z_i) \right) \text{ is linear.} \]

Then there exists \((\bar{x}, \bar{z}, \bar{p})\) with \(\bar{p}(\xi) \neq 0, \forall \xi \in D\) such that \((\bar{x}, \bar{z}, \bar{p}, \bar{q})\) is a weak-equilibrium.

\(^{10}\)Given a convex set \(Y \subset \mathbb{R}^n\), the asymptotic cone of \(Y\) is \(A(Y) := \{t \in \mathbb{R}^n \mid t + y \in Y, \forall y \in Y\}\).
The proof of Theorem 3.1 is then a consequence of the following proposition due to Martins da-Rocha and Triki [6].

**Proposition 4.1** Existence of a weak-equilibrium \((\bar{x}, \bar{z}, \bar{p}, \bar{q})\) implies the existence of an equilib-rium if we have:

\[-\sum_{i \in I} Z_i \cap Ker W(\bar{q}) \subset \sum_{i \in I} (A(Z_i) \cap Ker W(\bar{q})) \]

The condition in Proposition 4.1 holds if \(Ker W \subset \bigcup_{i \in I} A(Z_i)\).

**Theorem 4.2** (a) Suppose the financial exchange economy \((E, F)\) satisfies \(C\) and \(F\). Let \(\bar{q} \in \mathbb{R}^J\) be a aggregate arbitrage-free asset price such that closed cone \((\bigcup_{i \in I} W(\bar{q})(Z_i))\) is convex. Then there exists \((\bar{x}, \bar{z}, \bar{p})\) with \(\bar{p} \neq 0\) such that \((\bar{x}, \bar{z}, \bar{p}, \bar{q})\) is a quasi-weak-equilibrium.

(b) If we additionally assume that closed cone \((\bigcup_{i \in I} W(\bar{q})(Z_i))\) is linear then there exists \((\bar{x}, \bar{z}, \bar{p})\) with \(\bar{p}(\xi) \neq 0\) for all \(\xi \in D\) such that \((\bar{x}, \bar{z}, \bar{p}, \bar{q})\) is a weak-equilibrium.

5 Proof of existence under additional assumptions.

5.1 Additional assumptions (K):

We will first provide a proof of Theorem 4.2 under the following additional assumptions (together with those already made in Theorem 4.2).

(i) The sets \(X_i\) and \(W(\bar{q})Z_i\) for a given \(\bar{q} \in \mathbb{R}^J\), are bounded;

(ii) [Local Non-Satiation] for every \(\bar{x} \in \prod_{i \in I} X_i\), for every \(x_i \in P_i(\bar{x})\), \([x_i, \bar{x}_i) \subset P_i(\bar{x})\).

Then in the next section we will give the proof of Theorem 4.2, that is without assuming Assumption K.

5.2 Preliminary lemma

Before entering the proof of Theorem 4.2 we will state and prove the following:

**Lemma 5.1** Let \(\bar{q} \in \mathbb{R}^J\), let \(W\) be the closed convex cone spanned by \(W(\bar{q})(\bigcup_{i \in I} Z_i)\) and assume that

\[< W > \cap \mathbb{R}_+^D = \{0\}.\]

Then there exists \(\lambda \in \mathbb{R}_+^D\) such that:

\[\langle W \rangle \subset \lambda^+ := \{t \in \mathbb{R}^D | \lambda \cdot t = 0\}\]
Given $\lambda \in \mathbb{R}^D_+$ as in Lemma 5.1, let  
\[ B_L = \{ p \in \mathbb{R}^L \mid ||\lambda \cdot \square p|| \leq 1 \}. \]

Given $p \in B_L$, let  
\[ \rho(p) \in \mathbb{R}^D \text{ with } \rho(p, \xi) = (1 - ||\lambda \cdot \square p||) \text{ for all } \xi \in \mathbb{D}. \]

Given $p \in B_L$ and $\gamma : B_L \rightarrow \mathbb{R}^D$, for all $i \in \mathcal{I}$ define  
\[ B_i^{\gamma}(p) = \left\{ (x_i, w_i) \in X_i \times WZ_i : \exists \tau_i \in [0, 1], p \cdot (x_i - e_i) \leq w_i + \tau_i \gamma(p) + \rho(p) \right\}, \]
\[ \tilde{B}_i^{\gamma}(p) = \left\{ (x_i, w_i) \in X_i \times WZ_i : \exists \tau_i \in [0, 1], p \cdot (x_i - e_i) \ll w_i + \tau_i \gamma(p) + \rho(p) \right\}. \]

We state a lemma, which extends a previous result by Da-Rocha and Triki [6], the proof of which is in the appendix.

**Lemma 5.2**

(i) For every $\varepsilon > 0$, there exists a continuous mapping $\gamma : B_L \rightarrow \mathbb{R}^D$ such that,  
\[ \forall p \in B_L, \lambda \cdot \square \gamma(p) = 0 \text{ and } \forall w \in W, w \cdot \square \gamma(p) \leq 0 \text{ and } ||\gamma(p)|| \leq \varepsilon. \]

(ii) Assume that the closed cone spanned by $W(q)(\cup_{i \in I} Z_i)$ is convex, then  
\[ \forall p \in B_L, \exists i \in \mathcal{I}, \text{ such that } \tilde{B}_i^{\gamma}(p) \neq \emptyset. \]

### 5.2.1 The fixed point argument

Let $\gamma : B_L \rightarrow \mathbb{R}^D$ be the continuous mapping associated by Lemma 5.2 to some $\varepsilon$. For $(x, w, p) \in \prod_{i \in I} X_i \times \prod_{i \in I} WZ_i \times B_L$, we define the correspondences $\Phi_i$ for $i \in \mathcal{I}_0 := \{0\} \cup \mathcal{I}$ as follows:  
\[ \Phi^0(x, w, p) = \left\{ p' \in B_L \mid (\lambda \cdot \square (p' - p)) \cdot \sum_{i \in I} (x_i - e_i) > 0 \right\}, \]
and for every $i \in \mathcal{I}$,  
\[ \Phi_i(x, w, p) = \begin{cases} 
\{(e_i, 0)\} & \text{if } (x_i, w_i) \notin \tilde{B}^{\gamma}_i(p) \text{ and } \tilde{B}_i^{\gamma}(p) = \emptyset, \\
B_i^{\gamma}(p) & \text{if } (x_i, w_i) \notin \tilde{B}^{\gamma}_i(p) \text{ and } \tilde{B}_i^{\gamma}(p) \neq \emptyset, \\
\tilde{B}_i^{\gamma}(p) \cap (P_i(x) \times WZ_i) & \text{if } (x_i, w_i) \in \tilde{B}^{\gamma}_i(p). 
\end{cases} \]

The existence proof relies on the following fixed-point-type theorem due to Gale and MasCollel ([12]).

---

\[ ^{11} \text{For } x \in \mathbb{R}^n, ||x|| \text{ denotes the euclidean norm. } \]
Theorem 5.1 Let $I_0$ be a finite set, let $C_i (i \in I_0)$ be a nonempty, compact, convex subset of some Euclidean space, let $C = \prod_{i \in I_0} C_i$ and let $\Phi_i (i \in I_0)$ be a correspondence from $C$ to $C_i$, which is lower semicontinuous and convex-valued. Then, there exists $c \in C$ such that, for every $i \in I_0$ either $\bar{c}_i \in \Phi_i(c)$ or $\Phi_i(c) = \emptyset$.

We now show the sets $C_0 = B_{\mathbb{L}}, C_i = X_i \times WZ_i (i \in I)$ and the above defined correspondences $\Phi_i (i \in I_0)$ satisfy the assumptions of Theorem 5.1.

Claim 5.1 For every $c := (\bar{x}, \bar{w}, \bar{p}) \in \prod_{i \in I} X_i \times \prod_{i \in I} WZ_i \times B_{\mathbb{L}}$

(i) $\Phi_i(c)$ is convex (possibly empty);

(ii) $\bar{p} \notin \Phi_0(c)$, and for all $i \in I$, $(\bar{x}_i, \bar{w}_i) \notin \Phi_i(c)$;

(iii) for every $i \in I_0$, the correspondence $\Phi_i$ is lower semicontinuous at $c$.

Proof of Claim 5.1: Let $c := (\bar{x}, \bar{w}, \bar{p}) \in \prod_{i \in I} X_i \times \prod_{i \in I} WZ_i \times B_{\mathbb{L}}$ be given.

Proof of (i): Clearly $\Phi_0(c)$ is convex. For every $i \in I$, recalling that $P_i(\bar{x})$ and $WZ_i$ are convex sets, by Assumption $C$ and $F$, we have $\Phi_i(c)$ is a convex set.

Proof of (ii): Clearly, $\bar{p} \notin \Phi_0(c)$ and $(\bar{x}_i, \bar{w}_i) \notin \Phi_i(c)$ follows from the definitions of these sets and the fact that $\bar{x}_i \notin P_i(\bar{x})$ (from Assumption $C$).

Proof of (iii): We need to show that $\Phi_i$ is lower semicontinuous for all $i \in I_0$. Since $\Phi_0$ has an open graph, clearly it is lower semicontinuous. To show lower semicontinuity of $\Phi_i$ for $i \in I$, let $U$ be an open subset of $X_i \times WZ_i$ such that $\Phi(c) \cap U \neq \emptyset$ and we will distinguish three cases:

Case (1): $(\bar{x}_i, \bar{w}_i) \notin B_i^{\rho_0}(\bar{p})$ and $\bar{B}_i^{\rho_0}(\bar{p}) = \emptyset$. Then $\Phi_i(c) = \{(e_i, 0)\} \subset U$. The set $\Omega_i = \{(x_i, w_i, p) \mid (x_i, w_i) \notin B_i^{\rho_0}(p)\}$ is an open subset of $X_i \times WZ_i \times B_{\mathbb{L}}$ (by Assumptions $C$ and $F$). To see this, let $\{(x_i, w_i, p)\}$ be such that $(x_i, w_i) \in B_i^{\rho_0}(p)$ and $(x_i, w_i, p) \rightarrow (x, w, p)$.

Since for all $n, (x_i, w_i) \in B_i^{\rho_0}(p)$, there exists $\tau_i \in [0, 1]$ such that

$$p \square (x_i - e_i) \leq w_i + \tau_i \gamma(p) + \rho(p)$$

In the limit we have

$$p \square (x - e_i) \leq w + \tau \gamma(p) + \rho(p)$$

Where $\tau = \lim_{n \rightarrow \infty} \tau_i \in [0, 1]$. Thus $(x, w) \in B_i^{\rho_0}(p)$.

Thus $\Omega_i$ contains an open neighborhood $O$ of $c$. Now, let $c = (x, w, p) \in O$. If $\bar{B}_i^{\rho_0}(p) = \emptyset$ then $\Phi_i(c) = \{(e_i, 0)\} \subset U$ and so $\Phi_i(c) \cap U$ is nonempty. If $\bar{B}_i^{\rho_0}(p) \neq \emptyset$ then $\Phi_i(c) = B_i^{\rho_0}(p)$.

But Assumptions $C$ and $F$ imply that $(e_i, 0) \in X_i \times WZ_i$, hence $(e_i, 0) \in B_i^{\rho_0}(p)$ with $\tau_i = 0$ and noticing that $\rho(p) \geq 0$. So $\{(e_i, 0)\} \subset \Phi_i(c) \cap U$ which is also nonempty.
Case (2): \( \bar{c} = (\bar{x}_i, \bar{w}_i, \bar{p}) \in \Omega := \{ c = (x_i, w_i, p) : (x_i, w_i) \notin B_i^{\rho}(p) \text{ and } \bar{B}_i^{\rho}(p) \neq \emptyset \} \). Then the set \( \Omega \) is clearly open (since its complement is closed).

On the set \( \Omega \), one has \( \Phi_i(c) = B_i^{\rho}(p) \). We recall that \( \emptyset \neq \Phi_i(\bar{c}) \cap U = \bar{B}_i^{\rho}(\bar{p}) \cap U \). We notice that \( B_i^{\rho}(p) = \text{cl} \bar{B}_i^{\rho}(\bar{p}) \) since \( \bar{B}_i^{\rho}(\bar{p}) \neq \emptyset \). Consequently, \( \bar{B}_i^{\rho}(\bar{p}) \cap U \neq \emptyset \) and we chose a point \( (x_i, w_i) \in \bar{B}_i^{\rho}(\bar{p}) \cap U \), that is, \( (x_i, w_i) \in [X_i \times WZ_i] \cap U \) and for some \( \tau_i \in [0, 1] \),

\[
\bar{p} \Box (x_i - e_i) \leq w_i + \tau_i \gamma(\bar{p}) + \rho(\bar{p}).
\]

Clearly the above inequality is also satisfied for the same point \( (x_i, w_i) \) and the same \( \tau_i \) when \( p \) belongs to a neighborhood \( O \) of \( \bar{p} \) small enough (using the continuity of \( \rho(\cdot) \) and \( \gamma(\cdot) \)). This shows that on \( O \) one has \( \emptyset \neq \bar{B}_i^{\rho}(p) \cap U \subset B_i^{\rho}(p) \cap U = \Phi(c) \cap U \).

Case (3): \( (\bar{x}_i, \bar{w}_i) \in \bar{B}_i^{\rho}(\bar{p}) \). By assumption we have

\[
\emptyset \neq \Phi_i(c) \cap U = B_i^{\rho}(p) \cap [P_i(\bar{x}) \times WZ_i] \cap U.
\]

By an argument similar to what is done above, one shows that there exists an open neighborhood \( N \) of \( \bar{p} \) and an open set \( M \) such that, for every \( p \in N \), one has \( \emptyset \neq M \subset \bar{B}_i^{\rho}(p) \cap U \).

Since \( P_i \) is lower semicontinuous at \( \bar{c} \) (by Assumption C), there exists an open neighborhood \( \Omega \) of \( \bar{c} \) such that, for every \( c \in \Omega \), \( \emptyset \neq [P_i(x) \times WZ_i] \cap M \), hence

\[
\emptyset \neq [P_i(x) \times WZ_i] \cap \bar{B}_i^{\rho}(p) \cap U \subset B_i^{\rho}(p) \cap U \text{ for every } c \in \Omega.
\]

Consequently, from the definition of \( \Phi_i \), we get \( \emptyset \neq \Phi_i(c) \cap U \), for every \( c \in \Omega \).

The correspondence \( \Psi_i := B_i^{\rho} \cap (P_i \times WZ_i) \) is lower semicontinuous on the whole set, being the intersection of an open graph correspondence and a lower semicontinuous correspondence. Then there exists an open neighborhood \( O \) of \( \bar{c} := (\bar{x}, \bar{w}, \bar{p}) \) such that, for every \( (x, w, p) \in O \), then \( U \cap \Psi_i(x, w, p) \neq \emptyset \) hence \( \emptyset \neq U \cap \Phi_i(x, w, p) \) (since we always have \( \Psi_i(x, w, p) \subset \Phi_i(x, w, p) \)). \( \Box \)

Given \( \varepsilon > 0 \), in view of Claim 5.1, we can apply the fixed-point Theorem 5.1. Thus there exists \( \bar{c} = (\bar{x}, \bar{w}, \bar{p}) \in \prod_{i \in \mathcal{I}} X_i \times \prod_{i \in \mathcal{I}} WZ_i \times B_L \) such that, for every \( i \in \mathcal{I}_0 \), \( \Phi_i(\bar{x}, \bar{w}, \bar{p}) = \emptyset \).

Written coordinatewise, this is equivalent to saying that:

\[
\forall p \in B_L, (\lambda \Box p) \bullet \sum_{i \in \mathcal{I}} (\bar{x}_i - e_i) \leq (\lambda \Box \bar{p}) \bullet \sum_{i \in \mathcal{I}} (\bar{e}_i - e_i) 
\]

(5.1)

\[
\forall i \in \mathcal{I}, (\bar{x}_i, \bar{w}_i) \in \bar{B}_i^{\rho}(\bar{p}) \text{ and } \bar{B}_i^{\rho}(\bar{p}) \cap (P_i(\bar{x}) \times WZ_i) = \emptyset.
\]

(5.2)

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5.2.2 Properties of \( (\bar{x}, \bar{w}, \bar{p}, \bar{q}) \).

We first prove that \( \bar{x} = (\bar{x}_i)_{i \in I} \) satisfies the market clearing condition.

Claim 5.2 \( \sum_{i \in I} \bar{x}_i = \sum_{i \in I} e_i. \)

**Proof of Claim 5.2:** Suppose \( \sum_{i \in I} (\bar{x}_i - e_i) \neq 0. \) From the fixed-point assertion (5.1) we deduce that \( (\lambda \boxminus \bar{p}) = \frac{\sum_{i \in I} (\bar{x}_i - e_i)}{||\sum_{i \in I} (\bar{x}_i - e_i)||} \) and \( ||\lambda \boxminus \bar{p}|| = 1. \) So

\[
(\lambda \boxminus \bar{p}) \bullet \sum_{i \in I} (\bar{x}_i - e_i) > 0.
\]

(5.3)

Recalling that \( (\bar{x}_i, \bar{w}_i) \in B^\rho_i(\bar{p}) \), for all \( i \in I \), by the fixed-point assertion (5.2), hence there exists \( \bar{\tau}_i \in [0,1] \) such that

\[
\bar{p} \boxminus (\bar{x}_i - e_i) \leq \bar{w}_i + \bar{\tau}_i \gamma(\bar{p}) + \rho(\bar{p}).
\]

Summing up over \( i \) we get:

\[
\bar{p} \boxminus \sum_{i \in I} (\bar{x}_i - e_i) \leq \sum_{i \in I} \bar{w}_i + (\sum_{i \in I} \bar{\tau}_i) \gamma(\bar{p}) + (\#I) \rho(\bar{p}).
\]

Taking the scalar product with \( \lambda \) we get,

\[
(\lambda \boxminus \bar{p}) \bullet \sum_{i \in I} (\bar{x}_i - e_i) \leq \lambda \bullet \sum_{i \in I} \bar{w}_i + (\sum_{i \in I} \bar{\tau}_i) \lambda \bullet_D \gamma(\bar{p}) + (\#I) \lambda \bullet_D \rho(\bar{p}).
\]

On the right hand side, we have \( \lambda \bullet_D \sum_{i \in I} \bar{w}_i = 0 \) (by Lemma 5.1), \( \lambda \bullet_D \gamma(\bar{p}) = 0 \) (by Lemma 5.2), and \( \rho(\bar{p}) = 0 \) (since \( ||\lambda \boxminus \bar{p}|| = 1 \)). Thus \( (\lambda \boxminus \bar{p}) \bullet \sum_{i \in I} (\bar{x}_i - e_i) \leq 0 \), which contradicts (5.3). \( \square \)

**Claim 5.3 The following conditions hold:**

(i) If for some \( i \in I \), \( B^\rho_i(\bar{p}) \neq \emptyset \) then \( (\bar{x}_i, \bar{w}_i) \in B^\rho_i(\bar{p}) \) and \( B^\rho_i(\bar{p}) \cap (P_i(\bar{x}_i) \times WZ_i) = \emptyset; \)

(ii) For all \( \xi \in \mathbb{R}^D, \bar{p}(\xi) \neq 0; \)

(iii) For all \( i \in I, (\bar{x}_i, \bar{w}_i) \in B^\rho_i(\bar{p}) \) and \( B^\rho_i(\bar{p}) \cap (P_i(\bar{x}_i) \times WZ_i) = \emptyset. \)

**Proof of Claim 5.3: Proof of (i):** The first assertion that \( (\bar{x}_i, \bar{w}_i) \in B^\rho_i(\bar{p}) \) is a consequence of the Fixed-Point Assertion (5.2). We now show that \( B^\rho_i(\bar{p}) \cap (P_i(\bar{x}_i) \times WZ_i) = \emptyset. \) Indeed suppose that \( B^\rho_i(\bar{p}) \cap (P_i(\bar{x}_i) \times WZ_i) \) contains an element \( (x_i, w_i) \). Since \( B^\rho_i(\bar{p}) \neq \emptyset \), we let \( (\bar{x}_i, \bar{w}_i) \in B^\rho_i(\bar{p}) \).

Suppose first that \( \bar{x}_i = x_i \), then, from above \( (x_i, w_i) \in [P_i(\bar{x}_i) \times WZ_i] \cap B^\rho_i(\bar{p}), \) which contradicts the fact that this set is empty by Assertion (5.2). Suppose now that \( \bar{x}_i \neq x_i, \) from Assumption C (iii), (recalling that \( x_i \in P_i(\bar{x}_i) \)) the set \( [\bar{x}_i, x_i] \cap P_i(\bar{x}_i) \) is nonempty,
hence contains a point \( x_i(\lambda) := (1 - \lambda)\bar{x}_i + \lambda x_i \) for some \( \lambda \in [0, 1] \). We let \( w_i(\lambda) := (1 - \lambda)\bar{w}_i + \lambda w_i \) and we check that \( (x_i(\lambda), w_i(\lambda)) \in \mathcal{B}_i^\gamma(\bar{p}) \) (since \( (x_i, w_i) \in \mathcal{B}_i^\gamma(\bar{p}) \) and \( (\bar{x}_i, \bar{w}_i) \in \mathcal{B}_i^\gamma(\bar{p})) \). Consequently, \( \mathcal{B}_i^\gamma(\bar{p}) \cap (P_i(\bar{x}) \times WZ_i) \neq \emptyset \), which contradicts again Assertion (5.2).

Thus

\begin{equation}
(\bar{x}_i, \bar{w}_i) \in \mathcal{B}_i^\gamma(\bar{p}) \quad \text{and} \quad \mathcal{B}_i^\gamma(\bar{p}) \cap (P_i(\bar{x}) \times WZ_i) = \emptyset
\end{equation}

**Proof of (ii):** From Lemma 5.2 (ii), there exists \( i_0 \in \mathcal{I} \) such that \( \mathcal{B}_{i_0}^\gamma(\bar{p}) \neq \emptyset \). Suppose there exists \( \xi \in \mathcal{D} \) such that \( p(\xi) = 0 \). From Claim 5.2, \( \sum_{i \in \mathcal{I}} x_i = \sum_{i \in \mathcal{I}_e} e_i \) and from the Non-Satiation Assumption at node \( \xi \) (for Consumer \( i_0 \)) there exists \( x_{i_0} \in P_{i_0}(\bar{x}) \) such that \( x_{i_0}(\xi') = \bar{x}_{i_0}(\xi') \) for every \( \xi' \neq \xi \); from Assertion (5.2), \( (\bar{x}_{i_0}, \bar{w}_{i_0}) \in \mathcal{B}_{i_0}^\gamma(\bar{p}) \) and, recalling that \( \bar{p}(\xi) = 0 \), one deduces that \( (x_{i_0}, \bar{w}_{i_0}) \in \mathcal{B}_{i_0}^\gamma(\bar{p}) \). Consequently,

\[ \mathcal{B}_{i_0}^\gamma(\bar{p}) \cap [P_{i_0}(\bar{x}) \times WZ_{i_0}] \neq \emptyset, \]

which contradicts Condition 5.4.

**Proof of (iii):** From Part (ii) of this Claim we have, for all \( \xi \in \mathbb{R}^\mathcal{D}, \bar{p}(\xi) \neq 0 \), hence \( \bar{p}(\xi) \cdot \bar{p}(\xi) > 0 \) and \( \bar{p} \cap \bar{p} > 0 \). Taking \( \bar{w}_i = 0, \bar{r}_i = 0 \) and \( \bar{x}_i = e_i - t\bar{p}, \) for \( t \) > 0 small enough, we get \( \bar{x}_i \in B_1(e_i, r) \subset X_i \) (from the Survival Assumption). Then \( \bar{p} \cap (\bar{x}_i - e_i) = -t(\bar{p} \cap \bar{p}) < 0 \leq 0 + \rho(\bar{p}) \). This shows that \( (x_i, 0) \in \mathcal{B}_i^\gamma(\bar{p}) \neq \emptyset. \)

**Claim 5.4** The following conditions hold:

(i) \( \rho(\bar{p}) = 0 \)

(ii) If closed cone \( \left( \bigcup_{i \in \mathcal{I}} W(q)Z_i \right) \) is linear then for all \( i \in \mathcal{I}, \bar{r}_i = 0 \) and \( \sum_{i \in \mathcal{I}} \bar{w}_i = 0 \).

(iii) If closed cone \( \left( \bigcup_{i \in \mathcal{I}} W(q)Z_i \right) \) is linear then \( ||\sum_{i \in \mathcal{I}} \bar{w}_i|| \leq (\#I)||\gamma(\bar{p})|| \)

**Proof of Claim 5.4: Proof of (i):** We first prove that the modified budget constraints are binding, that is

\begin{equation}
\bar{p} \cap (\bar{x}_i - e_i) = \bar{w}_i + \bar{r}_i \gamma(\bar{p}) + \rho(\bar{p}), \quad \forall i \in \mathcal{I}
\end{equation}

Indeed, suppose that there exists \( i \in \mathcal{I} \) such that

\[ \bar{p} \cap (\bar{x}_i - e_i) < \bar{w}_i + \bar{r}_i \gamma(\bar{p}) + \rho(\bar{p}) \]

That is there exist \( \xi \in \mathcal{D} \) such that

\[ \bar{p}(\xi) \bullet_H (\bar{x}_i(\xi) - e_i(\xi)) < \bar{w}_i(\xi) + \bar{r}_i \gamma(\bar{p})(\xi) + \rho(\bar{p}) \]
But by Claim 5.2, $\sum_{i \in I} \bar{x}_i = \sum_{i \in I} e_i$ and by the nonsatiation assumption $C (\psi)$ for consumer $i$, there exists $x_i \in P_i(\bar{x})$ such that $x_i(\xi^*) = \bar{x}_i(\xi^*)$ for every $\xi^* \neq \xi$. Consequently, we can choose $x \in [x_i, \bar{x}_i)$ close enough to $\bar{x}_i$ so that $(x, \bar{w}_i) \in B^{\rho}(\bar{p})$. But, from the local non-satiation (Assumption $K (iii)$), $[x_i, \bar{x}_i) \subset P_i(\bar{x})$. Consequently, $B^{\rho}(\bar{p}) \cap (P_i(\bar{x}) \times W Z_i) \neq \emptyset$ which contradicts Claim 5.3. This ends the proof of Equation (5.5).

Summing up over $i \in I$ in the equalities (5.5) and using the market clearing condition (Claim 5.2) we get:

\begin{equation}
0 = \sum_{i \in I} \bar{w}_i + (\sum_{i \in I} \bar{\tau}_i) \gamma(\bar{p}) + (\#I) \rho(\bar{p})
\end{equation}

Taking above the scalar product with $\lambda$ yields:

\begin{equation}
0 = \lambda \cdot \mathbb{D} (\sum_{i \in I} \bar{w}_i) + (\sum_{i \in I} \bar{\tau}_i) \lambda \cdot \mathbb{D} \gamma(\bar{p}) + (\#I) \lambda \cdot \mathbb{L} \rho(\bar{p})
\end{equation}

But $\lambda \cdot \mathbb{D} (\sum_{i \in I} \bar{w}_i) = 0$ since $\sum_{i \in I} \bar{w}_i \in W < W \subset \lambda^\perp$ by Lemma 5.1. Moreover $0 = \lambda \cdot \mathbb{D} \gamma(\bar{p})$ by Lemma 5.2. Consequently $\rho(p) = 0$. □

**Proof of (ii):** From Assertion (5.6) and the fact that $\rho(p) = 0$ we get

\begin{equation}
\sum_{i \in I} \bar{w}_i = -(\sum_{i \in I} \bar{\tau}_i) \gamma(\bar{p})
\end{equation}

Taking the scalar product with $\sum_{i \in I} \bar{w}_i$ on both sides and using the fact the closed cone $(\bigcup_{i \in I} W(q) Z_i)$ is linear in Lemma 5.2 we get $\sum_{i \in I} \bar{w}_i = 0$. Again from Assertion 5.7 we get

\[(\sum_{i \in I} \bar{\tau}_i) \gamma(\bar{p}) = 0.\]

Since each $\bar{\tau}_i \geq 0$ we can take the scalar product in the above equation with $\gamma(\bar{p})$ to conclude that for all $i \in I, \bar{\tau}_i = 0$.

**Proof of (iii):** From Assertion 5.7, taking scalar product on both sides with $\sum_{i \in I} \bar{w}_i$ on both sides and recalling that $\tau_i \in [0, 1]$ we get $\sum_{i \in I} \bar{\tau}_i \leq \#I$ hence

\[||\sum_{i \in I} \bar{w}_i|| \leq (\#I)||\gamma(\bar{p})||\]

If closed cone $(\bigcup_{i \in I} W(q) Z_i)$ is a linear space then $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is a weak-equilibrium as in Theorem 3.1 as consequence of Claim 5.2, Claim 5.3 and Claim 5.4 (i and ii).

If however, closed cone $(\bigcup_{i \in I} W(q) Z_i)$ is a convex set then we can go to the limit as $\varepsilon$ goes to zero and verify that the limit of the sequence will be quasi-weak-equilibrium.
5.2.3 Limit argument when \( \varepsilon = \frac{1}{n} \) converges to zero and quasi-weak-equilibrium.

In the previous section we have associated to every \( \varepsilon > 0 \), the list \((\bar{x}, \bar{w}, \bar{p})\) and we will now take \( \varepsilon = 1/n \) and denote the associated list by \((\bar{x}^n, \bar{w}^n, \bar{p}^n)\), to make the dependence on \( n \) explicit. Since \( B_L \) and each of \( X_i \) and \( WZ_i \) are compact, without any loss of generality we can assume that the sequence \((\bar{x}^n, \bar{w}^n, \bar{p}^n)\) converges to some element \((\bar{x}, \bar{w}, \bar{p})\). Let \( \tilde{z}_i \) be such that \( \bar{w}_i = W(q)\tilde{z}_i \).

We first recall the following sets:

\[
B_i(p, q) = \{(x_i, z_i) \in X_i \times Z_i : p \square (x_i - e_i) \leq W(q)z_i\}.
\]

\[
\tilde{B}_i(p, q) = \{(x_i, z_i) \in X_i \times Z_i : p \square (x_i - e_i) < W(q)z_i\}.
\]

Note that we have

\textbf{Claim 5.5}

(i) \( \gamma(\bar{p}^n) \to 0 \).

(ii) \( \rho(\bar{p}) = 0 \), that is \( \|\lambda \square \bar{p}\| = 1 \).

(iii) \( \sum_{i \in I} \bar{x}_i = \sum_{i \in I} e_i \).

(iv) \( \sum_{i \in I} W\tilde{z}_i = 0 \).

\textbf{Proof.} The first assertion \((i)\) holds since \( \|\gamma(\bar{p}^n)\| \leq (#I)/n \) by Lemma 5.2. The second assertion \((ii)\) comes from the fact that \( \rho(\bar{p}) = \lim \rho(\bar{p}^n) \) (by continuity of \( \rho \)) and that \( \rho(\bar{p}^n) = 0 \) by Claim 5.4.

Proof of \((iii)\) and \((iv)\). From Claim 5.2 we have \( \sum_{i \in I} \bar{x}_i^n = \sum_{i \in I} e_i \) and from Claim 5.4 we have \( \|\sum_{i \in I} \bar{w}_i^n\| \leq (#I)\|\gamma(\bar{p}^n)\| \leq (#I)1/n \). Taking the limit, when \( n \to \infty \) we get Assertions \((iv)\) and \((v)\).

In view of the previous Claim 5.5, for \((\bar{x}, \bar{z}, \bar{p}, \bar{q})\) to be a quasi-weak-equilibrium it only remain to prove that each agent’s preferred set does not intersect the interior of her budget set, that is, the following claims holds true.

\textbf{Claim 5.6}

(i) For all \( i \in I \) \((\bar{x}_i, \bar{z}_i) \in B_i(\bar{p}, \bar{q}) \) and \( \tilde{B}_i(\bar{p}, \bar{q}) \cap (P_i(\bar{x}) \times Z_i) = \emptyset \).

(ii) For all \( \xi \in D \), \( x_i = (\bar{x}_i(\xi), x_i(\xi)) \in P_i(\bar{x}) \) implies \( \bar{p}(\xi) \bullet \bar{x}_i(\xi) \geq \bar{p}(\xi) \bullet \bar{x}(\xi) \).

\textbf{Proof of (i):} To prove the first part, from the fixed-point Condition (5.2), we have \((\bar{x}_i^n, \bar{w}_i^n) \in B_i^{\gamma^n p^n}(\bar{p}^n)\), that is,

\[
(\bar{x}_i^n, \bar{w}_i^n) \in X_i \times W(q)Z_i \text{ and there is } \bar{\tau}_i^n \in [0, 1], \bar{p}^n \square (\bar{x}_i^n - e_i) \leq \bar{w}_i^n + \bar{\tau}_i^n \gamma(\bar{p}^n) + \rho(\bar{p}^n).
\]

Taking the limit, when \( n \to \infty \), recalling that \( \gamma(\bar{p}^n) \to 0 \) and \( \rho(\bar{p}^n) \to \rho(\bar{p}) = 0 \), we deduce that \((\bar{x}_i, \bar{w}_i) \in B_i(\bar{p}) \) and \((\bar{x}_i, \tilde{z}_i) \in B_i(\bar{p}, \bar{q})\).

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We prove the second part by contradiction. Assume that there is some \((x_i, z_i) \in \tilde{B}_i(\bar{p}, \bar{q}) \cap (P_i(\bar{x}) \times Z_i) \neq \emptyset\), then, \((x_i, z_i) \in X_i \times Z_i\) and \(\bar{p} \square (x_i - e_i) << W(\bar{q})z_i\). Then, recalling that \(\bar{p}^n \to \bar{p}, \gamma(\bar{p}^n) \to 0\) and \(\rho(\bar{p}^n) \to 0\) (from Claim 5.5) for \(n\) large enough one has
\[
\bar{p}^n \square (x_i - e_i) << W(\bar{q})z_i + \gamma(\bar{p}^n) + \rho(\bar{p}^n),
\]
that is, \((x_i, W(\bar{q})z_i) \in \tilde{B}_i^{\gamma, \rho}(\bar{p}^n)\). Moreover, from the lower semicontinuity of \(P_i\) and the fact that \(P_i\) has open and convex values, we deduce that, for every \(x_i \in X_i\), the set \((P_i)^{-1}(x_i) := \{x \in \Pi_{\ell \in J}X_{\ell} | x_i \in P_i(x)\}\) is open (in \(X_i\) for its relative topology). Therefore for \(n\) large enough, \(x_i \in P_i(\bar{x}^n)\) (since, from above, \(x_i \in P_i(\bar{x})\) and \(\bar{x}^n \to \bar{x}\)). Hence, for \(n\) large enough, \((x_i, W(\bar{q})z_i) \in \tilde{B}_i^{\gamma, \rho}(\bar{p}^n) \cap (P_i(\bar{x}^n) \times W(Z_i))\), which is empty from Claim 5.3 (iii). A contradiction. \(\square\)

Thus in view of Claims 5.5 and 5.6 we have \((\bar{x}, \bar{z}, \bar{p}, \bar{q})\) is a weak-quasi-equilibrium.

**Proof of (ii):** Let \(x_i = (\bar{x}_i(-\xi), x_i(\xi)) \in P_i(\bar{x})\), then for \(n\) large enough \((\bar{x}_i^n(-\xi), x_i(\xi)) \in P_i(\bar{x}^n)\). Note that \(x_i \in P_i(\bar{x}^n)\) implies \(\bar{p}^n(\xi) \bullet_{\bar{H}} x_i(\xi) > \bar{p}^n(\xi) \bullet_{\bar{H}} \bar{x}_i^n(\xi)\). Suppose not, then \((\bar{x}_i^n(-\xi), x_i(\xi)) \in P_i(\bar{x}^n)\) and \(\bar{p}^n(\xi) \bullet_{\bar{H}} x_i(\xi) \leq \bar{p}^n(\xi) \bullet_{\bar{H}} \bar{x}_i^n(\xi)\). Thus
\[
\bar{p}^n \square (x_i - e_i) \leq \bar{p}^n \square (\bar{x}_i^n - e_i) \leq W(\bar{q})\bar{z}_i^n + \bar{p}_i^n \gamma(\bar{p}^n).
\]
This contradicts with the optimality of \((\bar{x}_i^n, \bar{z}_i^n)\) in \(\tilde{B}_i^{\gamma, \rho}(\bar{p}^n, \bar{q})\). Thus \(\bar{p}^n(\xi) \bullet_{\bar{H}} x_i(\xi) > \bar{p}^n(\xi) \bullet_{\bar{H}} \bar{x}_i^n(\xi)\). Since we have \(\bar{p}^n\) and \(\bar{x}_i^n\) converge to \(\bar{p}\) and \(\bar{x}_i\) respectively, thus in the limit we have \(\bar{p}(\xi) \bullet_{\bar{H}} x_i(\xi) \geq \bar{p}(\xi) \bullet_{\bar{H}} \bar{x}_i(\xi)\). \(\square\)

5.3 **Proof in the general case (without additional assumptions)**

We now give the proof of Theorem 4.2, without considering the additional Assumption K, as in the previous section. We will first enlarge the strictly preferred sets as in Gale-Mas Colell, and then truncate the economy \(\mathcal{E}\) by a standard argument to define a new economy \(\mathcal{E}^r\), which satisfies all the assumptions of \(\mathcal{E}\), together with the additional Assumption K.

From the previous section, there exists a weak equilibrium of \(\mathcal{E}^r\) and we will then check that it is also a weak equilibrium of \(\mathcal{E}\).

5.3.1 **Enlarging the preferences as in Gale and Mas-Colell**

The original preferences \(P_i\) are replaced by the “enlarged” ones \(\hat{P}_i\) defined as follows. For every \(i \in I\), \(\bar{x} \in \prod_{\ell \in I} X_{\ell}\), we let
\[
\hat{P}_i(\bar{x}) := \bigcup_{x_i \in P_i(\bar{x})} [\bar{x}_i, x_i] = \{\bar{x}_i + t(x_i - \bar{x}_i) \mid t \in [0, 1], \ x_i \in P_i(\bar{x})\}.
\]
The next proposition shows that $\hat{P}_i$ satisfies the same properties as $P_i$, for every $i \in I$, together with the additional Local Non-satiation Assumption K (ii).

**Proposition 5.1** Under (C), for every $i \in I$ and every $\bar{x} \in \prod_{i \in I} X_i$ one has:

(i) $P_i(\bar{x}) \subset \hat{P}_i(\bar{x}) \subset X_i$;

(ii) the correspondence $\hat{P}_i$ is lower semicontinuous at $\bar{x}$ and $\hat{P}_i(\bar{x})$ is convex;

(iii) for every $y_i \in \hat{P}_i(\bar{x})$ for every $(x')_i \in X_i, (x')_i \neq y_i$ then $[(x')_i, y_i] \cap \hat{P}_i(\bar{x}) \neq \emptyset$;

(iv) $\bar{x} \notin \hat{P}_i(\bar{x})$;

(v) (Non-Satiation at Every Node) if $\sum_{i \in I} \bar{x}_i = \sum_{i \in I} e_i$, for every $\xi \in D$, there exists $x \in \prod_{i \in I} X_i$ such that, for each $\xi' \neq \xi$, $x_i(\xi') = x_i(\xi')$ and $x_i \in \hat{P}_i(\bar{x})$;

(vi) for every $y_i \in \hat{P}_i(\bar{x})$, then $[y_i, \bar{x}_i] \subset \hat{P}_i(\bar{x})$.

**Proof.** Let $\bar{x} \in \prod_{i \in I} X_i$ and let $i \in I$.

**Part (i).** It follows by the convexity of $X_i$, for every $i \in I$.

**Part (ii).** Let $y_i \in \hat{P}_i(\bar{x})$ and consider a sequence $(\bar{x}^{n})_i \subset \prod_{i \in I} X_i$ converging to $\bar{x}$. Since $y_i \in \hat{P}_i(\bar{x})$, then $y_i = x_i + t(x_i - \bar{x}_i)$ for some $x_i \in P_i(\bar{x})$ and some $t \in [0, 1]$. Since $P_i$ is lower semicontinuous, there exists a sequence $(x_i)$ converging to $x_i$ such that $x_n \in P_i(\bar{x}^n)$ for every $n \in \mathbb{N}$. Now define $y_n^i := \bar{x}_i + t(x_i - \bar{x}_i) \in [x_n^i, x_i]$: then $y_n^i \in \hat{P}_i(\bar{x}^n)$ and obviously the sequence $(y_n^i)$ converges to $y_i$. This shows that $\hat{P}_i$ is lower semicontinuous at $\bar{x}$.

To show that $\hat{P}_i(\bar{x})$ is convex, let $y_1^i, y_2^i \in \hat{P}_i(\bar{x})$, let $\lambda_1 \geq 0, \lambda_2 \geq 0$, such that $\lambda_1 + \lambda^2 = 1$, we show that $\lambda_1 y_1^i + \lambda_2 y_2^i \in \hat{P}_i(\bar{x})$. Then $y_k^i = \bar{x}_i + t_k(x_k^i - \bar{x}_i)$ for some $t_k \in [0, 1]$ and some $x_k^i \in P_i(\bar{x})$ ($k = 1, 2$). One has

$$\lambda_1 y_1^i + \lambda_2 y_2^i = \bar{x}_i + (\lambda_1 t_1^i + \lambda_2 t_2^i)(x_i - \bar{x}_i),$$

where $x_i := (\lambda_1 t_1^i x_i^1 + \lambda_2 t_2^i x_i^2)/(\lambda_1 t_1 + \lambda_2 t_2) \in P_i(\bar{x})$ (since $P_i(\bar{x})$ is convex, by Assumption C) and $\lambda_1 t_1^i + \lambda_2 t_2 \in [0, 1]$. Hence $\lambda_1 y_1^i + \lambda_2 y_2^i \in \hat{P}_i(\bar{x})$.

**Part (iii).** Let $y_i \in \hat{P}_i(\bar{x})$ and let $(x')_i \in X_i, (x')_i \neq y_i$. From the definition of $\hat{P}_i$, $y_i = \bar{x}_i + t(x_i - \bar{x}_i)$ for some $x_i \in P_i(\bar{x})$ and some $t \in [0, 1]$. Suppose first that $x_i = (x')_i$, then $y_i \in [\bar{x}_i, x_i] \subset \hat{P}_i(\bar{x})$. Consequently, $[(x')_i, y_i] \cap \hat{P}_i(\bar{x}) \neq \emptyset$. Suppose now that $x_i \neq (x')_i$; since $P_i$ satisfies Assumption C (iii), there exists $\lambda \in [0, 1]$ such that $x_i(\lambda) = (x')_i + \lambda(x_i - (x')_i) \in P_i(\bar{x})$. We let

$$z := [\lambda(1 - t)\bar{x}_i + t(1 - \lambda)(x')_i + t\lambda x_i]/\alpha \quad \text{with} \quad \alpha := t + \lambda(1 - t),$$

and we check that $z = [\lambda(1 - t)\bar{x}_i + tx_i(\lambda)]/\alpha \in [\bar{x}_i, x_i(\lambda)]$, with $x_i(\lambda) \in P_i(\bar{x})$, hence $z \in \hat{P}_i(\bar{x})$. Moreover, $z := [\lambda y_i + t(1 - \lambda)(x')_i]/\alpha \in [(x')_i, y_i]$. Consequently, $[(x')_i, y_i] \cap \hat{P}_i(\bar{x}) \neq \emptyset$, which ends the proof of (iii)).
Proof of Lemma 5.3: They follow immediately by the definition of \( \hat{P}_t \) and the properties satisfied by \( p \) in (C).

5.3.2 Truncating the economy

Given \( q \in \mathbb{R}^d \) the set of admissible consumptions and income transfers, \( K(q) \) is given by:

\[
K(q) := \{(x, w) \in \prod_{i \in T} X_i \times \prod_{i \in T} W(q)Z_i : \exists p \in B_L(0, 1), \quad (x_i, w_i) \in B_i(p, q) \text{ for every } i \in I, \quad \sum_{i \in T} x_i = \sum_{i \in T} e_i, \quad \sum_{i \in T} w_i = 0 \}.
\]

Lemma 5.3 \( K(q) \) is bounded.

Proof of Lemma 5.3: Given \( q \in \mathbb{R}^d \), for every \( i \in I \) define the following:

\[
\hat{X}_i(q) := \left\{ x_i \in X_i : \exists (x^j)_{j \neq i} \in \prod_{j \neq i} X_j, \exists w \in \prod_{i \in I} W Z_i, (x, w) \in K(q) \right\}
\]

and

\[
\hat{W}_i(q) := \left\{ w_i \in W Z_i : \exists (w^j)_{j \neq i} \in \prod_{j \neq i} W Z_j, \exists x \in \prod_{i \in I} X_i, (x, w) \in K(q) \right\}.
\]

We need to show that \( \hat{X}_i(q) \) and \( \hat{W}_i(q) \) are bounded. Since \( \hat{X}_i \) is compact (by Assumption C (i)) clearly \( \hat{X}_i(q) \) is bounded.

To show \( \hat{W}_i(q) \) is bounded, let \( w_i \in \hat{W}_i(q) \). Since

\[
(x_i, w_i) \in \{(x, w) \in X_i \times W(q)Z_i : p \boxtimes (x - e_i) \leq w \}
\]

and \( (x_i, p) \in \hat{X}_i(q) \times B_L(0, 1) \), a compact set from above, there exists \( \alpha_i \in \mathbb{R}^d \), such that

\[
\alpha_i \leq p \boxtimes (x_i - e_i) \leq w_i
\]

Using the fact that \( \sum_{i \in I} w_i = 0 \) we also have

\[
w_i = -\sum_{j \neq i} w^j \leq -\sum_{j \neq i} \alpha^j.
\]

Thus \( \hat{W}_i(q) \) is bounded for every \( i \in I \).

We now define the “truncated economy” as follows.

Since \( \hat{X}_i(q) \) and \( \hat{W}_i(q) \) are bounded subsets of \( \mathbb{R}^l \) and \( \mathbb{R}^d \), respectively (by Lemma 5.3), there exists a real number \( r > 0 \) such that, for every agent \( i \in I \), \( \hat{X}_i(q) \subset \text{int}B_L(0, r) \) and \( \hat{W}_i(q) \subset \text{int}B_\mathbb{D}(0, r) \). The truncated economy \( (\hat{E}^r, \hat{F}^r) \) is the collection

\[
(\hat{E}^r, \hat{F}^r) = [D, \mathcal{H}, I, (X^*_i, \hat{p}, e_i)_{i \in I}; \mathcal{J}, (\xi(j), V^j)_{j \in \mathcal{J}}, (Z^*_i)_{i \in I}],
\]
where,
\[ X'_i = X_i \cap B_L(0, r), \quad Z'_i = \{ z \in Z_i \mid W(\bar{q})z \in B_\Delta(0, r) \} \] and \( \hat{P}_i(x) = \hat{P}_i(x) \cap \text{int}B_L(0, r). \)

The existence of a weak equilibrium of \((\hat{\mathcal{E}}^r, \mathcal{F}^r)\) is then a consequence of Section 5.1, that is, Theorem 4.2 with the additional Assumption \(K\). We just have to check that Assumption \(K\) and all the assumptions of Theorem 4.2 are satisfied by \((\hat{\mathcal{E}}^r, \mathcal{F}^r)\). In view of Proposition 5.1, this is clearly the case for all the assumptions but the Survival Assumption \(C\) (vi) that is proved via a standard argument (that we recall hereafter).

Indeed we first notice that \((e_i, 0)_{i \in \mathcal{I}}\) belongs to \(K(q)\), hence, for every \(i \in \mathcal{I}, e_i \in \hat{X}_i(q) \subset \text{int}B_L(0, r)\). Recalling that \(e_i \in \text{int}X_i\) (from the Survival Assumption), we deduce that \(e_i \in \text{int}X_i \cap \text{int}B_L(0, r) \subset \text{int}[X_i \cap B_L(0, r)] = \text{int}X'_i\).

**Proposition 5.2** Given \(\bar{q} \in \mathbb{R}^d\), if \((\bar{x}, \bar{z}, \bar{p}, \bar{q})\) is a weak-equilibrium of \((\hat{\mathcal{E}}^r, \mathcal{F}^r)\) then it is also a weak-equilibrium of \((\mathcal{E}, \mathcal{F})\).

**Proof of proposition 5.2.** Let \((\bar{x}, \bar{z}, \bar{p}, \bar{q})\) be a weak-equilibrium of the economy \((\hat{\mathcal{E}}^r, \mathcal{F}^r)\). In view of the definition of a weak-equilibrium, to prove that it is also a weak-equilibrium of \((\mathcal{E}, \mathcal{F})\) we only have to check, for every \(i \in \mathcal{I}, \) \([P_i(\bar{x}) \times Z_i] \cap B_i(\bar{p}, \bar{q}) = \emptyset\), where \(B_i(\bar{p}, \bar{q})\) denotes the budget set of agent \(i\) in the economy \((\mathcal{E}, \mathcal{F})\).

Assume, on the contrary, that, for some \(i \in \mathcal{I}\) the set \([P_i(\bar{x}) \times Z_i] \cap B_i(\bar{p}, \bar{q})\) is nonempty, hence contains a couple \((x_i, z_i)\). Clearly the allocation \((\bar{x}, W(\bar{q})\bar{z})\) belongs to the set \(K(\bar{q})\), hence for every \(i \in \mathcal{I}, \bar{x}_i \in \hat{X}_i(q) \subset \text{int}B_L(0, r)\) and \(\bar{w}_i = W(\bar{q})\bar{z}_i \in \hat{W}_i(\bar{q}) \subset \text{int}B_\Delta(0, r)\). Thus, for \(t \in [0, 1]\) sufficiently small, \(x_i(t) := \bar{x}_i + t(x_i - \bar{x}_i) \in \text{int}B_L(0, r)\) and \(w_i(t) := \bar{w}_i + t(w_i - \bar{w}_i) \in \text{int}B_\Delta(0, r)\). Clearly \((x_i(t), w_i(t))\) is such that \(w_i(t) = W(\bar{q})z_i(t)\) where \(z_i(t) = (\bar{z}_i + t(z_i - \bar{z}_i)) \in Z_i\) and \((x_i(t), z_i(t))\) belongs to the budget set \(B_i(\bar{p}, \bar{q})\) of agent \(i\) (for the economy \((\mathcal{E}, \mathcal{F})\)) and since \(x_i(t) \in X'_i := X_i \cap B_L(0, r)\), \(z_i(t) \in Z'_i := \{ z \in Z_i \mid W(\bar{q})z \in B_\Delta(0, r) \}\), the couple \((x_i(t), z_i(t))\) belongs also to the budget set \(B_i(\bar{p}, \bar{q})\) of agent \(i\) (in the economy \((\hat{\mathcal{E}}^r, \mathcal{F}^r)\)). From the definition of \(\hat{P}_i\), we deduce that \(x_i(t) \in \hat{P}_i(\bar{x})\) (since from above \(x_i(t) := \bar{x}_i + t(x_i - \bar{x}_i)\) and \(x_i \in P_i(\bar{x})\)), hence \(x_i(t) \in \hat{P}_i^r(\bar{x}) \cap \text{int}B_L(0, r)\). We have thus shown that, for \(t \in [0, 1]\) small enough, \((x_i(t), z_i(t)) \in [\hat{P}_i^r(\bar{x}) \times Z'_i] \cap B_i(\bar{p}, \bar{q})\).

This contradicts the fact that this set is empty, since \((\bar{x}, \bar{y}, \bar{p}, \bar{q})\) is a weak-equilibrium of the economy \((\hat{\mathcal{E}}^r, \mathcal{F})\). \(\square\)
6 Appendix: Proof of lemma 5.2

Proof of Lemma 5.2: Part (i): We recall that, from Assumption C, there exists \( r > 0 \) such that, for all \( i \in I, B_{L}(e_{i}, r) \subset X_{i} \), and we define the correspondence \( \Gamma \) from \( B_{L} \) to \( \Delta \) by

\[
\Gamma(p) = \{ \gamma \in \lambda^{\perp} \cap \mathcal{W}^{o} \cap \mathbb{B}_{D}(0, \varepsilon) \mid \exists u \in B_{L}(0, r), \exists w \in \mathcal{W}, p \circ [u << w + \gamma + \rho(p)] \}.
\]

We will show that there is a continuous selection of \( \Gamma \), that is, a continuous mapping \( \gamma : B_{L} \rightarrow \Delta \) such that, for all \( p \in B_{L}, \gamma(p) \in \Gamma(p) \). This will be deduced from Michael’s theorem (see Prop. 1.5.3 in Florenzano [11]) and we only need to show that for all \( p \in B_{L}, (i) \Gamma(p) \) is convex, (ii) \( \Gamma(p) \) is nonempty, and (iii) \( \Gamma \) is lower semi-continuous at \( p \).

(i) \( \Gamma(p) \) is convex. To see this, let \( \delta^{1} \in \Gamma(p) \) and \( \delta^{2} \in \Gamma(p) \), then

\[
\exists u^{1} \in B_{L}(0, r), \exists w^{1} \in \mathcal{W} : p \circ u^{1} << w^{1} + \gamma^{1} + \rho(p)
\]

\[
\exists u^{2} \in B_{L}(0, r), \exists w^{2} \in \mathcal{W} : p \circ u^{2} << w^{2} + \gamma^{2} + \rho(p)
\]

Then for all \( \alpha \in [0, 1], \alpha \gamma^{1} + (1 - \alpha)\gamma^{2} \in \lambda^{\perp} \cap \mathcal{W}^{o} \cap \mathbb{B}_{D}(0, \varepsilon), \alpha u^{1} + (1 - \alpha)u^{2} \in \mathbb{B}_{L}(0, r), \alpha w^{1} + (1 - \alpha)w^{2} \in \mathcal{W}, and

\[
p \circ (\alpha u^{1} + (1 - \alpha)u^{2}) << (\alpha w^{1} + (1 - \alpha)w^{2}) + (\alpha \gamma^{1} + (1 - \alpha)\gamma^{2}) + \rho(p)
\]

Thus \( \alpha \gamma^{1} + (1 - \alpha)\gamma^{2} \in \Gamma(p) \). This shows that \( \Gamma(p) \) is convex.

(ii) \( \Gamma(p) \) is nonempty. We will distinguish the two cases \( \rho(p) > 0 \) and \( \rho(p) = 0 \).

If \( \rho(p) > 0 \), taking \( u = 0 = \alpha \) and \( w = 0 \) we see that \( \gamma = 0 \in \Gamma(p) \), since \( 0 << 0 + \rho(p) \).

If \( \rho(p) = 0 \), i.e., \( ||\lambda \circ p||^{2} = 1 \). Since \( \lambda \in \mathbb{R}_{++}^{D} \), there exists \( \xi \in \mathcal{D} \) such that \( p(\xi) \neq 0 \).

Thus there exists \( u \in B_{L}(0, r) \) such that \( p \circ u < 0 \). Moreover, there exists \( t \in \lambda^{\perp} \) such that \( p \circ u << t \); take \( t = p \circ u - \frac{\lambda \circ (p \circ u)}{||\lambda||} \lambda \) (recalling that \( \lambda >> 0 \)).

Since \( \mathcal{W} \) is a closed, convex cone, one has \( \mathbb{R}^{D} = \mathcal{W} + \mathcal{W}^{o} \) (see Rockafellar). Hence, there exists \( w \in \mathcal{W} \) and \( \gamma \in \mathcal{W}^{o} \) such that \( t = w + \gamma \). But \( w \in \lambda^{\perp} \) since from Lemma 5.1, \( w \in \mathcal{W} \subset \lambda^{\perp} \). Recalling that \( t \in \lambda^{\perp} \), we thus deduce that \( \gamma \in \lambda^{\perp} \). Consequently, for every \( \tau \in (0, 1] \)

\[
p \circ \tau u << \tau w + \tau \gamma \quad \text{with} \quad \tau u \in B_{L}(0, r), \tau w \in \mathcal{W} \text{ and } \tau \gamma \in \lambda^{\perp} \cap \mathcal{W}^{o}
\]

and for \( \tau > 0 \) small enough, \( \tau \gamma \in \mathbb{B}_{D}(0, \varepsilon) \), which shows that \( \tau \gamma \in \Gamma(p) \). Thus \( \Gamma(p) \neq \emptyset \).

(iii) \( \Gamma \) is lower semicontinuous at \( p \). Let \( G_{\Gamma} := \{(p, \gamma) \in B_{L} \times \Delta \mid \delta \in \Gamma(p)\} \), denote the graph of \( \Gamma \). To show that \( \Gamma \) is lower semicontinuous it is sufficient to show that \( G_{\Gamma} \) is
open or equivalently that \((B_L \times \Delta) \setminus G_\Gamma\) is closed. Let \(\{(p^k, \delta^k)\} \in (B_L \times \Delta) \setminus G_\Gamma\) such that \((p^k, \delta^k) \to (p, \delta)\), we show that \((p, \delta) \notin G_\Gamma\) by contradiction. Indeed if \((p, \delta) \in G_\Gamma\), that is, \(\delta \in \Gamma(p)\), there exists \(\bar{u} \in B_L(0, r)\) and \(\bar{w} \in \mathcal{W}\) such that

\[
\forall \xi \in \mathcal{D}, \quad p(\xi) = \bar{u}(\xi) < \bar{w}(\xi) + \delta(\xi) + \rho(p)
\]

Also for all \(k, \delta^k \notin \Gamma(p^k)\) thus for all \(u \in B_L(0, r)\) and for all \(w \in \mathcal{W}\) one has

\[
\exists \xi^k \in \mathcal{D}, p^k(\xi^k) \cdot u(\xi^k) \geq w(\xi^k) + \gamma^k(\xi^k) + \rho(p^k)
\]

in particular, there exists \(\xi \in \mathcal{D}\) such that \(\{k \in \mathbb{N} | \xi^k = \xi\}\) is infinite. Without any loss of generality, by considering a subsequence, we can say that

\[
\exists \xi \in \mathcal{D}, p(\xi) \cdot u(\xi) \geq w(\xi) + \delta(\xi) + \rho(p)
\]

Since \(p^k \to p\) and \(\delta^k \to \delta\), taking the limit when \(k \to \infty\) we get

\[
\exists \xi \in \mathcal{D}, p(\xi) \cdot u(\xi) \geq w(\xi) + \delta(\xi) + \rho(p)
\]

a contradiction with the inequality 6.1. Thus \(\Gamma\) is l.s.c. at \(p\). Moreover in the above we have shown that for all \(p \in B_L, ||\gamma(p)|| \leq \varepsilon\). □

**Part (ii):** Let \(\gamma\) be the continuous selection of \(\Gamma\) obtained in Part (i) above. We want to show that, for every \(p \in B_L\), there exists \(i \in \mathcal{I}\) such that \(\bar{B}_i(\gamma(p)) \neq \emptyset\). To see this, we let \(p \in B_L\) and we successively consider the two cases \(\rho(p) > 0\) and \(\rho(p) = 0\).

If \(\rho(p) > 0\), for all \(i \in \mathcal{I}\), taking \(\tau_i = 0\), we deduce that \((x_i, w_i) = (e_i, 0) \in \bar{B}_i(\gamma(p))\).

If \(\rho(p) = 0\), since \(\gamma(p) \in \Gamma(p)\), there exists \(w \in B_L(0, r)\), and \(w \in \mathcal{W}\) such that

\[
p \quad u \ll w + \gamma(p)
\]

By assumption made in the lemma, the closed cone spanned by \(W(\bar{q})(\cup_{i \in \mathcal{I}} Z_i)\) is convex, hence \(\mathcal{W} \subset \text{cl cone} \ W(\bar{q})(\cup_{i \in \mathcal{I}} Z_i)\) and \(w = \lim t^n v^n\) for some sequence \((t^n) \subset \mathbb{R}_+\) and \((v^n) \subset W(\bar{q})(\cup_{i \in \mathcal{I}} Z_i)\). Without any loss of generality (by eventually considering a sequence) we can assume that \(v^n = w^n_i\) belongs to some given set \(W(\bar{q})Z_i\) (for some given \(i\) independent of \(n\)).

Suppose first that \(w = 0\), then clearly \((e_i + u, 0) \in \bar{B}_i(\gamma(p))\) for all \(i \in \mathcal{I}\).

Suppose now that \(w \neq 0\), then without any loss of generality (by eventually considering a subsequence) we can assume that \(t^n > 0\) for every \(n\). From the above inequality, recalling that \(w = \lim t^n w^n_i\) we have \(p \quad u \ll t^n w^n_i + \gamma(p)\) for \(n\) large enough.
Then for $\tau > 0$ small enough ($\tau \leq 1$ and $\tau \leq t^n$) we have

$$p \bigotimes \left( \frac{\tau}{t^n} u \right) \ll \tau w_i^n + \frac{\tau}{t^n} \gamma(p) \quad \text{with} \quad \frac{\tau}{t^n} u \in B_L(0, r)$$

Moreover $\tau w_i^n \in W(\bar{q})Z_i$ since $0 \in W(\bar{q})Z_i$ (and $0 \leq \tau \leq 1$).

This shows that, for $n$ large enough, $(e_i + \frac{\tau}{t^n} u, \tau w_i^n) \in \mathcal{B}_i^{\gamma(p)}$. □
**Additional comments:**

(1) \( \langle W \sum_{i \in I} Z_i \cap \mathbb{R}_+^I \{0\} \rangle \);

(2) \( \text{Cone} \ (\bigcup_{i \in I} WZ_i) \) is linear.

(1') \( W \sum_{i \in I} Z_i \cap \mathbb{R}_+^I \{0\} \);

(2') \( \langle W \sum_{i \in I} Z_i \rangle = \text{Cone} \ (\bigcup_{i \in I} WZ_i) \).

Note that

(1) and (2) \( \Rightarrow \) (1') and (2').

The proof in the thesis considers assumptions (1') and (2').
References


