Understanding the effects of government spending on Consumption and Output in the presence of Incomplete Credit Market: Reconciling theory with Empirical evidence

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Abstract

1 Introduction

will be written later

2 The Model

Time is taken in discrete intervals, t = 1, 2,..... The economy is composed of a continuum of household in the interval of (0,1). There are two types of households, named Borrowers and Savers, of measure of (1 – F) and F. Each household is endowed with one unit of time. Each household derives utility from three sources: consumption of durable goods (C_{t}^{a,p}), consumption of non-durable goods(D_{t}^{a,p}) and leisure(1 – L_{t}^{a,p}) where the superscript a refers to savers and p refers to borrowers. They also incur disutility from working and making decisions on a durable goods purchase. The household faces three saving opportunities, invest in physical capital, buy government/public bond and serve as a lender in the private debt market. Households also has the opportunity to borrow from the private market by issuing one period state contingent bond. The state-contingent claims are assumed to be unbacked and are unenforceable. As a result, the private credit market is incomplete. The collateralized value of the durable goods stock is generally less than its replace cost. For a stock of durable good D_{t+1}, it is given by:

\[ V_{t+1}^{a,p} = (1 - \pi) \sum_{j=1}^{\infty} (1 - \phi) \left[ D_{t-j+1}^{a,p} - (1 - \delta_D) D_{t-j}^{a,p} \right] \quad (1) \]

Here \( \pi \) is the fraction of a new durable good that cannot serve as collateral and \( \phi \) is the rate at which a good’s collateral value depreciates. Following Campbell and Hercowitz (2005), we assume that \( \phi \geq \delta_D \), so that the good’s value to a creditor declines at least as rapidly as its value to its owner. Collateral requirement limits household borrowing. That is:

\[ B_{t+1}^{a,p} \leq V_{t+1}^{a,p} \quad (2) \]
Here $B_{t+1}^{a,p}$ refers to the outstanding debts of the two households at the end of period $t$ and $V_{t+1}^{a,p}$ are the collateralized value of their durable goods. Using (1) and (2), the borrowing constraint can be written recursively as:

$$B_{t+1}^{a,p} \leq (1 - \phi)B_{t}^{a,p} + (1 - \pi) \left[ D_{t+1}^{a,p} - (1 - \delta_D)D_{t}^{a,p} \right]$$

(3)

### 2.1 Utility Maximization by the Consumer

Let’s index the consumers by $h = a, p$. Then the general utility function of the agents look like:

$$U(.) = \left[ \left( C_t^h \right)^{1-\frac{1}{\beta}} + V_t^h (D_t^h)^{1-\frac{1}{\sigma}} \right]^{\frac{1}{1-\frac{1}{\beta}}} - \frac{\eta}{2} \left( \frac{D_t^h - D_{t-1}^h}{D_{t-1}^h} \right)^2 + \chi_t^h (1-L_t^h)^{1-\theta} - 1$$

(4)

Here $C_t^h = C_t^h - b_t^h C_{t-1}^h$, with $b_t^h \geq 0$ indicating the degree of internal habit persistence. $\tau$ and $\sigma$ are the elasticities of intertemporal and intratemporal substitution of consumption. $\theta$ is the inverse elasticity of intertemporal substitution of consumption. $\chi_t^h$ is the weight on leisure in the utility function. The quadratic term $\frac{\eta}{2} \left( \frac{D_t^h - D_{t-1}^h}{D_{t-1}^h} \right)^2$ is interpreted as the deliberation cost where $\eta$ captures the disutility of changing durable good stock. Household are also endowed with physical capital stocks, $k_{t-1}^h \geq 0$, initial private bonds, $B_{t-1}^{h^s} \geq 0$ and initial public bond $B_{t-1}^{h^p} \geq 0$ at time 0. Given their initial capital stock each household chooses a sequence of consumption of durable goods, non-durable goods and private debt, government debt, Investment in physical capital, physical capital itself and leisure; $\{C_t^h, D_t^h, B_t^h, K_{t-1}^{h^s}, L_t^h, I_t^h, \mu_t^h \}_{t=1}^{\infty}$ to maximize his expected lifetime utility:

$$E_0 \sum_{t=1}^{\infty} \beta^{-t-1} U(C_t^h, D_t^h, L_t^h)$$

(5)

Subject to the budget constraint:

$$C_t^h + D_t^h + X_t^d + I_t^h \leq (1 - \tau_t^L W_t L_t^{h^s} + B_{t-1}^h - B_{t-1}^h R_{t-1} + (1 - \delta_D)D_{t-1}^h$$

$$+ \beta \tau_t^k K_{t-1}^{h^s} + (1 - \tau_t^k) r_t \mu_t^h K_{t-1}^{h^s} + X_{t-1}^{h^d} R_{t-2} + TR_t^h$$

(6)

and the borrowing constraint:

$$B_t^h \leq (1 - \phi)B_{t-1}^h + (1 - \pi) \left[ D_t^h - (1 - \delta_D)D_{t-1}^h \right]$$

(7)

and the law of motion of capital stock:

$$K_t^{h^s} \leq \left\{ 1 - s \left( \frac{I_t^h}{I_{t-1}^h} \right) \right\} I_t^h + (1 - \delta_t^h)K_{t-1}^{h^s}$$

(8)

where:

$$S(1) = S'(1) = 0, S''(1) = \gamma > 0$$

(9)

and:
\[ \delta^h_t = \delta \left( \mu^h_t \right) \omega \] (10)

Also, we impose non-negativity restrictions: \( C^h_t \geq 0, D^h_t \geq 0, I^h_t \geq 0, K^h_t \geq 0, \mu^h_t \geq 0, L^h_t \geq 0, B^h_t \geq 0 \) for all \( t \). Note that \( K^h_t \) and \( L^h_t \) refers to the supply of labor and capital by the households. Finally, we assume that in the initial period, \( K^h_0 = I^h_0 \).

**Remark 1** \( U(.) \epsilon C^{(2)}(R_+, R) \), strictly concave, \( U_r(.) \succ 0, U_t(.) \succ 0, U_{1-L}(.)(.) \succ 0, U_0(.)(.) \prec 0 \), \( U_{Dh}(.) \prec 0, U_{1-Lh}(.) \prec 0 \) for all \( C^h \geq 0, D^h \geq 0, L^h \geq 0, \) and \( \lim_{t \to 0} U_r(.) = + \infty \), \( \lim_{t \to 0} L^h \to 0 U_r(.) = + \infty \).

Let \( L \) be the lagrangian function and \( \lambda_{ht}, \psi_{ht} \) and \( \varphi_{ht} \) be the lagrangian multipliers associated with the budget constraint, borrowing constraint and the law of motion for capital. Then the Kuhn-Tucker conditions for this problem look like:

\[
\mathcal{L}_{\lambda_{ht}} = - \left\{ C^h_t + D^h_t + X^h_t + I^h_t - (1 - \tau^h_t) W_t L^h_t - B^h_t + B^h_{t-1} R_{t-1} - (1 - \delta^h) D^h_{t-1} - \delta^h_t K^h_{t-1} - (1 - \tau^h_t) r_t \mu^h_t K^h_{t-1} - X^h_{t-1} R_{t-1} + TR^h_t \right\} \geq 0, \lambda_{ht} \geq 0 \text{ with Complementary Slackness(CS) Condition} \tag{11}
\]

\[
\mathcal{L}_{\psi_{ht}} = - \left\{ B^h_t - (1 - \phi) B^h_{t-1} - (1 - \delta^h) D^h_{t-1} \right\} \geq 0, \psi_{ht} \geq 0 \text{ with "CS"} \tag{12}
\]

\[
\mathcal{L}_{\varphi_{ht}} = - \left\{ K^h_t - \left\{ 1 - s \left( \frac{I^h_t}{I^h_{t-1}} \right) \right\} I^h_t - (1 - \delta^h) K^h_{t-1} \right\} \geq 0, \varphi_{ht} \geq 0 \text{ with "CS"} \tag{13}
\]

\[
\mathcal{L}_{C^h_t} = \left[ \left( C^h_t \right)^{1 - \frac{1}{\beta}} + V^h \left( D^h_t \right)^{1 - \frac{1}{\beta}} \right] \left[ \frac{\tau^h_t}{\sigma - 1} \right] \left( C^h_t \right)^{-\frac{1}{\beta}} \tag{14}
\]

\[
-\beta^h_t b^h E_t \left\{ \left( C^h_{t+1} \right)^{1 - \frac{1}{\beta}} + V^h \left( D^h_{t+1} \right)^{1 - \frac{1}{\beta}} \right\} \left( C^h_{t+1} \right)^{-\frac{1}{\beta}} \right\} - \lambda_{ht} \leq 0, C^h_t \geq 0 \text{ with "CS"} \tag{15}
\]

\[
\mathcal{L}_{D^h_t} = \left[ \left( C^h_t \right)^{1 - \frac{1}{\beta}} + V^h \left( D^h_t \right)^{1 - \frac{1}{\beta}} \right] \left[ \frac{\tau^h_t}{\sigma - 1} \right] V^h \left( D^h_t \right)^{-\frac{1}{\beta}} - \eta \left( D^h_t \right)^{\frac{1}{\beta}} - \lambda_{ht} \psi_{ht} (1 - \pi) \beta^h_t b^h E_t \left\{ \left( \frac{D^h_{t+1} - D^h_t}{D^h_t} \right)^{2} - 1 \right\} + \lambda_{ht+1} (1 - \delta^h) \right\} \leq 0, D^h_t \geq 0 \text{ with "CS"} \tag{16}
\]

\[
\mathcal{L}_{B^h_t} = \lambda_{ht} - \psi_{ht} - \beta^h_t b^h E_t \left\{ \lambda_{ht+1} R_{t+1} - \psi_{ht+1} (1 - \phi) \right\} \leq 0, B^h_t \geq 0 \text{ with "CS"} \tag{17}
\]

\[
K^h_t \geq 0 \text{ with "CS"}
\]
2.2 Profit Maximization by the Firm

The production function used by the firm is defined as follows:

\[ Y_t = f \left( L_t^d, L_t^p, K_{t-1}^d, K_{t-1}^p; \mu_t^d, \mu_t^p \right) = \left\{ \left( F(\mu_t^d K_{t-1}^d) + (1 - F)(\mu_t^p K_{t-1}^p) \right) \right\} \alpha \left\{ FL_t^d + (1 - F)L_t^p \right\}^{1-\alpha} \]  

The Representative firm rents capital and labor from agents to maximize profit

\[ \text{Profit} = \left\{ \left( F(\mu_t^d K_{t-1}^d) + (1 - F)(\mu_t^p K_{t-1}^p) \right) \right\} \alpha \left\{ FL_t^d + (1 - F)L_t^p \right\}^{1-\alpha} - r_t \left\{ \left( F(\mu_t^d K_{t-1}^d) + (1 - F)(\mu_t^p K_{t-1}^p) \right) \right\} - W_t \left\{ FL_t^d + (1 - F)L_t^p \right\} \]  

\[ \text{Remark 2} \ f(.) \in C^2(R_+, R), \text{ strictly concave, } f_{L_t^d} \geq 0, f'_{L_t^d}(.) > 0, f'_{K_{t-1}^d}(.) > 0, f'_{K_{t-1}^d}(.) > 0, f_{L_t^p} \leq 0, f''_{L_t^p}(.) < 0, f''_{K_{t-1}^p}(.) < 0, f''_{K_{t-1}^p}(.) < 0 \text{ and } \lim_{L_t^d \to 0} \frac{f'}{L_t^d}(.) = +\infty, \lim_{K_{t-1}^d \to 0} f'(K_{t-1}^d)(.) = +\infty. \]

The first order condition of profit maximization are as follows:

\[ r_t = \frac{\alpha Y_t}{\left\{ \left( F(\mu_t^d K_{t-1}^d) + (1 - F)(\mu_t^p K_{t-1}^p) \right) \right\}} \]  

\[ W_t = \frac{(1 - \alpha) Y_t}{\left\{ FL_t^d + (1 - F)L_t^p \right\}} \]  

2.3 Government Budget Constraint

The government levies taxes on capital and labor income separately, sells one period government bond to the consumers, issues a depreciation allowance for capital and provide lump-sum transfers to the consumers to balance the budget. The government budget constraint is:

\[ G_t + X_{t-1}^s R_{t-1} + \bar{\delta} \tau_t^k \left\{ \left( F(\mu_t^d K_{t-1}^d) + (1 - F)(\mu_t^p K_{t-1}^p) \right) \right\} + TR_t = T_t + X_t^s \]
Where $T_t$ is the total tax collected defined as:

$$T_t = T^l_t + T^k_t$$  \hspace{1cm} (27)$$

$$T^l_t = F^t L_t^p + (1 - F) \tau^l_t W_t L_t^p$$  \hspace{1cm} (28)$$

$$T^k_t = \tau^k_t \left\{ \left( F (\mu_t^a K_t^{a-1}) + (1 - F) (\mu_t^p K_t^{p-1}) \right) \right\}$$  \hspace{1cm} (29)$$

Finally, the total transfer in the economy, $TR_t$ looks like:

$$TR_t = TR^a_t + TR^p_t$$  \hspace{1cm} (30)$$

The government also has to maintain intertemporal fiscal solvency. First, any equilibrium must satisfy the Transversality conditions:

$$E_t \lim_{T \to \infty} \beta^{t+T} \lambda_{h,t+T} K_{t+T}^{h^*} = 0$$  \hspace{1cm} (31)$$

$$E_t \lim_{T \to \infty} \beta^{t+T} \lambda_{h,t+T} B_{t+T}^h = 0$$  \hspace{1cm} (32)$$

$$E_t \lim_{T \to \infty} \beta^{t+T} \lambda_{h,t+T} X_{t+T}^{h^d} = 0$$  \hspace{1cm} (32.a)$$

Imposing the TVC on the flow budget constraint of the government, we derive the intertemporal budget constraint for the government:

$$B_t = \sum_{j=0}^{\infty} d_{t,t+j} \left[ \left( 1 - \alpha \right) \tau^L_{t+j} L_{t+j}^{s+T} + \alpha \tau^K_{t+j} K_{t+j}^{s+T} - s_{t+j} \right]$$  \hspace{1cm} (33)$$

Where $L_t$ is the aggregate labor supply in the economy, to be defined later and $d_{t,t+j} = \prod_{i=0}^{j-1} R_{t+i} Y_{t+i+1} / Y_{t+i}$. The fiscal policy rules that the government uses are summarized as follows:

$$\ln \left( \frac{s^T R^a_t}{s^T R^a} \right) = q_{T R^a} M - M \ln \left( \frac{s_{t-1}^B}{s^B} \right), q_{T R^a} \leq 0$$  \hspace{1cm} (34)$$

$$\ln \left( \frac{s^T R^p_t}{s^T R^p} \right) = q_{T R^p} N - N \ln \left( \frac{s_{t-1}^B}{s^B} \right), q_{T R^p} \leq 0$$  \hspace{1cm} (35)$$

$$\ln G_t = \rho_G \ln G_{t-1} + \varepsilon^G_t$$  \hspace{1cm} (36)$$

$$\ln \left( \frac{\tau^L_t}{\tau^L} \right) = q_L \ln \left( \frac{s_{t-1}^B}{s^B} \right), q_L \geq 0$$  \hspace{1cm} (37)$$

$$\ln \left( \frac{\tau^K_t}{\tau^K} \right) = q_K \left( \frac{s_{t-1}^B}{s^B} \right), q_K \geq 0$$  \hspace{1cm} (38)$$

Here $s^T R^h_t = \frac{TR^h_t}{Y_t}$ and $\varepsilon^G_t \sim iid. N(0, \sigma^2_G)$. Also:

$$M - M = \frac{TR^a_t}{TR^a}$$ if distributionally neutral transfer adjustment and 1 otherwise

$$N - N = \frac{TR^p_t}{TR^p}$$ if distributionally neutral transfer adjustment and 1 otherwise
2.4 Aggregation

We will aggregate the economy as follows:

\[
I_t = FI_t^a + (1 - F)I_t^p \tag{39}
\]

\[
B_t^d = FB_t^{ad} + (1 - F)B_t^{pd} \tag{40}
\]

\[
X_t^d = FX_t^{ad} + (1 - F)X_t^{pd} \tag{40.a}
\]

\[
X_t^s = FX_t^{as} + (1 - F)X_t^{ps} \tag{40.b}
\]

\[
K_t^d = FK_t^{ad} + (1 - F)K_t^{pd} \tag{41}
\]

\[
K_t^s = FK_t^{as} + (1 - F)K_t^{ps} \tag{41.a}
\]

\[
L_t^d = FL_t^{ad} + (1 - F)L_t^{pd} \tag{42}
\]

\[
L_t^s = FL_t^{as} + (1 - F)L_t^{ps} \tag{42.a}
\]

\[
C_t = FC_t^a + (1 - F)C_t^p \tag{43}
\]

\[
D_t = FD_t^a + (1 - F)D_t^p \tag{44}
\]

\[
TR_t^a = Ftr_t^a \tag{45}
\]

\[
TR_t^p = (1 - F)tr_t^p \tag{46}
\]

The market clearing conditions are as follows:

Labor Market: \[L_t^d = L_t^s\] (47)

Capital Market: \[K_t^d = K_t^s\] (47.a)

Private Debt Market: \[B_t = 0\] (47.b)

Public Debt Market: \[X_t^d = X_t^s\] (47.c)

Goods Market: \[C_t + I_t + G_t + D_t = Y_t + (1 - \delta_D)D_{t-1}\] (47.d)

In addition, we will define several other variables that will be used for simulation. First, the aggregate budget constraints look like:
\[ FC^a_t + FD^a_t + FX_t^d + FI_t^a = (1 - \tau_t^d) W_t FL_t^a + FB^a_t - FB^a_{t-1} R_{1t-1} + (1 - \delta_D) FD_{t-1}^a + \tilde{\tau}_t^d FK^a_{t-1} \]
\[ +(1 - \tau_t^d) r_t F \mu^a_t K^a_{t-1} + FX_t^{d} R_{2t-1} + TR_t^a \]

\[ (1 - F)C_p^a + (1 - F)D_t^p + (1 - F)X_t^d + (1 - F)I_t^p = (1 - \tau_t^d) W_t (1 - F) L_t^p + (1 - F) B_t^p - (1 - F) B_{t-1}^p R_{1t-1} \]
\[ +(1 - \delta_D)(1 - F) D_{t-1}^p + \tilde{\tau}_t^k (1 - F) K_t^p + (1 - \tau_t^k) r_t (1 - F) \mu_t^p K_t^p + (1 - F) X_t^p R_{2t-1} + TR_t^p \]

Second, we will define the flow of durable goods service as:

\[ \text{Durable Service} = D_S_t = D_t - (1 - \delta_D) D_{t-1} \]

Since the NIPA data reports the flow of durable goods, equation (50) will be used for calibration purpose.

Finally, the aggregate consumption in the economy is defined as follows:

\[ \text{Aggregate Consumption} = AD_C = C_t + D_S_t \]

**Definition 3** A perfect foresight competitive equilibrium (PFCE) is a pair of sequence of prices \( \{p_t, W_{t}\}_{t=1}^{\infty} \), a sequence of a set of consumers’ decisions \( \{C^p_t, D_t^p, B_t^h, K_t^h, X_t^h, I_t^h, L_t^h, \mu_t^h\}_{t=1}^{\infty} \), a sequence of firm’s decisions \( \{K_t^h, L_t^h\}_{t=1}^{\infty} \), a sequence of policy variables, \( \{X_t^p, G_t, \tau_t^K, \tau_t^L, \tau_t^P, TR_t\}_{t=1}^{\infty} \) such that, given initial levels of capital stock, private and public debt, \( K_{t-1}, B_{t-1}, X_{t-1} \), the optimization for the agents and firm’s are solved; the goods, capital, labor and the debt markets clear; the transversality conditions for capital and debts hold; the government budget constraint and at least one of the policy rules and all the aggregate conditions are satisfied. Furthermore, we will only consider the ranges of the fiscal adjustment parameters- the q’s- that are consistent with the existence of a rational expectations PFCE.

Note here that the existence of the PFCE follows from the usual inada condition of remarks 1 and 2. The uniqueness of the PFCE could also be shown to stem from the strict concavity of the utility and the production function.

**Definition 4** We say that \( r, W, C^a, D^a, I^a, L^a, K^a, X^a, B^a, C^p, D^p, I^p, L^p, K^p, X^p, B^p, Y, G \) is a stationary perfect foresight competitive equilibrium (SPFCE) if it is a perfect foresight competitive equilibrium that exhibits the following properties:

1. Both \( r \) and \( W \) are constant over time.
2. \( I^{a,p} = \hat{a}_t^{a,p} K^{a,p} \) are constant over time.
3. \( C, G, K, X, B^a, B^p, \hat{p} \) are all constant over time.
4. \( K = FK^a + (1 - F) K^p \) are constant over time where \( K \) is the unique solution to \( r = f_k(.) \).
5. \( L = FL^a + (1 - F) L^p \) is constant over time where \( L \) is the unique solution to \( W = f_L(.) \).
Axiom 5  The savers are more patient than the borrowers; $\beta^a > \beta^p$.

Axiom 6  In the stationary steady state, there is a positive amount of public debt held by consumers; $X > 0$.

Axiom 7  In the steady state, the budget constraint of both type of consumers bind; $\lambda_h > 0, h = a, p$.

Axiom 8  In the steady, consumption of durable and non-durables are positive; $C^a, C^p, D^a, D^p > 0$.

Definition 9  Let $R^K$ be the after tax-after subsidy gross return on capital in the steady state.

Theorem 10  Let assumptions 5-8 are satisfied. Then given remarks 1,2 and the TVCs, there exists a unique SPFCE given by the following conditions:

1. $R^K = \frac{1}{\sigma^a} = R_1 = R_2$.
2. $I^p = 0, K^p = 0, X^p = 0$
3. The borrowing constraint binds for the borrower and does not bind for the saver; $\psi_a = 0, \psi_p > 0$. Also, $B^a > 0, B^p < 0$

Proof. As mentioned earlier, the strict concavity assumptions will ensure that the SPFCE is unique while the inada conditions will ensure the existence. We will prove the rest of the theorems as follows:

(1) Proof of (1) is an extension of Becker(1980). In any other candidate for SPFCE, either $R^K > \frac{1}{\sigma^a}$ or $R^K < \frac{1}{\sigma^a}$ must hold.

If $R^K > \frac{1}{\sigma^a}$ the then the saver has a feasible policy over increasing accumulation and consumption yielding arbitrary large stock of capital; by diminishing returns and the TVC, this cannot be a SPFCE.

If $R^K < \frac{1}{\sigma^a} < \frac{1}{\sigma^p}$, then following Becker(1980), for any initial distribution of capital, it can be shown to be optimal for each household to decumulate from its initial stocks to zero in finite time. This would violate the stationary condition of the SPFCE.

(2) For the borrower, the cost of investing on physical capital at $t$ is given by $U'(t)$. The benefit of investment at $t+1$ is given by $U'(t+1)R^K_t$, the discounted value of which at $t$ is $\beta^p U'(t+1)R^K_t$. In steady state, since $R^K = \frac{1}{\sigma^a} < \frac{1}{\sigma^p}$, the cost exceeds the benefit and hence, $K^p = 0$. Next, from the definition of investment in steady state, it follows that $I^p = 0$. Finally, the CS condition for $X^p$ in steady state looks like:

$$\lambda_p \left( \beta^p R_2 - 1 \right) \geq 0, X^p \geq 0 \text{ with } X^p \lambda_p \left( \beta^p R_2 - 1 \right) = 0$$

Since by assumption $\lambda_1 > 0$ and $\beta^p R_2 - 1 < 0$, it follows that $X^p = 0$.

(3) The CS condition for $B^p$ in steady state looks like:

$$\lambda_p \left( 1 - \beta^p R_2 \right) - \psi_p \left( 1 - \beta^p (1 - \phi) \right) \geq 0, B^p \geq 0 \text{ with } B^p \left[ \lambda_p \left( 1 - \beta^p R_2 \right) - \psi_p \left( 1 - \beta^p (1 - \phi) \right) \right] = 0$$

From the first part of the CS condition, we see:

$$\psi_p \left( 1 - \beta^p (1 - \phi) \right) \geq \lambda_p \left( 1 - \beta^p R_2 \right) \Rightarrow \psi_p \geq 0$$

From equation(12), for the borrower, in steady state we see that the since $\psi_p > 0 \Rightarrow B^p(\phi) - (1 - \pi)\delta_D D^p = 0 \Rightarrow B^p > 0$ because of assumption.

Finally from the private debt market clearing condiiton, we see that $FB^a = -(1 - F)B^a \Rightarrow B^a < 0$.