BARGAINING IN STATIONARY NETWORKS

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Abstract. We study an infinite horizon game in which pairs of players connected in a network are randomly matched to bargain over a unit surplus. Players that reach agreement are replaced by new players at the same positions in the network. We prove that for each discount factor all equilibria are payoff equivalent. The equilibrium payoffs and the set of equilibrium agreement links converge as players become patient. Several new concepts–mutually estranged sets, partners, and shortage ratios–provide insights into the relative strengths of the positions in the network. We develop a procedure to determine the limit equilibrium payoffs by iteratively applying the following results. Limit payoffs are lowest for the players in the largest mutually estranged set that minimizes the shortage ratio, and highest for the corresponding partners. In equilibrium, for high discount factors, the partners act as an oligopoly for the estranged players. In the limit, surplus within the induced oligopoly subnetwork is divided according to the shortage ratio. We characterize equitable networks, stable networks, and non-discriminatory buyer-seller networks. The results extend to heterogeneous discount factors and general matching technologies.

1. Introduction

Competitive equilibrium theory assumes large and anonymous markets, in which every buyer can trade with every seller. Underlying these assumptions are standard goods and services that may be traded at low transaction costs by agents who are not in specific relationships with one another. However, in many markets goods and services are heterogeneous (e.g., cars, apartments) or need to be tailored to particular needs (e.g., manufacturing inputs, technical support). Furthermore, trading opportunities may depend on transportation costs, social relationships, technological compatibility, joint business opportunities, free trade

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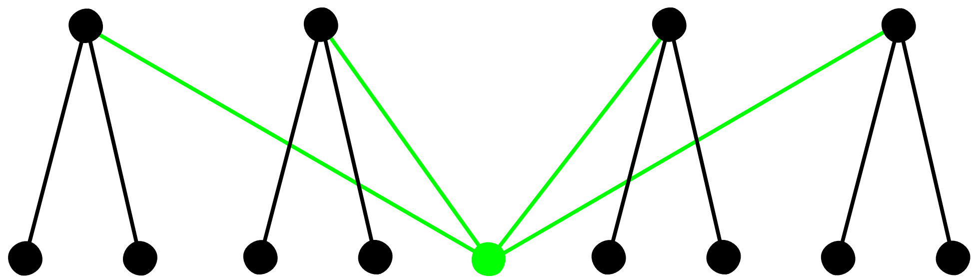

FIGURE 1. The green position is weak despite having the largest number of connections.

agreements, etc. In such cases it is natural to model the market using a *network*, where only pairs of connected agents can engage in exchange. New theories are needed to explore the influence of the *network structure* on *market outcomes*. Many questions arise: How does an agent's position in the network determine his bargaining power and the local prices he faces? Who trades with whom and on what terms? Are trading outcomes equitable or non-discriminatory? Which networks are stable?

One possible conjecture is that an agent's bargaining power is determined by his (relative) number of connections in the network. However, this simple theory is not very plausible. Consider the network of four sellers (located at the top nodes) and nine buyers (located at the bottom nodes) depicted in Figure 1. The buyer located at the position colored green has the largest number of links in the network, as he is connected to each of the four sellers. Yet every seller has monopoly power over two other buyers whom he can extort, even if trade with the green buyer is unattainable. Hence the green buyer is not able to extract a large fraction of the surplus from any seller despite his relatively large number of connections. This example illustrates that the relative strengths of the positions in a network are highly interdependent. An agent's bargaining power does not depend only on the number of his partners, but also on the identities and bargaining power of his partners. Each partner's bargaining power depends in turn on the bargaining power of his corresponding partners, and so forth. An adequate measure of bargaining power in networks needs to take this interdependence into account.

In a recent book [15], Jackson surveys the emerging field of social and economic networks and concludes that several central issues remain unsolved.

> There are important open questions regarding how network structure affects the distribution of the benefits that accrue to different actors in a network.

In particular, Jackson notes that analyzing "a non-cooperative game that completely models the bargaining process through which ultimate payoffs are determined [...] is usually intractable." The present paper attempts to fill this gap using a *non-cooperative* model of *decentralized bilateral bargaining* in networks. Our model is tractable and provides answers to the questions listed in the first paragraph.

The setting is as follows. We consider a network where each pair of players connected by a link can jointly produce a unit surplus. The network generates the following infinite horizon discrete time bargaining game. Each period a link is randomly selected, and one of the two matched players is randomly chosen to make an offer to the other player specifying a division of the unit surplus between themselves. If the offer is accepted, the two players exit the game with the shares agreed on. We make the following steady state assumption. The two players who reached agreement are replaced in the next period by two new players at the same positions in the network. If the offer is rejected, the two players remain in the game for the next period. All players have a common discount factor.

The steady state assumption captures the idea that in many trading environments agents face stationary distributions of bargaining opportunities, and some agents take similar positions in relationships and transactions at different points in time. In the benchmark model this assumption entails that every period an exogenous inflow of agents matches the *stochastic* endogenous outflow of agents who reach agreements in equilibrium. Nevertheless, the results extend to a model in the spirit of Gale (1987), where the steady state analysis involves a *deterministic* inflow of agents. In that model every period a *continuum* of players are present at each node in the network, and a positive measure of player pairs are matched to bargain across each link (see footnote 9).

In Abreu and Manea (2008) we drop the stationarity assumption, and analyze the situation in which players that reach agreements are removed from the network without replacement. The bargaining protocol is identical to the one of the present paper. Our findings, along with the key differences between the two models, are discussed in the literature review.

1.1. **Outline of the paper.** We assume that all players have perfect information about all the events preceding any of their decision nodes in the game. The equilibrium concept we use is subgame perfect equilibrium.[1]

In Section 3, we prove that for every discount factor the equilibrium payoff of every player present at the beginning of any period is uniquely determined by his position in the network (Theorem 1). For all but a finite number of discount factors, there exists a partition of the set of links into equilibrium agreement and disagreement links (Proposition 1). In every equilibrium, after any history, a pair of players connected by an equilibrium agreement link reaches an agreement when matched to bargain, and the division agreed on is uniquely determined by the positions in the network of the proposer and the responder. Players connected by equilibrium disagreement links never reach agreements when matched to bargain.

We prove that there exists a limit equilibrium agreement network that describes the set of equilibrium agreement links for sufficiently high discount factors (Theorem 2). Also, there is a limit equilibrium payoff vector to which the equilibrium payoffs converge as the discount factor goes to 1.

For instance, consider the network G_1 with 5 players illustrated in Figure 2.[2] For every discount factor there is a unique equilibrium, with agreement network equal to G_1. In equilibrium every match ends in agreement because players 4 and 5 cannot be monopolized by either player 1 or 2. The limit equilibrium payoffs are 3/5 for players 1 and 2, and 2/5 for players 3, 4, and 5. The limit equilibrium agreement network coincides with G_1.

Consider next the network G_2, obtained from G_1 by removing the link $(2, 4)$. For low discount factors there exists a unique equilibrium, and the agreement network is the entire G_2. However, for high discount factors, players 1 and 5 do not reach an equilibrium agreement when matched to bargain. The intuition is that player 1 can extort players 3 and 4, since these two players do not have other bargaining partners. Player 1 cannot extract as much surplus from player 5, since player 5 has monopoly over the bargaining opportunities of player 2. The limit equilibrium payoffs are 2/3 for player 1, 1/3 for players 3 and 4, and

[1]Section 3 discusses the robustness of the results to some features of the information structure and the equilibrium requirements.

[2]In all figures, limit equilibrium payoffs for each player are represented next to the corresponding node, and limit equilibrium agreement and disagreement links are drawn as thick and thin line segments, respectively.

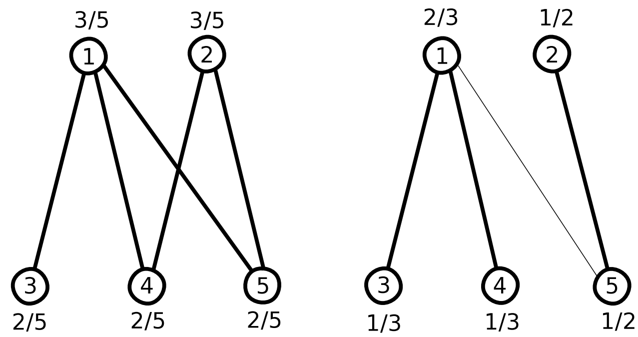

FIGURE 2. Networks G_1 (left) and G_2

1/2 for players 2 and 5. The limit equilibrium agreement network consists of all links of G_2 except $(1, 5)$. The equilibria of the bargaining games on the networks G_1 and G_2 for all discount factors are described in Example 1 from Section 3.

The main objective of our analysis is to determine the limit equilibrium payoffs for every network. The following essential observation is presented in Section 4. Consider a set of players who are pairwise disconnected in the limit equilibrium agreement network, and the set of players with whom these players share limit equilibrium agreement links. We refer to players in the former set as mutually estranged, and to ones in the latter set as partners.[3] Basically, as players become patient, the partners have control over the (equilibrium) relevant bargaining opportunities of the mutually estranged players. For high discount factors, since the estranged players can only reach equilibrium agreements in pairwise matchings with the partners, the mutually estranged set is weak if the partners are relatively scarce. The appropriate measure of the strength of a mutually estranged set proves to be the simplest that springs to mind–the shortage ratio, which is defined as the ratio of the numbers of partners and estranged players.

For example, in the network G_1 the shortage ratios of the mutually estranged sets $\{3, 4\}$ and $\{3, 4, 5\}$ are 1 and 2/3, respectively, since the partner set is $\{1, 2\}$ in either case. In the network G_2 the shortage ratios of the mutually estranged sets $\{3, 4\}$ and $\{3, 4, 5\}$ are 1/2 and 2/3, respectively, since the corresponding partner sets are $\{1\}$ and $\{1, 2\}$, respectively. The determination of the partners for the mutually estranged sets considered here is based

[3] To illustrate the definition, note that the list of all mutually estranged sets and corresponding partner sets in the limit equilibrium agreement network for the bargaining game on the network G_2 is $(\{1\}, \{3, 4\})$, $(\{2\}, \{5\})$, $(\{3\}, \{1\})$, $(\{4\}, \{1\})$, $(\{5\}, \{2\})$, $(\{1, 2\}, \{3, 4, 5\})$, $(\{1, 5\}, \{2, 3, 4\})$, $(\{2, 3\}, \{1, 5\})$, $(\{2, 4\}, \{1, 5\})$, $(\{3, 4\}, \{1\})$, $(\{3, 5\}, \{1, 2\})$, $(\{4, 5\}, \{1, 2\})$, $(\{2, 3, 4\}, \{1, 5\})$, and $(\{3, 4, 5\}, \{1, 2\})$.

on the aforementioned limit equilibrium agreement subnetworks for the bargaining games on the networks G_1 and G_2.

The concepts of mutually estranged sets, partners and shortage ratios play key roles in the prediction of bargaining power. Formally, the shortage ratio measures the strength of a mutually estranged set in the following sense. For every set of mutually estranged players and their partners the ratio of the limit equilibrium payoffs of the worst-off estranged player and the best-off partner is not larger than the shortage ratio of the mutually estranged set (Theorem 3). The proof is based on the fact that a player's equilibrium payoff is the expected present value of his stream of first mover advantage. Since first mover advantage in a bilateral encounter is symmetric for the two players, the sum of equilibrium payoffs for every set of mutually estranged players is not larger than for the corresponding partners. The result yields an upper (lower) bound for the limit equilibrium payoff of the worst-off estranged player (best-off partner).

There may be a multitude of mutually estranged sets, and it is not immediately clear which, if any, of the corresponding bounds for the limit equilibrium payoffs are binding. One delicate step toward the main result (Theorem 4) is the idea that the bounds generated by a set of mutually estranged players and their partners need to be binding unless the worst-off estranged player is part of an even weaker mutually estranged set, and the best-off partner is part of an even stronger partner set. Based on this intuition, we prove that the *extreme* bounds–the ones derived from the (largest) mutually estranged set that *minimizes* the shortage ratio and the corresponding partners–must bind.[4] The two sets of players associated with these bounds have extremal limit equilibrium payoffs, and induce an oligopoly subnetwork enclosing all their limit equilibrium agreement links. Thus, for high discount factors, the partners act as an oligopoly that corners and extorts the estranged players. In the equilibrium limit, surplus within the oligopoly subnetwork is divided according to the shortage ratio of the mutually estranged players with respect to their partners, with all players on each side receiving identical payoffs. The limit equilibrium payoffs for the networks G_1 and G_2 are obtained by computing that the lowest shortage ratio in G_1 is 2/3,

[4]Our analysis reveals that the lowest shortage ratio, when smaller than 1, may be computed by considering sets of players that are pairwise disconnected (and their neighbors) in the entire network rather than in the (a priori unknown) limit equilibrium agreement network.

attained for the mutually estranged set $\{3,4,5\}$ with the oligopoly $\{1,2\}$, while in G_2 it is $1/2$, attained for the mutually estranged set $\{3,4\}$ with the oligopoly $\{1\}$.

Section 5 defines an algorithm that sequentially determines the limit equilibrium payoffs of all players based on the ideas above. At each step, the algorithm determines the union of all mutually estranged sets with the lowest shortage ratio, and removes the corresponding estranged players and their partners.[5] Within the identified extremal oligopoly subnetwork surplus is divided between the two sides according to the shortage ratio. The algorithm stops when all players have been removed, or when the lowest shortage ratio is greater than or equal to 1, corresponding to limit equilibrium payoffs for the remaining players of $1/2$.

We use the algorithm to address a number of questions about the uniformity of payoffs and the stability of the network. In Section 6 we characterize the class of equitable networks, i.e., networks for which the limit equilibrium payoffs of all players are identical (equal to $1/2$). A network is equitable if and only if it is quasi-regularizable; another equivalent condition is that the network can be covered by a match and odd cycles disjoint union (Theorem 5).

Section 7 studies the networks that are stable with respect to the equilibrium payoffs. A network is unilaterally stable if no player benefits from severing one of his links. A network is pairwise stable if it is unilaterally stable and no pair of players benefit from forming a new link. We prove that every network is unilaterally stable, but only equitable networks are pairwise stable, with respect to the limit equilibrium payoffs (Theorem 6). The same conclusions hold for approximate stability with respect to the equilibrium payoffs for high discount factors smaller than 1 (Corollary 2).

In Section 8 we show that restricting attention to buyer-seller networks permits a more transparent characterization of limit equilibrium oligopoly subnetworks and a more straightforward procedure to compute the limit equilibrium payoffs (Theorem 4^{BS}). Limit equilibrium payoffs of $1/2$ play no special role in the results for buyer-seller networks. We also analyze non-discriminatory buyer-seller networks, i.e., networks for which the limit equilibrium payoffs of all buyers are identical. If the buyer-seller ratio is an integer, then the network is non-discriminatory if and only if it can be covered by a disjoint union of clusters

[5] An important observation, which ensures that the algorithm identifies and removes all residual players with extremal limit equilibrium payoffs simultaneously, is that the set of mutually estranged sets with the lowest shortage ratio is closed with respect to unions, as long as the lowest shortage ratio is less than 1 (Lemma 5).

formed by one seller connected to a number of buyers equal to the buyer-seller ratio (Theorem 7). We adapt the definition of pairwise stability to buyer-seller networks and show that a buyer-seller network is two-sided pairwise stable with respect to the limit equilibrium payoffs if and only if it is non-discriminatory (Theorem 6.ii^{BS}).

One consequence of the analysis is that submarkets endogenously emerge in equilibrium. A market described by a connected network, which cannot be decomposed into non-overlapping submarkets, may induce a disconnected limit equilibrium agreement network where players are partitioned into oligopoly subnetworks. In the equilibrium limit, each oligopoly subnetwork describes an independent submarket since no transactions occur across distinct oligopoly subnetworks. Each player self-selects into the most favorable submarket he is linked to. The limit equilibrium prices are uniform within every submarket. In particular, an outside observer who is only aware of the equilibrium outcomes, but is unfamiliar with the underlying network structure, may attempt to analyze the ensuing submarkets separately, failing to notice that a priori they are interconnected.

Section 9 extends the main results to the cases of heterogeneous discount factors and general matching technologies. Section 10 reviews the related literature, and Section 11 concludes.

2. Framework

Let N denote the set of n **players**, $N = \{1, 2, \ldots, n\}$. A **network** is an **undirected graph** $H = (V, E)$ with set of **vertices** $V \subset N$ and set of **edges** (also called **links**) $E \subset \{(i, j)|i \neq j \in V\}$ such that $(j, i) \in E$ whenever $(i, j) \in E$. We identify the pairs (i, j) and (j, i), and use the shorthand ij or ji instead. We say that player i is **connected** in H to player j, or i has an H link to j, if $ij \in E$. We often abuse notation and write $ij \in H$ for $ij \in E$. A network $H' = (V', E')$ is a **subnetwork** of H if $V' \subset V$ and $E' \subset E$. A network $H' = (V', E')$ is the subnetwork of H **induced** by V' if $E' = E \cap (V' \times V')$.

Let G be a fixed network with vertex set N. A link ij in G is interpreted as the ability of players i and j to jointly generate a unit surplus.[6] Consider the following infinite horizon **bargaining game** generated by the network G. Each period $t = 0, 1, \ldots$ a link ij in G

[6]For simplicity, we assume everywhere except Section 7 that each player has at least one G link.

is selected randomly (with equal probability),[7] and one of the players (the proposer) i and j is chosen randomly (with equal probability) to make an offer to the other player (the responder) specifying a division of the unit surplus between themselves. If the responder accepts the offer, the two players exit the game with the shares agreed on. In period $t+1$ two new players assume the same positions in the network as i and j, respectively. If the responder rejects the offer, the two players remain in the game for the next period. In period $t+1$ the game is repeated with the set of n players, consisting of the ones from period t, with the departing players replaced by new players if an agreement obtains in period t. All players share a discount factor $\delta \in (0,1)$.[8] The game is denoted Γ^δ.

Formally, there exists a sequence $i_0, i_1, \ldots, i_\tau, \ldots$ of players of type $i \in N$ (a player's type is defined by his position in the network). When player i_τ exits the game (following an agreement with another player), player $i_{\tau+1}$ replaces him for the next period.[9] We assume that players have perfect information of all the events preceding any of their decision nodes in the game. Possible relaxations of the information structure are discussed in the next section.

There are three types of histories. We denote by h_t a history of the game up to (not including) time t, which is a sequence of $t-1$ pairs of proposers and responders connected in G, with corresponding proposals and responses. We call such histories, and the subgames that follow them, **complete**. For simplicity, we assume that for every history players are only labeled by their type without reference to the index of their copy. The index τ of the copy of i playing the game at time t following the history h_t can be recovered by counting the number of bargaining agreements involving i along h_t. Therefore, a history h_t uniquely determines the copy i_τ of player i present in the game at time t, and when there is no risk of confusion we suppress the index of i_τ. We denote by $\mathcal{H}(i_\tau)$ the set of complete histories, or subgames, where i_τ is the copy of player i present in the game. We denote by $(h_t; i \to j)$

[7] The analysis is not sensitive to the specification of the matching technology. See Subsection 9.2.

[8] The case of heterogeneous discount factors is considered in Subsection 9.1.

[9] The results translate to an alternative specification of the model in the spirit of Gale (1987). Suppose that there exists a continuum of players of each type in N. Measure μ_i of players of type i are present in the game at each period. The matching technology is such that, for each link ij, measure μ_{ij} of players i are matched to bargain with one of the players j (through random selection of proposer). It is assumed that $\mu_i > 0, \mu_{ij} = \mu_{ji} > 0$ and $\mu_i > \sum_{\{j|ij\in G\}} \mu_{ij}$. The probability that a player i is matched to a player j is μ_{ij}/μ_i, and no player is involved in more than one match at once. The set of players of each type who reach agreements is immediately replaced by a set of players of the same type of equal measure.

the history consisting of h_t followed by nature selecting i to propose to j. We denote by $(h_t; i \to j; x)$ the history consisting of $(h_t; i \to j)$ followed by i offering $x \in [0,1]$ to j.

A **strategy** σ_{i_τ} for player i_τ specifies, for all j connected to i in G and all $h_t \in \mathcal{H}(i_\tau)$, the **offer** $\sigma_{i_\tau}(h_t; i \to j)$ that i makes to j after the history $(h_t; i \to j)$, and the **response** $\sigma_{i_\tau}(h_t; j \to i; x)$ that i gives to j after the history $(h_t; j \to i; x)$. We allow for mixed strategies, hence $\sigma_{i_\tau}(h_t; i \to j)$ and $\sigma_{i_\tau}(h_t; j \to i; x)$ are probability distributions over $[0,1]$ and {Yes, No}, respectively. In the context of our game, we say that two strategy profiles are **payoff equivalent** if they induce identical payoffs for any player i_τ, when payoffs are evaluated as follows. A player's **payoff** is the expected value of his gains from all bargaining agreements discounted relative to the time when the player entered the game (rather than period 0 of the game). A **strategy profile** $(\sigma_{i_\tau})_{i \in N, \tau \geq 0}$ is a **subgame perfect equilibrium** of Γ^δ if it induces Nash equilibria in subgames following every history $(h_t; i \to j)$ and $(h_t; i \to j; x)$.

3. Essential Equilibrium Uniqueness and Discounting Asymptotics

We first show that across all equilibria of the bargaining game the expected payoff of every player present in any complete subgame is uniquely determined by his position in the network. The unique and stationary equilibrium payoffs associated with each player type may be used to describe the possible equilibrium outcomes of every bargaining encounter.

Theorem 1. *For every $\delta \in (0,1)$, there exists a payoff vector $(v_i^{*\delta})_{i \in N}$ such that for every subgame perfect equilibrium of Γ^δ the expected payoff of player i_τ in any $\mathcal{H}(i_\tau)$ subgame is uniquely given by $v_i^{*\delta}$ for all $i \in N, \tau \geq 0$. For every equilibrium of Γ^δ, in any subgame where i_τ is selected to make an offer to $j_{\tau'}$, the following statements are true with probability one:*

(1) *if $\delta(v_i^{*\delta} + v_j^{*\delta}) < 1$ then i_τ offers $\delta v_j^{*\delta}$ and $j_{\tau'}$ accepts;*
(2) *if $\delta(v_i^{*\delta} + v_j^{*\delta}) > 1$ then i_τ makes an offer that $j_{\tau'}$ rejects.*

We can extend the conclusions of Theorem 1 to settings in which players do not have perfect information about all past bargaining encounters. It may be that players only know their own history of interactions, or know the history of all pairs matched to bargain but only see the outcomes of their own interactions. Players who reach agreements and exit the game may pass down information to the players who take their positions in the network. Some players may be informed of the exact identities of their bargaining partners, or only about their positions in the network. In such settings, players need to form beliefs about the unrevealed bargaining outcomes. Extending the proof to show uniqueness of the sequential equilibrium payoffs for each player type under various information structures is straightforward.

Furthermore, Theorem 1 generalizes to **security equilibria** (Binmore and Herrero 1988b). The definition of a security equilibrium is related to the notion of rationalizability introduced by Bernheim (1984) and Pearce (1984). The requirement that it be common knowledge that a player never uses a strategy which is not a best response to a rational strategy profile of his opponents is replaced by a similar requirement concerning security levels. A security equilibrium entails common knowledge of the fact that no player takes an action under any contingency which yields a payoff smaller than his security level for that contingency. The requirements of security equilibrium are weaker than those of sequential equilibrium.

The proof proceeds in two steps. First, we show that if attention is restricted to steady state stationary strategies then all equilibria are payoff equivalent. Second, we argue that all subgame perfect equilibria yield the unique steady state stationary equilibrium payoffs. A strategy profile $(\sigma_{i_\tau})_{i\in N,\tau\geq 0}$ is **steady state stationary** if each player's strategy at any time t depends exclusively on his position in the network and the play of the game in period t, that is, $\sigma_{i_\tau}(h_t; i \to j) = \sigma_{i_{\tau'}}(h'_{t'}; i \to j)$ and $\sigma_{i_\tau}(h_t; j \to i; x) = \sigma_{i_{\tau'}}(h'_{t'}; j \to i; x)$ for all $ij \in G, x \in [0,1], \tau, \tau' \geq 0, h_t \in \mathcal{H}(i_\tau), h'_{t'} \in \mathcal{H}(i_{\tau'})$. A **steady state stationary equilibrium** is a subgame perfect equilibrium in steady state stationary strategies.

The following two lemmas are essential to the proof. Lemma 1 is a simple algebraic observation, and Lemma 2 is the statement of the first part of Theorem 1 restricted to steady state stationary equilibria. Lemma 1 is invoked in the proofs of Lemma 2 and Theorem 1.

Lemma 1. *For all $w_1, w_2, w_3, w_4 \in \mathbb{R}$,*

$$|\max(w_1, w_2) - \max(w_3, w_4)| \leq \max(|w_1 - w_3|, |w_2 - w_4|).$$

Lemma 2. *There exists a payoff vector $(\tilde{v}_i^{*\delta})_{i\in N}$ such that in every steady state stationary equilibrium of Γ^δ the expected payoff of i_τ at the beginning of any $\mathcal{H}(i_\tau)$ subgame is uniquely given by $\tilde{v}_i^{*\delta}$ for all $i \in N, \tau \geq 0$.*

The proofs of Lemmata 1 and 2 appear in the Appendix. We sketch the proof of Lemma 2 here. Let σ be a steady state stationary equilibrium of Γ^δ. Under σ, for every $i \in N$, each player i_τ receives the same expected payoff, $\tilde{v}_i$, in any $\mathcal{H}(i_\tau)$ subgame. We argue that $\tilde{v}$ is a fixed point of the function $f^\delta = (f_1^\delta, f_2^\delta, \ldots, f_n^\delta) : [0,1]^n \to [0,1]^n$ defined by

$$f_i^\delta(v) = \frac{2e - e_i}{2e}\delta v_i + \frac{1}{2e} \sum_{\{j|ij \in G\}} \max(1 - \delta v_j, \delta v_i), \tag{3.1}$$

where e denotes the total number of links in G and e_i denotes the number of links player i has in G. Next we use Lemma 1 to show that f^δ is a contraction with respect to the sup norm on $\mathbb{R}^n$, hence it has a unique fixed point $(\tilde{v}_i^{*\delta})_{i \in N}$. Therefore, $\tilde{v} = \tilde{v}^{*\delta}$.

Remark 1. The set of steady state stationary pure strategy equilibria of Γ^δ is non-empty. The following is an element. When i is selected to propose to j, he offers $\min(1 - \delta\tilde{v}_i^{*\delta}, \delta\tilde{v}_j^{*\delta})$, and when i has to respond, he accepts any offer of at least $\delta\tilde{v}_i^{*\delta}$ and rejects smaller offers (regardless of the proposer). However, uniqueness of the steady state stationary equilibrium payoffs does not imply uniqueness of the equilibrium strategies. For instance, when i is selected to make an offer to j and $\delta(\tilde{v}_i^{*\delta} + \tilde{v}_j^{*\delta}) > 1$, we can replace i's behavior by any mixed strategy over the interval $[0, \delta\tilde{v}_j^{*\delta})$ or $[0, \delta\tilde{v}_j^{*\delta}]$ (depending on whether j's strategy is to accept offers of $\delta\tilde{v}_j^{*\delta}$ from i with positive probability).

Proof of Theorem 1. Consider the (non-empty) set of all subgame perfect equilibria of Γ^δ (including those which are not steady state stationary). For each $i \in N$, let $\underline{v}_i^\delta$ and $\overline{v}_i^\delta$ be the infimum and respectively supremum of the expected payoff of i_τ in any $\mathcal{H}(i_\tau)$ subgame, over all $\tau \geq 0$, and across all subgame perfect equilibria of Γ^δ.[10]

Fix an $i \in N$ and a subgame perfect equilibrium of Γ^δ. No copy of player j will accept an offer smaller than $\delta\underline{v}_j^\delta$, so i can obtain a payoff of at most $1 - \delta\underline{v}_j^\delta$ from an agreement with j, when i is the proposer.[11] Player i accepts any offer larger than $\delta\overline{v}_i^\delta$ since he receives at most $\delta\overline{v}_i^\delta$ in the continuation subgame after a rejection, so no player j offers him more than $\delta\overline{v}_i^\delta$ in equilibrium. If there is no agreement involving i in some period, his continuation payoff is at

[10] The approach is similar to the Shaked and Sutton (1984) proof of equilibrium uniqueness for the Rubinstein (1982) alternating offer bargaining game. For our bargaining game, the steps are complicated by the a priori unknown set of pairs of players who reach agreements in equilibrium when the possible bargaining partners for each player are determined by the network.

[11] Throughout this argument i (j) should be read as "copy of i (j)."

most $\delta\overline{v}_i^\delta$. It follows that for all $\tau \geq 0$ the equilibrium payoff v_{i_τ} of i_τ in an $\mathcal{H}(i_\tau)$ subgame satisfies

$$v_{i_\tau} \leq \frac{2e-e_i}{2e}\delta\overline{v}_i^\delta + \frac{1}{2e}\sum_{\{j|ij\in G\}}\max(1-\delta\underline{v}_j^\delta, \delta\overline{v}_i^\delta).$$

By the definition of $\overline{v}_i^\delta$, the inequality above implies that

$$\overline{v}_i^\delta \leq \frac{2e-e_i}{2e}\delta\overline{v}_i^\delta + \frac{1}{2e}\sum_{\{j|ij\in G\}}\max(1-\delta\underline{v}_j^\delta, \delta\overline{v}_i^\delta). \tag{3.2}$$

Again, fix an $i \in N$ and a subgame perfect equilibrium of Γ^δ. Consider the following deviation for player i from the equilibrium strategy. Player i offers $\delta\overline{v}_j^\delta + \varepsilon$ ($\varepsilon > 0$) to any j such that $\delta\underline{v}_i^\delta + \delta\overline{v}_j^\delta + \varepsilon \leq 1$. Since each player j receives at most $\delta\overline{v}_j^\delta$ in the continuation subgame, the offer is accepted in equilibrium. Player i makes unreasonable offers (say, offers 0) to all other players, and rejects any offer he receives. It must be that for all $\tau \geq 0$ the equilibrium payoff v_{i_τ} of i_τ in an $\mathcal{H}(i_\tau)$ subgame is not smaller than the expected payoff from the deviation,

$$v_{i_\tau} \geq \frac{2e-e_i}{2e}\delta\underline{v}_i^\delta + \frac{1}{2e}\sum_{\{j|ij\in G\}}\max(1-\delta\overline{v}_j^\delta - \varepsilon, \delta\underline{v}_i^\delta), \forall\varepsilon > 0,$$

and taking the limit $\varepsilon \to 0$,

$$v_{i_\tau} \geq \frac{2e-e_i}{2e}\delta\underline{v}_i^\delta + \frac{1}{2e}\sum_{\{j|ij\in G\}}\max(1-\delta\overline{v}_j^\delta, \delta\underline{v}_i^\delta).$$

By the definition of $\underline{v}_i^\delta$, the inequality above implies that

$$\underline{v}_i^\delta \geq \frac{2e-e_i}{2e}\delta\underline{v}_i^\delta + \frac{1}{2e}\sum_{\{j|ij\in G\}}\max(1-\delta\overline{v}_j^\delta, \delta\underline{v}_i^\delta). \tag{3.3}$$

Let $D = \max_{k\in N}\overline{v}_k^\delta - \underline{v}_k^\delta$. If $i \in \arg\max_{k\in N}\overline{v}_k^\delta - \underline{v}_k^\delta$, then from 3.2, 3.3, and Lemma 1,

$$\begin{aligned}
D = \overline{v}_i^\delta - \underline{v}_i^\delta &\leq \frac{2e-e_i}{2e}\delta(\overline{v}_i^\delta - \underline{v}_i^\delta) + \frac{1}{2e}\sum_{\{j|ij\in G\}}(\max(1-\delta\underline{v}_j^\delta, \delta\overline{v}_i^\delta) - \max(1-\delta\overline{v}_j^\delta, \delta\underline{v}_i^\delta)) \\
&\leq \frac{2e-e_i}{2e}\delta D + \frac{1}{2e}\sum_{\{j|ij\in G\}}\max(|1-\delta\underline{v}_j^\delta - (1-\delta\overline{v}_j^\delta)|, |\delta\overline{v}_i^\delta - \delta\underline{v}_i^\delta|) \\
&\leq \frac{2e-e_i}{2e}\delta D + \frac{1}{2e}\sum_{\{j|ij\in G\}}\delta\max(\overline{v}_j^\delta - \underline{v}_j^\delta, \overline{v}_i^\delta - \underline{v}_i^\delta) \\
&\leq \delta D.
\end{aligned}$$

Since $D \geq 0$ and $\delta \in (0,1)$, it follows that $D = 0$. Therefore, $\underline{v}_k^\delta = \overline{v}_k^\delta$ for all $k \in N$.

Then 3.2 and 3.3 imply that

$$\overline{v}_i^\delta = \frac{2e - e_i}{2e}\delta\overline{v}_i^\delta + \frac{1}{2e}\sum_{\{j|ij\in G\}} \max(1-\delta\overline{v}_j^\delta, \delta\overline{v}_i^\delta), \forall i \in N,$$

which means that $\overline{v}^\delta$ is identical to the unique fixed point $\tilde{v}^{*\delta}$ of f^δ. Hence $\underline{v}^\delta = \overline{v}^\delta = \tilde{v}^{*\delta}$. It follows that i_τ obtains an expected payoff of $\tilde{v}_i^{*\delta} =: v_i^{*\delta}$ in any $\mathcal{H}(i_\tau)$ subgame in every subgame perfect equilibrium. The second part of the theorem follows immediately. □

The following description of the equilibria for two simple networks illustrates the conclusions of Theorem 1. Equilibria for less trivial networks are analyzed in Example 2.

Example 1. Consider the network G_1 illustrated in Figure 2. The equilibrium payoffs are

$$v_1^{*\delta} = \frac{150 - 250\delta + 103\delta^2}{5(100 - 220\delta + 158\delta^2 - 37\delta^3)}, \qquad v_2^{*\delta} = \frac{100 - 160\delta + 63\delta^2}{5(100 - 220\delta + 158\delta^2 - 37\delta^3)}$$

$$v_3^{*\delta} = \frac{2(25 - 40\delta + 16\delta^2)}{5(100 - 220\delta + 158\delta^2 - 37\delta^3)}, \qquad v_4^{*\delta} = v_5^{*\delta} = \frac{100 - 165\delta + 67\delta^2}{5(100 - 220\delta + 158\delta^2 - 37\delta^3)},$$

converging to $v_1^* = v_2^* = 3/5$ and $v_3^* = v_4^* = v_5^* = 2/5$ as $\delta \to 1$. There exists a unique equilibrium in which, for all $ij \in G_1$, when i is selected to propose to j, he offers $\delta v_j^{*\delta}$, and when i has to respond to a proposal from j, he accepts any offer of at least $\delta v_i^{*\delta}$ and rejects smaller offers. In equilibrium every match ends in agreement.

Consider next the network G_2, also illustrated in Figure 2. The equilibrium payoffs when players have discount factor $\delta \le 10(9-\sqrt{2})/79 \approx 0.9602 =: \underline{\delta}$ are

$$v_1^{*\delta} = \frac{300 - 520\delta + 223\delta^2}{5(200 - 460\delta + 346\delta^2 - 85\delta^3)}, \qquad v_2^{*\delta} = \frac{100 - 160\delta + 63\delta^2}{5(200 - 460\delta + 346\delta^2 - 85\delta^3)}$$

$$v_3^{*\delta} = v_4^{*\delta} = \frac{2(50 - 85\delta + 36\delta^2)}{5(200 - 460\delta + 346\delta^2 - 85\delta^3)}, \qquad v_5^{*\delta} = \frac{2(100 - 170\delta + 71\delta^2)}{5(200 - 460\delta + 346\delta^2 - 85\delta^3)}.$$

For $\delta < \underline{\delta}$, there is a unique equilibrium with a description similar to the case of G_1.

A payoff irrelevant equilibrium multiplicity arises for the discount factor $\underline{\delta}$. For $\delta = \underline{\delta}$, it is true that $\delta(v_1^{*\delta} + v_5^{*\delta}) = 1$. Any behavior of player 1 in bargaining encounters with player 5 that satisfies the following conditions is part of an equilibrium. Player 1's offer is an arbitrary probability distribution over $[0, \delta v_5^{*\delta}]$. Player 1 rejects offers smaller than $\delta v_1^{*\delta}$, accepts with some arbitrary probability an offer of $\delta v_1^{*\delta}$, and accepts with probability 1 larger offers.[12]

[12]Note that the probability of agreement between 1_τ and $5_{\tau'}$ does not influence their own payoffs, but affects the length of time that future copies of players 1 and 5 need to wait before entering the game. However, the equilibrium payoffs of these players are not affected by the induced delay since, as already mentioned,

The equilibrium payoffs when players have discount factor $\delta > \underline{\delta}$ are

$$v_1^{*\delta} = \frac{2}{10-7\delta},\ v_3^{*\delta} = v_4^{*\delta} = \frac{1}{10-7\delta},\ v_2^{*\delta} = v_5^{*\delta} = \frac{1}{2(5-4\delta)},$$

converging to $v_1^* = 2/3, v_3^* = v_4^* = 1/3$ and $v_4^* = v_5^* = 1/2$ as $\delta \to 1$. For $\delta > \underline{\delta}$, in every equilibrium agreement obtains across all links except $(1,5)$. The equilibrium requirements do not pin down the disagreement offer in an encounter between players 1 and 5, and there exist multiple payoff equivalent equilibria as explained in Remark 1. However, in every equilibrium the strategies for bargaining across the links $(1,3),(1,4),(2,5)$ need to be as specified by Remark 1.

We call $(v_i^{*\delta})_{i\in N}$ the **equilibrium payoff vector** at δ. The **equilibrium agreement network** at δ, denoted $G^{*\delta}$, is defined as the subnetwork of G with the link ij included if and only if $\delta(v_i^{*\delta} + v_j^{*\delta}) \leq 1$. For δ such that $\delta(v_i^{*\delta} + v_j^{*\delta}) \neq 1, \forall ij \in G$, the agreements and disagreements in any subgame across all equilibria are entirely characterized as in the second part of Theorem 1. We show that the condition $\delta(v_i^{*\delta} + v_j^{*\delta}) \neq 1, \forall ij \in G$ holds for all but a finite set of discount factors δ, hence the description of equilibrium agreements and disagreements is complete generically.

Proposition 1. *The condition $\delta(v_i^{*\delta} + v_j^{*\delta}) \neq 1, \forall ij \in G$ holds for all but a finite set of δ.*

The proof appears in the Appendix. We outline the approach here since some of the ideas resurface in the proof of the next result.

For every $\delta \in (0,1)$ and every subnetwork H of G, consider the $n \times n$ linear system

$$v_i = \frac{2e - e_i^H}{2e}\delta v_i + \frac{1}{2e}\sum_{\{j|ij\in H\}}(1-\delta v_j), \forall i = \overline{1,n}, \tag{3.4}$$

where e_i^H denotes the number links that player i has in H (e denotes the total number of links in G, as in the proof of Theorem 1). We showed that $v^{*\delta}$ solves the system for $H = G^{*\delta}$. It is easy to check that the system 3.4 has a unique solution $v^{\delta,H}$. In particular, $v^{*\delta} = v^{\delta,G^{*\delta}}$.

All entries in the augmented matrix of the linear system 3.4 are linear functions of δ. Then for each $i \in N$ the solution $v_i^{\delta,H}$ is given by Cramer's rule, as the ratio of two determinants

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a player's payoff is evaluated by discounting relative to the time when the player entered the game rather than period 0 of the game.

that are polynomials in δ of degree at most n,

$$v_i^{\delta,H} = P_i^H(\delta)/Q_i^H(\delta). \tag{3.5}$$

We can then argue that every δ for which there exist i, j with $\delta(v_i^{*\delta} + v_j^{*\delta}) = 1$ is a root of one of a finite family of non-zero polynomials in δ.

Denote by Δ the finite set of δ for which the condition $\delta(v_i^{*\delta} + v_j^{*\delta}) \neq 1, \forall ij \in G$ does not hold. As established by Theorem 1, for $\delta \notin \Delta$, in every equilibrium of Γ^δ, in any subgame where i_τ is chosen to make an offer to $j_{\tau'}$, with probability one: (1) if $ij \in G^{*\delta}$ then i_τ offers $\delta v_j^{*\delta}$ and $j_{\tau'}$ accepts; (2) if $ij \notin G^{*\delta}$ then i_τ makes an offer that $j_{\tau'}$ rejects.

Theorem 2. *There exist $\underline{\delta} \in (0,1)$ and a subnetwork G^* of G such that the equilibrium agreement network $G^{*\delta}$ is equal to G^* for all $\delta > \underline{\delta}$. The equilibrium payoff vector $v^{*\delta}$ converges to a vector v^* as δ goes to 1. The rate of convergence of $v^{*\delta}$ to v^* is $O(1-\delta)$.*

Proof. To establish the first part, recall that the proof of Proposition 1 shows that the set $\bar{\Delta}$ of δ for which there exist a link $ij \in G$ and a subnetwork H of G such that $\delta(v_i^{\delta,H} + v_j^{\delta,H}) = 1$ is finite. Fix $\delta_0 > \max \bar{\Delta} =: \underline{\delta}$. Let $G^* = G^{*\delta_0}$. We show that $G^{*\delta} = G^*$ for all $\delta > \underline{\delta}$.

Fix $ij \in G$. The function $1 - \delta(v_i^{\delta,G^*} + v_j^{\delta,G^*})$ is continuous in δ for $\delta \in (0,1)$ as it is a **rational function** (ratio of two polynomials) by 3.5,[13] and it has no roots δ outside $\bar{\Delta}$. Then the sign ε_{ij}^δ of $1 - \delta(v_i^{\delta,G^*} + v_j^{\delta,G^*})$ is strict and constant for all $\delta > \underline{\delta}$. In particular, $\varepsilon_{ij}^\delta = \varepsilon_{ij}^{\delta_0}$ for $\delta > \underline{\delta}$.

Since $\delta_0 > \underline{\delta}$ and $G^* = G^{*\delta_0}$, the following conditions hold (1) $\varepsilon_{ij}^{\delta_0} = 1 \iff ij \in G^*$, and (2) $\varepsilon_{ij}^{\delta_0} = -1 \iff ij \notin G^*$. For all $\delta > \underline{\delta}$ we have $\varepsilon_{ij}^\delta = \varepsilon_{ij}^{\delta_0}$, hence the following conditions must also be true: (1) $\varepsilon_{ij}^\delta = 1 \iff ij \in G^*$, and (2) $\varepsilon_{ij}^\delta = -1 \iff ij \notin G^*$. For $\delta > \underline{\delta}$, it follows that $G^{*\delta} = G^*$.

To establish the second part, fix $i \in N$. From the first part, $v_i^{*\delta} = P_i^{G^*}(\delta)/Q_i^{G^*}(\delta)$ for $\delta > \underline{\delta}$. Rewrite $P_i^{G^*}/Q_i^{G^*} = \bar{P}_i^{G^*}/\bar{Q}_i^{G^*}$, with $\bar{P}_i^{G^*}$ and $\bar{Q}_i^{G^*}$ relatively prime polynomials. Since $v_i^{*\delta} \in [0,1]$ for all $\delta \in (\underline{\delta}, 1)$, it must be that $\bar{Q}_i^{G^*}(1) \neq 0$. For, if $\bar{Q}_i^{G^*}(1) = 0$ then $\bar{P}_i^{G^*}(1) \neq 0$ and $\bar{P}_i^{G^*}(\delta)/\bar{Q}_i^{G^*}(\delta)$ diverges at $\delta = 1$. Consequently, $\bar{P}_i^{G^*}(\delta)/\bar{Q}_i^{G^*}(\delta)$ converges to $\bar{P}_i^{G^*}(1)/\bar{Q}_i^{G^*}(1)$ as δ tends to 1. Therefore, $v_i^{*\delta} = \bar{P}_i^{G^*}(\delta)/\bar{Q}_i^{G^*}(\delta)$ has a finite limit, $v_i^* := \bar{P}_i^{G^*}(1)/\bar{Q}_i^{G^*}(1)$, at $\delta = 1$.

[13]For all $\delta \in (0,1)$, $Q_i^{G^*}(\delta) \neq 0$ because the system 3.4 is non-singular.

To show that the rate of convergence of $v^{*\delta}$ to v^* is $O(1-\delta)$, write

$$v_i^{*\delta} - v_i^* = \frac{\bar{P}_i^{G^*}(\delta)}{\bar{Q}_i^{G^*}(\delta)} - \frac{\bar{P}_i^{G^*}(1)}{\bar{Q}_i^{G^*}(1)} = \frac{\bar{Q}_i^{G^*}(1)\bar{P}_i^{G^*}(\delta) - \bar{P}_i^{G^*}(1)\bar{Q}_i^{G^*}(\delta)}{\bar{Q}_i^{G^*}(1)\bar{Q}_i^{G^*}(\delta)}.$$

The latter rational function has a vanishing numerator and a non-vanishing denominator at $\delta = 1$, hence it can be rewritten as $(1-\delta)R_i^{G^*}(\delta)$ where $R_i^{G^*}$ is a rational function with a finite limit at $\delta = 1$. □

We call G^* the **limit equilibrium agreement network**, and v^* the **limit equilibrium payoff vector**. Our main objective is to determine the limit equilibrium payoff vector. The following preliminary observations are proven in the Appendix.

Proposition 2. *If $ij \in G$, then $v_i^* + v_j^* \geq 1$. If $ij \in G^*$, then $v_i^* + v_j^* = 1$. In particular, if $v_i^* + v_j^* > 1$, then $ij \notin G^*$.*

Lemma 3. *Every player has at least one link in G^* (under the assumption in footnote 6).*

4. Bounds for Limit Equilibrium Payoffs

For every network H, and a non-empty subset of players M, let $L^H(M)$ denote the set of players that have H-links to players in M, $L^H(M) = \{j | ij \in H, i \in M\}$. A set M is **H-independent** if there exists no H-link between two players in M, $M \cap L^H(M) = \emptyset$. A set is **mutually estranged** if it is G^*-independent. The set of **partners** for a mutually estranged set M is defined as $L^{G^*}(M)$.

Fix a mutually estranged set M with partner set L. Basically, as players become patient, the players in L have control over the relevant bargaining opportunities of the players in M. For high discount factors, since players in M can only reach equilibrium agreements in pairwise matchings with players in L, the set M is weak if the set L is relatively small. This intuition is formalized by the shortage ratio, which measures the strength of the mutually estranged players in a sense made precise later. The **shortage ratio** of M is defined as the ratio of the numbers of partners and estranged players, $|L|/|M|$.

The next result is essential for developing a procedure to determine the limit equilibrium payoffs. For every mutually estranged set M with partner set L, the ratio of the limit equilibrium payoffs of the worst-off estranged player, $\min_{i \in M} v_i^*$, and the best-off partner, $\max_{j \in L} v_j^*$, is not larger than the shortage ratio of M.

Theorem 3. *For every mutually estranged set M with partner set L, the following bounds on limit equilibrium payoffs hold*

$$\min_{i \in M} v_i^* \leq \frac{|L|}{|M| + |L|}$$

$$\max_{j \in L} v_j^* \geq \frac{|M|}{|M| + |L|}.$$

The proof of Theorem 3 is relegated to the Appendix. It uses the following result.

Lemma 4. *For every δ, the equilibrium payoff of each player in Γ^δ is equal to the expected present value of his stream of first mover advantage, i.e.,*

$$v_i^{*\delta} = \frac{1}{1-\delta} \sum_{\{j|ij \in G\}} \frac{1}{2e} \max(1 - \delta v_i^{*\delta} - \delta v_j^{*\delta}, 0), \forall i \in N, \forall \delta \in (0, 1).$$

Proof. Fix a discount factor δ and a player i. The expected payoff of i is $\delta v_i^{*\delta}$ in any subgame where he is not the proposer, and $\max(1 - \delta v_j^{*\delta}, \delta v_i^{*\delta})$–which represents a net gain of $\max(1 - \delta v_i^{*\delta} - \delta v_j^{*\delta}, 0)$ over $\delta v_i^{*\delta}$–in any subgame where he is selected to make an offer to player j. Hence, $\max(1 - \delta v_i^{*\delta} - \delta v_j^{*\delta}, 0)$ measures the net **first mover advantage** that i gains from making an offer to j. Therefore, the expected loss from not being the proposer at some round, measured by $(1 - \delta)v_i^{*\delta}$, is equal to the sum of the first mover advantage $\max(1 - \delta v_i^{*\delta} - \delta v_j^{*\delta}, 0)$ that i enjoys when selected to make an offer to j weighted by the probability of this event in that round. □

The intuition for the proof of Theorem 3 is as follows. Suppose that M is a mutually estranged set with partner set L. Fix a discount factor $\delta > \underline{\delta}$, with $\underline{\delta}$ specified as in Theorem 2. Thus, $G^{*\delta} = G^*$. In every equilibrium, in any subgame, a player i in M only reaches agreements with players in L with whom he shares G^* links. Any net first mover advantage that a player i in M gains from making an offer to a player j in L is mapped to an equal net first mover advantage that j gains from making an offer to i ($\max(1 - \delta v_i^{*\delta} - \delta v_j^{*\delta}, 0)$ is symmetric in i and j). It follows that the sum of the expected present values of the streams of first mover advantage enjoyed by all players in M is not larger than the same expression evaluated for the players in L.[14] Hence, by Lemma 4, $\sum_{j \in L} v_j^{*\delta} \geq \sum_{i \in M} v_i^{*\delta}$. Taking the

[14]Players in M only gain first mover advantage from players in L, while players in L gain first mover advantage from the corresponding players in M, and possibly from players that are not in M.

limit $\delta \to 1$ we obtain that $\sum_{j \in L} v_j^* \geq \sum_{i \in M} v_i^*$. Then the proof is completed by repeated use of Proposition 2 to establish that $\min_{i \in M} v_i^* + \max_{j \in L} v_j^* = 1$.

While Theorem 3 invokes knowledge we do not have a priori about the limit equilibrium agreement network G^*, it has an immediate corollary that involves exclusively properties of G.[15] It is sufficient to note that since G^* is a subnetwork of G, if M is G-independent then M is also G^*-independent, and $L^{G^*}(M) \subset L^G(M)$, so $|L^{G^*}(M)| \leq |L^G(M)|$.

Corollary 1. *For every G-independent set of players M, the following bounds on limit equilibrium payoffs hold*

$$\begin{aligned} \min_{i \in M} v_i^* &\leq \frac{|L^G(M)|}{|M| + |L^G(M)|} \\ \max_{j \in L^G(M)} v_j^* &\geq \frac{|M|}{|M| + |L^G(M)|}. \end{aligned}$$

5. Limit Equilibrium Payoff Computation

Theorem 3 suggests that it may be useful to study the mutually estranged sets M that minimize the upper bound $|L^{G^*}(M)|/(|M| + |L^{G^*}(M)|)$ for the limit equilibrium payoff of the worst-off player in M, or equivalently, minimize the shortage ratio $|L^{G^*}(M)|/|M|$. The next lemma (applied to the network G^*) shows that the set of such minimizers is closed with respect to unions if the attained minimum is less than 1. It is useful to generalize this conclusion to all networks. For every network H let $\mathcal{I}(H)$ denote the set of non-empty H-independent sets.

Lemma 5. *Let H be a network. Suppose that*

$$\min_{M \in \mathcal{I}(H)} \frac{|L^H(M)|}{|M|} < 1,$$

and that M' and M'' are two H-independent sets achieving the minimum. Then $M' \cup M''$ is also H-independent, and

$$M' \cup M'' \in \arg \min_{M \in \mathcal{I}(H)} \frac{|L^H(M)|}{|M|}.$$

The proof, which is a conjunction of combinatorial arguments and brute force algebra, is relegated to the Appendix.

[15] However, for the results of the next section we need the full strength of Theorem 3.

We show that the bounds on limit equilibrium payoffs corresponding to a set of mutually estranged players and their partners provided by Theorem 3 need to be binding unless the worst-off estranged player is part of an even weaker mutually estranged set, and the best-off partner is part of an even stronger partner set. The intuition is that each player is part of a limit equilibrium oligopoly subnetwork where, for high δ, some mutually estranged players and their partners share the unit surplus according to the shortage ratio. Consequently, the limit equilibrium payoff of any player cannot be smaller than the upper bound for the worst-off player from a mutually estranged set with the lowest shortage ratio. Therefore, the bounds for the limit equilibrium payoffs of the worst-off estranged player and the best-off partner corresponding to a mutually estranged set with the lowest shortage ratio must be binding.

Suppose that the lowest shortage ratio $r_1 = \min_{M \in \mathcal{I}(G)} |L^G(M)|/|M|$ is smaller than 1.[16] Let M_1 be the union of all G-independent sets M minimizing the shortage ratio. By Lemma 5, M_1 is also a G-independent set with minimal shortage ratio. Let $L_1 = L^G(M_1)$ be the corresponding set of partners.[17] We argued above that $\min_{i \in M_1} v_i^* = r_1/(r_1+1)$ and $\max_{j \in L_1} v_j^* = 1/(r_1+1)$. We set out to show that the limit equilibrium payoffs equal $r_1/(r_1+1)$ for all players in M_1 and $1/(r_1+1)$ for all players in L_1; that is, all players in $M_1 \cup L_1$, not only the worst-off estranged player and the best-off partner, have extremal limit equilibrium payoffs. The following algorithm sequentially iterates this hypothesis in order to determine the limit equilibrium payoffs of all players.

Definition 1 (Algorithm $\mathcal{A}(G) = (r_s, x_s, M_s, L_s, N_s, G_s)_{s=1,2,\ldots,\bar{s}}$)**.** Define the sequence $(r_s, x_s, M_s, L_s, N_s, G_s)_s$ recursively as follows. Let $N_1 = N$ and $G_1 = G$. For $s \geq 1$, if $N_s = \emptyset$ then **stop**. Else, let[18]

$$r_s = \min_{M \subset N_s,\ M \in \mathcal{I}(G)} \frac{|L^{G_s}(M)|}{|M|}. \tag{5.1}$$

[16]Note that G^* is a priori unknwon. Our proof reveals that the lowest shortage ratio, when smaller than 1, may be computed by restricting attention to sets that are G-independent rather than G^*-independent.
[17]One implication of our analysis is that $L^{G^*}(M_1) = L^G(M_1)$.
[18]It can be shown that each player in N_s has at least one link in G_s, hence r_s is well-defined and positive.

If $r_s \geq 1$ then **stop**. Else, set $x_s = r_s/(1+r_s)$. Let M_s be the union of all minimizers M in 5.1.[19] Denote $L_s = L^{G_s}(M_s)$. Let $N_{s+1} = N_s \setminus (M_s \cup L_s)$, and G_{s+1} be the subnetwork of G induced by the players in N_{s+1}. Denote by $\overline{s}$ the finite step at which the algorithm stops.[20]

At each step, the algorithm $\mathcal{A}(G)$ determines the largest cardinality mutually estranged set minimizing the shortage ratio in the subnetwork induced by the remaining players (Lemma 5), and removes the corresponding estranged players and partners (the remarks in footnotes 16 and 17 are essential). We are going to show that for high discount factors the removed players can only reach agreements among themselves in equilibrium; the oligopoly subnetwork they form encloses all their limit equilibrium agreement links. The definition ensures that $\mathcal{A}(G)$ identifies and removes all residual players with extremal limit equilibrium payoffs simultaneously. The algorithm stops when every G-independent set formed by the remaining players has shortage ratio greater than or equal to 1, or when all players have been removed.[21]

The next lemma, which is used in the proofs of Proposition 3 and Theorem 4 below, follows immediately from the definition of the algorithm $\mathcal{A}(G)$. The proof of Proposition 3 is provided in the Appendix.

Lemma 6. *$\mathcal{A}(G)$ satisfies the following conditions for all $1 \leq s \leq s' < \overline{s}$*

$$\begin{aligned} L^{G_s}(M_s \cup M_{s+1} \cup \ldots \cup M_{s'}) &= L_s \cup L_{s+1} \cup \ldots \cup L_{s'} \\ L^G(N_{s+1}) \cap (M_1 \cup M_2 \cup \ldots \cup M_s) &= \emptyset \\ M_s \cup M_{s+1} \cup \ldots \cup M_{s'} & \text{ is } G\text{-independent.} \end{aligned}$$

Proposition 3. *The sequences $(r_s)_s$ and $(x_s)_s$ defined by $\mathcal{A}(G)$ are strictly increasing.*

Note that the sets $M_1, L_1, \ldots, M_{\overline{s}-1}, L_{\overline{s}-1}, N_{\overline{s}}$ partition N. The limit equilibrium payoff of each player is uniquely determined by the partition set he belongs to.

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[19]By Lemma 5, since $r_s < 1$, M_s is also a minimizer in 5.1.

[20]In some cases the (irrelevant) variables $r_{\overline{s}}, x_{\overline{s}}, M_{\overline{s}}, L_{\overline{s}}$ are left undefined.

[21]As an illustration, the algorithm $\mathcal{A}(G_2)$, for the network G_2 introduced in Section 1, ends in $\overline{s} = 2$ steps. The relevant outcomes are $r_1 = 1/2, x_1 = 1/3, M_1 = \{3,4\}, L_1 = \{1\}$ at the first step, and $r_2 = 1, N_2 = \{2,5\}$ at the second step.

Theorem 4. *Let* $(r_s, x_s, M_s, L_s, N_s, G_s)_{s=1,2,\ldots,\overline{s}}$ *be the outcome of the algorithm* $\mathcal{A}(G)$. *The limit equilibrium payoffs for* Γ^δ *as* $\delta \to 1$ *are given by*

$$\begin{aligned} v_i^* &= x_s, \forall i \in M_s, \forall s < \overline{s} \\ v_j^* &= 1 - x_s, \forall j \in L_s, \forall s < \overline{s} \\ v_k^* &= \frac{1}{2}, \forall k \in N_{\overline{s}}. \end{aligned}$$

Proof. We prove the theorem by induction on s. Suppose we proved the assertion for all lower values, and we proceed to proving it for s $(1 \le s \le \overline{s})$.[22] We treat the case $s = \overline{s}$ separately, in the Appendix.

Let $s < \overline{s}$ and define $\underline{x}_s = \min_{i \in N_s} v_i^*$. Denote by $\underline{M}_s = \arg\min_{i \in N_s} v_i^*$ the set of players in N_s whose limit equilibrium payoffs equal $\underline{x}_s$, and set $\underline{L}_s = L^{G_s}(\underline{M}_s)$. We first show that $\underline{x}_s = x_s$ by arguing that $\underline{x}_s \le x_s$ and $\underline{x}_s \ge x_s$.

Claim 4.1. $\underline{x}_s \le x_s$

We proceed by contradiction. Suppose that $\underline{x}_s > x_s$. Then $v_j^* \ge 1 - x_{s-1} > 1 - x_s > 1 - \underline{x}_s$ for all j in $L_1 \cup L_2 \cup \ldots \cup L_{s-1}$.[23] The first inequality follows from the induction hypothesis and Proposition 3, and the second from Proposition 3. But $v_i^* \ge \underline{x}_s$ for all i in M_s. Thus, $v_i^* + v_j^* > 1, \forall i \in M_s, \forall j \in L_1 \cup L_2 \cup \ldots \cup L_{s-1}$. By Proposition 2 no player $i \in M_s$ has G^* links to players $j \in L_1 \cup L_2 \cup \ldots \cup L_{s-1}$, or $L^{G^*}(M_s) \cap (L_1 \cup L_2 \cup \ldots \cup L_{s-1}) = \emptyset$.

By Lemma 6, $L^G(M_s) \cap (M_1 \cup M_2 \cup \ldots \cup M_{s-1}) = \emptyset$.

It follows that $L^{G^*}(M_s) \subset L^{G_s}(M_s) = L_s$. Theorem 3 implies that

$$\min_{i \in M_s} v_i^* \le \frac{|L_s|}{|M_s| + |L_s|} = x_s,$$

a contradiction with $\min_{i \in N_s} v_i^* = \underline{x}_s > x_s$.

Claim 4.2. $\underline{x}_s \ge x_s$ and $v_j^* = 1 - \underline{x}_s, \forall j \in \underline{L}_s$

We proved that $\underline{x}_s \le x_s$. Since $r_s < 1$ it follows that $x_s < 1/2$. By Proposition 2 and Claim 4.1,

$$v_j^* \ge 1 - \underline{x}_s \ge 1 - x_s > 1/2, \forall j \in \underline{L}_s.$$

[22]The following technical detail is used in order to avoid analogous arguments proving the base case and the inductive step. Append step 0 to the algorithm, with $(r_0, x_0, M_0, L_0, N_0, G_0) = (0, 0, \emptyset, \emptyset, N, G)$. Then the base case $s = 0$ follows trivially, and the inductive steps, $s = 1, 2, \ldots, \overline{s} - 1$, involve analogous arguments.
[23]This argument is only necessary for $s > 1$.

Then Proposition 2 implies that $\underline{L}_s$ is a G^*-independent set.

Fix $j \in \underline{L}_s$. By Proposition 2 there exist no G^* links from j to players $k \in N_s \setminus \underline{M}_s$, since for these players $v_k^* > \underline{x}_s$ (by the definition of $\underline{M}_s$) and we already argued that $v_j^* \geq 1 - \underline{x}_s$. Also, there exist no G links from j to players in $M_1 \cup M_2 \cup \ldots \cup M_{s-1}$ by Lemma 6.

By Proposition 2, there exist no G^* links from j to players $k \in L_1 \cup L_2 \cup \ldots \cup L_{s-1}$ since for these players $v_k^* \geq 1 - x_{s-1} > 1/2$, and we need $v_j^* > 1/2$.

Therefore, any $j \in \underline{L}_s$ only has G^* links to players in $\underline{M}_s$. By Proposition 2 and Lemma 3, any player $j \in \underline{L}_s$ must have limit equilibrium payoff $v_j^* = 1 - \underline{x}_s$.

Hence $\underline{L}_s$ is G^*-independent and $L^{G^*}(\underline{L}_s) \subset \underline{M}_s$,[24] so it follows from Theorem 3 that

$$\underline{x}_s = \max_{i \in L^{G^*}(\underline{L}_s)} v_i^* \geq \frac{|\underline{L}_s|}{|L^{G^*}(\underline{L}_s)| + |\underline{L}_s|} \geq \frac{|\underline{L}_s|}{|\underline{M}_s| + |\underline{L}_s|}.$$

Recall that $\underline{L}_s = L^{G_s}(\underline{M}_s)$. We can rewrite the inequality above as

$$\frac{\underline{x}_s}{1 - \underline{x}_s} \geq \frac{|L^{G_s}(\underline{M}_s)|}{|\underline{M}_s|}. \tag{5.2}$$

Yet by the definition of r_s and x_s,

$$\frac{|L^{G_s}(\underline{M}_s)|}{|\underline{M}_s|} \geq r_s = \frac{x_s}{1 - x_s}, \tag{5.3}$$

and the last two inequalities imply $\underline{x}_s \geq x_s$.

Claims 4.1 and 4.2 establish that $\underline{x}_s = x_s$. Hence $v_i^* = x_s, \forall i \in \underline{M}_s$ and $v_j^* = 1 - x_s, \forall j \in \underline{L}_s$. Moreover, we need to have equalities in the weak inequalities 5.2-5.3, so $|\underline{L}_s|/|\underline{M}_s| = r_s$.

Claim 4.3. $\underline{M}_s \subset M_s$

Since M_s is the union of all G-independent $M \subset N_s$ with $|L^{G_s}(M)|/|M| = r_s$ and $|L^{G_s}(\underline{M}_s)|/|\underline{M}_s| = r_s$, it follows that $\underline{M}_s \subset M_s$. ($\underline{M}_s$ is G-independent by Proposition 2, as the limit equilibrium payoff of each player in $\underline{M}_s$ is $x_s < 1/2$.)

Claim 4.4. $\underline{M}_s = M_s$

We show that $\underline{M}_s = M_s$ by contradiction. Fix $i \in M_s \setminus \underline{M}_s$. Since $i \in M_s$ and $L^{G_s}(M_s) = L_s$, i has no G links to players in $N_s \setminus L_s$. By Lemma 6, i has no G links to $M_1 \cup M_2 \cup \ldots \cup M_{s-1}$.

By Proposition 2, i has no G^* links to players $j \in L_1 \cup L_2 \cup \ldots \cup L_{s-1} \cup \underline{L}_s$ as for such players $v_j^* \geq 1 - x_s$, and $v_i^* > \underline{x}_s = x_s$ by the definition of $\underline{M}_s$.

[24] It can be easily argued that the inclusion holds with equality.

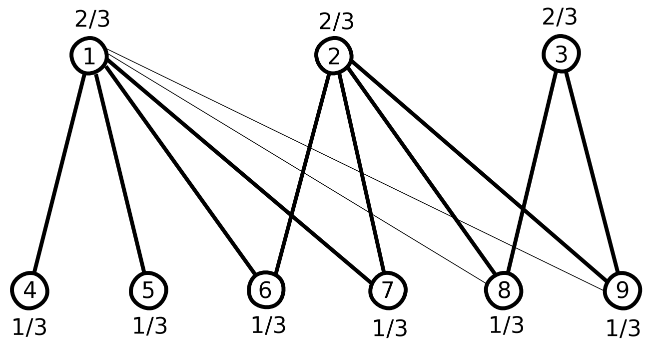

FIGURE 3. Network G_3

It follows that i may only have G^* links to players in $L_s \setminus \underline{L}_s$. Therefore, $L^{G^*}(M_s \setminus \underline{M}_s) \subset L_s \setminus \underline{L}_s$, implying that $|L^{G^*}(M_s \setminus \underline{M}_s)| \leq |L_s \setminus \underline{L}_s| = |L_s| - |\underline{L}_s|$.

Note that

$$\frac{|L_s|}{|M_s|} = r_s \ \& \ \frac{|\underline{L}_s|}{|\underline{M}_s|} = r_s \implies \frac{|L_s| - |\underline{L}_s|}{|M_s| - |\underline{M}_s|} = r_s.$$

Then by Theorem 3,

$$\min_{i \in M_s \setminus \underline{M}_s} v_i^* \leq \frac{|L^{G^*}(M_s \setminus \underline{M}_s)|}{|M_s \setminus \underline{M}_s| + |L^{G^*}(M_s \setminus \underline{M}_s)|} \leq \frac{|L_s| - |\underline{L}_s|}{|M_s| - |\underline{M}_s| + |L_s| - |\underline{L}_s|} = \frac{r_s}{1 + r_s} = x_s,$$

a contradiction with $v_i^* > x_s$ for all $i \in N_s \setminus \underline{M}_s$.

Therefore, $\underline{M}_s = M_s, \underline{L}_s = L_s$, and $v_i^* = x_s, \forall i \in M_s$ and $v_j^* = 1 - x_s, \forall j \in L_s$, completing the proof of the induction step for $s < \overline{s}$. □

Example 2 below shows that the limit equilibrium agreement network does not necessarily contain all the links from players in M_s to players in L_s. Although all players in M_s (L_s) have identical limit equilibrium payoffs, their relative bargaining strengths may vary, and the rates of convergence to the common limit are not identical across M_s (L_s). Moreover, it is possible that the players in $M_s \cup L_s$ induce a connected subnetwork in G, but a disconnected subnetwork in G^*.

Example 2. Consider the network G_3 with 9 players illustrated in Figure 3.[25] The algorithm $\mathcal{A}(G_3)$ ends in one step, with $r_1 = 1/2$, $M_1 = \{4, 5, 6, 7, 8, 9\}$ and $L_1 = \{1, 2, 3\}$. Therefore, the limit equilibrium payoffs are $1/3$ for all players in M_1 and $2/3$ for all players in L_1. However, it is not the case that the limit equilibrium agreement network contains all the links from players in M_1 to players in L_1, that is, all the links of G_3.

[25]See the legend in footnote 2.

Indeed, the limit equilibrium agreement network G_3^* excludes the links $(1,8)$ and $(1,9)$. The intuition is that, although $v_1^* = v_2^* = v_3^* = 2/3$ and $v_4^* = v_5^* = v_6^* = v_7^* = v_8^* = v_9^* = 1/3$, player 1 is relatively stronger than players 2 and 3 as he is connected to all players that 2 and 3 are connected to, and players 8 and 9 are relatively stronger than players 4, 5, 6, and 7 as they are connected to all players that 4, 5, 6, and 7 are connected to. For similar reasons, player 3 is relatively weaker than players 1 and 2, augmenting the relative strength of 8 and 9 over 4, 5, 6, and 7; and players 4 and 5 are relatively weaker than players 6, 7, 8, 9, augmenting the relative strength of 1 over 2 and 3. For high δ, the equilibrium payoffs of player 1, and also of players 8 and 9, will be sufficiently high so that 1 does not reach agreements with either 8 or 9.

By the proof of Theorem 1, to check that G_3^* is the limit equilibrium agreement network, we only need to show that v^{δ,G_3^*} is a fixed point of the corresponding f^δ for δ sufficiently large.[26] The payoff vector v^{δ,G_3^*} solves the $n \times n$ linear system

$$v_i = \frac{2e - e_i^{G_3^*}}{2e}\delta v_i + \frac{1}{2e}\sum_{\{j|ij\in G_3^*\}}(1-\delta v_j), \forall i = \overline{1,9}.$$

A closed form solution is immediately obtained, but is omitted for expositional brevity. For example,

$$v_1^{\delta,G_3^*} = \frac{2(576 - 1068\delta + 493\delta^2)}{3(2304 - 6048\delta + 5264\delta^2 - 1519\delta^3)},$$

and the other components of v^{δ,G_3^*} have similar rational function expressions.

Simple calculus shows that for all $\delta > \underline{\delta} := 6/251(45 - \sqrt{17}) \approx 0.977$, the following is true: $\forall ij \in G_3,\ \delta(v_i^{\delta,G_3^*} + v_j^{\delta,G_3^*}) \leq 1 \iff ij \in G_3^*$. Hence v^{δ,G_3^*} is a fixed point of f^δ, and $G_3^{*\delta} = G_3^*$. Therefore, G_3^* is the limit equilibrium agreement network. For $\delta < \underline{\delta}$, the equilibrium agreement network is the entire G_3, and for $\delta > \underline{\delta}$ the equilibrium agreement network is G_3^*. The set of equilibria admits a characterization similar to that for the network G_2 in Example 1.

An example where the players in $M_s \cup L_s$ induce a connected subnetwork in G, but a disconnected subnetwork in G^* is provided by the network G_4 from Figure 4.[27] G_4 essentially consists of two copies of G_3, with two additional links, $(8,10)$ and $(9,10)$. The limit equilibrium agreement network G_4^* excludes the links $(1,8), (1,9), (8,10), (9,10), (10,17), (10,18)$

[26] Recall the definitions 3.1 and 3.4.
[27] See the legend in footnote 2.

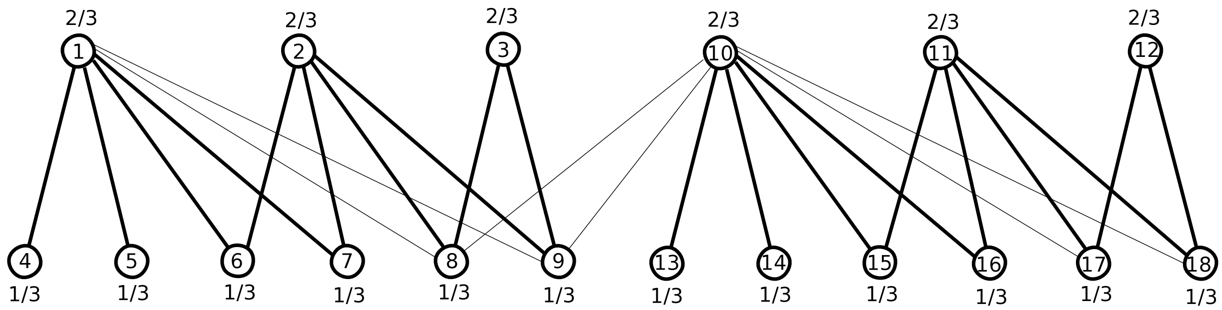

FIGURE 4. Network G_4

(by a logic analogous to the one suggesting that $(1,8)$ and $(1,9)$ are not limit equilibrium agreement links in G_3).[28]

6. Equitable Networks

We are interested in characterizing the class of **equitable** networks, that is, networks for which the limit equilibrium payoffs of all players are identical–equal to $1/2$, by Proposition 2. By Theorem 4, a network G is equitable if and only if $r_1 \geq 1$, so that the algorithm $\mathcal{A}(G)$ stops at the first step. Intuitively, this means that no oligopoly may emerge in equilibrium. Thus G is equitable if and only if $|L^G(M)| \geq |M|$ for every G-independent set M. Networks satisfying the latter property have been studied in graph theory. The following definitions are useful.

A graph is an **odd cycle** if its vertex set has odd cardinality and can be relabeled $\nu_1, \nu_2, \ldots, \nu_k$ such that the set of edges is $\{\nu_1\nu_2, \nu_2\nu_3, \ldots, \nu_{k-1}\nu_k, \nu_k\nu_1\}$. A graph is a **match** if its vertex set has even cardinality and can be relabeled $\nu_1, \nu_2, \ldots, \nu_k$ such that the set of edges is $\{\nu_1\nu_2, \nu_3\nu_4, \ldots, \nu_{k-1}\nu_k\}$. A graph is a **match and odd cycles disjoint union** if it is the union of a match and a number of odd cycles that are pairwise vertex-disjoint. A network **covers** a vertex if the vertex has at least one link in the network. A network H' **covers** a network H if H' is a subnetwork of H that covers each vertex of H. A graph is **regular** if all its vertices are incident to an identical number of edges.

A graph H is **quasi-regularizable** (Berge 1981) if there exists $d > 0$ and non-negative integer weights ω_{ij} associated with each edge $ij \in H$ such that the sum of the weights of the

[28]The bipartite nature of G_3 and G_4 is not critical to the asymptotic results. For instance, the limit equilibrium payoffs and agreement networks for G_3 and G_4 remain unchanged if the link $(2,3)$ is added to either network.

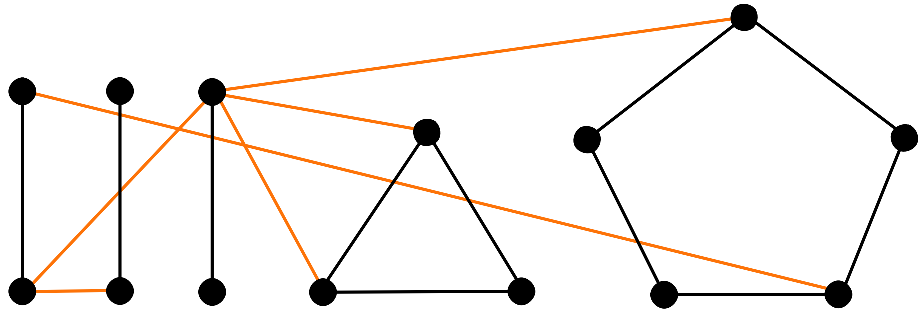

FIGURE 5. A network that can be covered by a match and odd cycles disjoint union

edges incident to any vertex i is d,

$$\sum_{\{j|ij\in H\}} \omega_{ij} = d.$$

Examples of quasi-regularizable graphs are regular graphs (set all the weights equal to 1), and match and odd cycles disjoint unions (set the weights of the edges along the odd cycles and in the match equal to 1 and 2, respectively). If a network is quasi-regularizable then so is any network it covers. Berge (1981) shows that [G satisfies $|L^G(M)| \geq |M|$ for every G-independent set M] if and only if [G is quasi-regularizable] if and only if [G can be covered by a match and odd cycles disjoint union]. Figure 5 depicts a network that can be covered by a match and odd cycles disjoint union. The links of the relevant covering subnetwork are drawn in black, while the rest of the links are colored orange. Berge's alternative characterizations, along with the discussion opening the section, establish the following result.

Theorem 5. *[G is equitable] if and only if [G is quasi-regularizable] if and only if [G can be covered by a match and odd cycles disjoint union].*

One important corollary is that regular networks, and also networks that can be covered by regular networks, are equitable.

Example 3. Consider the network G_5 with 6 players drawn in Figure 6.[29] By Theorem 5, G_5 is equitable since it is covered by the match $\{(1,4),(2,5),(3,6)\}$. By methods similar to those of Example 2 we can prove that the limit equilibrium agreement network excludes the link $(1,6)$.

[29]See the legend in footnote 2.

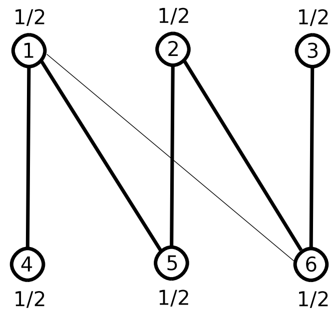

FIGURE 6. Network G_5

Limit equilibrium payoffs depend on the network structure in a more complex fashion than simply by way of the relative number of bargaining partners. In G_5 players may have 1, 2, or 3 links, but they all receive limit equilibrium payoffs of 1/2. Therefore, while regular networks are equitable, regularity is far from being a necessary condition for equitability. Quasi-regularizability is a necessary and sufficient condition.

7. STABLE NETWORKS

In our model the network structure is exogenously given, and we do not study network formation. Nonetheless, we may ask whether patient players can benefit in the bargaining game from forming new links or severing existing ones. The algorithm $\mathcal{A}(G)$ can be used to address this question. Fix the set of players N, and let $\mathcal{G}$ be the set of networks G with vertex set equal to N. A **payoff function** u assigns to each player $i \in N$ a payoff, denoted $u_i(G)$, for every network $G \in \mathcal{G}$. If $v_i^{*\delta}(G)$ and $v_i^*(G)$ denote the equilibrium payoff of player i in the bargaining game on the network G for discount factor δ, and respectively its limit as $\delta \to 1$, then the profiles $(v_i^{*\delta}(G))_{i\in N, G\in\mathcal{G}}$ and $(v_i^*(G))_{i\in N, G\in\mathcal{G}}$ define payoff functions. For every network G and any $i \neq j \in N$, let $G + ij$ $(G - ij)$ denote the network obtained by adding (deleting) the link ij to (from) G.

Definition 2 (Stability)**.** A network G is **unilaterally stable** with respect to the payoff function u if $u_i(G) \geq u_i(G - ij)$ for all $ij \in G$. A network G is **pairwise stable** with respect to the payoff function u if it is unilaterally stable with respect to u, and for all $ij \notin G$, $u_i(G + ij) > u_i(G)$ only if $u_j(G + ij) < u_j(G)$.

To rephrase, a network is unilaterally stable if no player benefits from severing one of his links. Pairwise stability requires additionally that no pair of players benefit from forming a new link. Jackson and Wolinsky (1996) motivate the definitions by the fact that the formation of the link ij necessitates the consent of both players i and j, but its severance can be done unilaterally by either i or j.

Theorem 6. *(i) Every network is unilaterally stable with respect to $(v_i^*(G))_{i\in N, G\in\mathcal{G}}$. (ii) A network is pairwise stable with respect to $(v_i^*(G))_{i\in N, G\in\mathcal{G}}$ if and only if it is equitable.*

Part (i) of the statement is not surprising, but its proof is involved, as the removal of a single link may create a long chain effect in the procedure for determining limit equilibrium payoffs. All proofs are in the Appendix.

While every network is unilateral stabile with respect to the equilibrium payoffs in the limit as players become patient, the conclusion does not necessarily apply before taking the limit. Indeed, not every network is unilaterally stable with respect to $(v_i^{*\delta}(G))_{i\in N, G\in\mathcal{G}}$ for $\delta < 1$. Consider the networks G_2 and G_2^* from Figure 2. G_2^* is obtained from G_2 by removing the link $(1,5)$. For every discount factor $\delta \in (10(9-\sqrt{2})/79, 1)$, players 1 and 5 receive higher equilibrium payoffs in the bargaining game on the network G_2^* than in that on G_2,

$$v_1^{*\delta}(G_2) = \frac{2}{10-7\delta} < \frac{2}{8-5\delta} = v_1^{*\delta}(G_2^*) \text{ and } v_5^{*\delta}(G_2) = \frac{1}{2(5-4\delta)} < \frac{1}{2(4-3\delta)} = v_5^{*\delta}(G_2^*).$$

Thus both players 1 and 5 benefit from removing the link connecting them and prefer playing the game on G_2^* rather than G_2.

The intuition for this observation is simple. For the range of discount factors considered, $(1,5)$ is an equilibrium disagreement link in the bargaining game on G_2, whence it becomes a source of delay for the possible agreements and deflates the equilibrium payoffs of all players. However, the gains to players 1 and 5 from severing the link $(1,5)$ vanish as δ approaches 1. If players only consider deleting or adding links when the ensuing gains are significant, we need to focus on **approximate stability**.

Definition 3 (ε-Stability)**.** A network G is **unilaterally ε-stable** with respect to the payoff function u if $u_i(G) + \varepsilon \geq u_i(G - ij)$ for all $ij \in G$. A network G is **pairwise ε-stable** with respect to the payoff function u if it is unilaterally ε-stable with respect to u, and for all $ij \notin G$, $u_i(G + ij) > u_i(G) + \varepsilon$ only if $u_j(G + ij) < u_j(G) + \varepsilon$.

For any sufficiently low $\varepsilon > 0$, there exists a discount factor threshold $\underline{\delta} < 1$ such that the two statements of Theorem 6 also hold for ε-stability with respect to the equilibrium payoffs for any $\delta > \underline{\delta}$. The next result is based on ideas from the proofs of Theorems 2 and 6.

Corollary 2. *There exists $\bar{\varepsilon} > 0$ such that for all $\varepsilon < \bar{\varepsilon}$ there exists $\underline{\delta} < 1$ such that for all $\delta > \underline{\delta}$ the following statements are true. (i) Every network is unilaterally ε-stable with respect to $(v_i^{*\delta}(G))_{i\in N, G\in\mathcal{G}}$. (ii) A network is pairwise ε-stable with respect to $(v_i^{*\delta}(G))_{i\in N, G\in\mathcal{G}}$ if and only if it is equitable.*

8. Buyer and Seller Networks

Suppose that $N = B \cup S$, where B and S denote the sets of buyers and sellers, respectively. Each seller owns one unit of a homogeneous indivisible good. Each buyer demands one unit of the good. The utilities of buyers and sellers for the good are normalized to 1 and 0, respectively. In a **buyer-seller network** every link is between a buyer and a seller, i.e., $i \in B \iff j \in S$ for every link ij. Fix a buyer-seller network G. Only buyer-seller pairs connected in G can engage in exchange. With this interpretation, pairs of agents connected in G can generate a unit surplus, as in the benchmark model. Buyers and sellers have a common discount factor δ, and play the bargaining game Γ^δ on the network G. Since buyer-seller networks form a special class of networks, all results of the paper apply, and some refinements are possible.

The bipartite nature of buyer-seller networks permits a more straightforward description of the bounds and the accompanying procedure for computing limit equilibrium payoffs. The restatements of Theorems 3 and 4 specialized to buyer-seller networks are based on the following observations. Fix a buyer-seller network H and a subset of players $K = M \cup L$ ($M \subset B, L \subset S$). Then $L^H(K) = L^H(M) \cup L^H(L)$ with $L^H(M) \subset S$ and $L^H(L) \subset B$. The set K is H-independent if there exists no H-link between buyers in M and sellers in L, or $L^H(M) \cap L = \emptyset$ (or $L^H(L) \cap M = \emptyset$). M and L are H-independent.

Theorem $3^{\mathbf{BS}}$. *For every set of buyers M, the following bounds on limit equilibrium payoffs hold*

$$\begin{aligned} \min_{i \in M} v_i^* &\leq \frac{|L^{G^*}(M)|}{|M| + |L^{G^*}(M)|} \\ \max_{j \in L^{G^*}(M)} v_j^* &\geq \frac{|M|}{|M| + |L^{G^*}(M)|}. \end{aligned}$$

If we restrict attention to sets of buyers M that minimize $|L^H(M)|/|M|$, Lemma 5 can be extended to show that the set of such minimizers is closed with respect to unions without the assumption that the attained minimum is strictly less than 1.

Lemma $5^{\mathbf{BS}}$. *Let H be a buyer-seller network. Suppose that*

$$M', M'' \in \arg\min_{M \subset B} \frac{|L^H(M)|}{|M|}.$$

Then

$$M' \cup M'' \in \arg\min_{M \subset B} \frac{|L^H(M)|}{|M|}.$$

In view of Theorem 3^{BS} and Lemma 5^{BS}, the algorithm computing limit equilibrium payoffs does not have to treat $r_s \geq 1$ as a stopping condition at step s if the focus is on sets of buyers M that minimize $|L^{G_s}(M)|/|M|$. While the algorithm $\mathcal{A}(G)$ is effective for buyer-seller networks, the adapted version $\mathcal{A}^{BS}(G)$ offers a simplified procedure.

Definition 4 (Algorithm $\mathcal{A}^{BS}(G) = (r_s, x_s, B_s, S_s, N_s, G_s)_{s=1,2,\ldots,\overline{s}}$)**.** Define the sequence $(r_s, x_s, B_s, S_s, N_s, G_s)_s$ recursively as follows. Set $N_1 = N$ and $G_1 = G$. For $s \geq 1$, let

$$r_s = \min_{M \subset N_s \cap B} \frac{|L^{G_s}(M)|}{|M|}. \tag{8.1}$$

Set $x_s = r_s/(1+r_s)$. Let B_s be the union of all minimizers M in 8.1.[30] Denote $S_s = L^{G_s}(B_s)$. If $N_s = B_s \cup S_s$ then **stop**. Else, set $N_{s+1} = N_s \setminus (B_s \cup S_s)$, and let G_{s+1} be the subnetwork of G induced by the players in N_{s+1}. Denote by $\overline{s}$ the finite step at which the algorithm stops.

Proposition $3^{\mathbf{BS}}$. *The sequences $(r_s)_s$ and $(x_s)_s$ defined by $\mathcal{A}^{BS}(G)$ are strictly increasing.*

Note that the sets $B_1, S_1, \ldots, B_{\overline{s}}, S_{\overline{s}}$ partition N.

[30]By Lemma 5^{BS}, B_s is also a minimizer in 8.1.

Theorem $4^{\mathbf{BS}}$. *Let $(r_s, x_s, B_s, S_s, N_s, G_s)_{s=1,2,\ldots,\overline{s}}$ be the outcome of the algorithm $\mathcal{A}^{BS}(G)$ for the buyer-seller network G. The limit equilibrium payoffs for Γ^δ as $\delta \to 1$ are given by*

$$\begin{aligned} v_i^* &= x_s, \forall i \in B_s, \forall s \leq \overline{s} \\ v_j^* &= 1 - x_s, \forall j \in S_s, \forall s \leq \overline{s}. \end{aligned}$$

Similarly to the study of equitable networks for the general model, we are interested in characterizing the class of **non-discriminatory** networks in the bipartite case. A buyer-seller network is non-discriminatory if the limit equilibrium payoffs of all buyers are identical. By Proposition 3^{BS} and Theorem 4^{BS}, G is non-discriminatory if and only if the algorithm $\mathcal{A}^{BS}(G)$ stops at the first step, hence $B_1 = B, S_1 = S, r_1 = |S|/|B|$. Let β denote the **buyer-seller ratio**, $\beta = |B|/|S|$. In a non-discriminatory network the common limit equilibrium payoffs of all buyers and all sellers are $1/(\beta+1)$ and $\beta/(\beta+1)$, respectively. Hence the limit equilibrium price is $\beta/(\beta+1)$.

Suppose that the buyer-seller ratio β is an integer. A **β-buyer-seller cluster** is a network formed by a seller connected to β buyers. A **β-buyer-seller cluster disjoint union** is a network that is a union of vertex-disjoint β-buyer-seller clusters. Set $B = \{4, 5, 6, 7, 8, 9\}$ and $S = \{1, 2, 3\}$ for the network G_3 from Example 2. The buyer-seller ratio is 2, and the network can be covered by the disjoint union of three 2-buyer-seller clusters, $\{(1, 4), (1, 5)\} \cup \{(2, 6), (2, 7)\} \cup \{(3, 8), (3, 9)\}$. The network G_3 is non-discriminatory, with a limit equilibrium price of 2/3. The intuition is that each seller enjoys a distinct 2-buyer base, so there is no differentiation in bargaining power across sellers. This observation can be generalized.

Theorem 7. *Suppose that the buyer-seller ratio β is an integer. A buyer-seller network is non-discriminatory if and only if it can be covered by a β-buyer-seller cluster disjoint union.*

Proof. Let G be a buyer-seller network. By Proposition 3^{BS} and Theorem 4^{BS}, G is non-discriminatory if and only if $B_1 = B, S_1 = S, r_1 = 1/\beta$ if and only if $|L^G(M)|/|M| \geq 1/\beta$ for all $M \subset B$.

Recall the definitions from Section 6. A network H' is a **perfect match** of a network H if H' is a match that covers H. Let H be the graph obtained from G by replacing each vertex corresponding to a seller with β identical copies (each copy is connected to all buyers whom the corresponding seller was connected to). Note that $|L^G(M)|/|M| \geq 1/\beta, \forall M \subset B$ is equivalent to $|L^H(M)|/|M| \geq 1, \forall M \subset B$. Since the numbers of buyers and sellers in H are equal, Hall (1935)'s theorem implies that the latter condition is equivalent to H having a perfect match. By construction, H has a perfect match if and only if G can be covered by a β-buyer-seller cluster disjoint union, which completes the proof. □

Corollary 3. *A buyer-seller network is equitable if and only if it has a perfect match.*

Remark 2. Corollary 3 also follows from Theorem 5 since bipartite networks contain no odd cycles, so any match and odd cycle disjoint union that covers such a network must be a match.

Remark 3. A symmetric set of results obtains if we interchange the roles of buyers and sellers in Theorems 4^{BS} and 7.

In the context of buyer-seller networks the definition of pairwise stability should account for the fact that only buyer-seller pairs may consider forming new links.

Definition 5 (Buyer-seller stability)**.** A buyer-seller network G is **two-sided pairwise stable** with respect to the payoff function u if it is unilaterally stable with respect to u, and for all $(i,j) \in (B \times S) \cup (S \times B)$, $u_i(G + ij) > u_i(G)$ only if $u_j(G + ij) < u_j(G)$.

The next result is the analogue of Theorem 6.ii.

Theorem 6.ii$^{\mathbf{BS}}$. *A buyer-seller network is two-sided pairwise stable with respect to $(v_i^*(G))_{i \in N, G \in \mathcal{G}}$ if and only if it is non-discriminatory.*

9. Robustness of the Results

9.1. **Heterogeneous discount factors.** All players are assumed to have a common discount factor hitherto. We can extend the results to the case of heterogeneous discount factors, where the copies of player i share a discount factor δ_i. The accumulation points of the equilibrium payoffs and agreement networks along a $(\delta_1, \delta_2, \ldots, \delta_n)$ sequence that converges to $(1, 1, \ldots, 1)$ depend on the choice of the sequence.[31] One condition that guarantees

[31]For example, in the game for the two player network, if discount factors are given by the pair (δ^a, δ^b) $(a, b > 0)$, then as $\delta \to 1$ the limit equilibrium payoffs are $(b/(a+b), a/(a+b))$. For different choices of (a, b)

convergence of the equilibrium payoffs and agreement network is that the relative rates of convergence of δ_i and δ_j to 1 be constant along the sequence of discount factors. That is, there exists $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ such that $\delta_i = \delta^{\alpha_i}$ for $\delta \in (0,1)$. Denote by $\Gamma^{\delta,\alpha}$ the bargaining game described in Section 2 with payoffs modified by the assumption that i_τ has discount factor δ^{α_i} for all $i \in N$ and $\tau \geq 0$.

For a fixed α, we are interested in the asymptotic equilibrium behavior in $\Gamma^{\delta,\alpha}$ as $\delta \to 1$. Theorems 1 and 2 generalize verbatim. The notation for $\Gamma^\delta, v^{*\delta}, \underline{\delta}, G^{*\delta}, G^*, v^*, \ldots$ needs to be replaced by $\Gamma^{\delta,\alpha}, v^{*\delta}(\alpha), \underline{\delta}(\alpha), G^{*\delta}(\alpha), G^*(\alpha), v^*(\alpha), \ldots$ to reflect the dependence of the variables on α.

Remark 4. For a subnetwork H of G, the analogue of the linear system 3.4 used in the proofs of Proposition 1 and Theorem 2 is

$$v_i = \frac{2e - e_i^H}{2e}\delta^{\alpha_i} v_i + \frac{1}{2e}\sum_{\{j|ij\in H\}}(1 - \delta^{\alpha_j} v_j), \forall i = \overline{1,n}.$$

As in the proof of Proposition 1 the unique solution $v_i^{\delta,H}(\alpha)$ is given by Cramer's rule, as the ratio of two determinants that are finite sums of positive real powers (which are not necessarily polynomials) of δ. In order to show that for fixed i, j, H there exists a finite number of δ solving the equation $\delta(v_i^{\delta,H}(\alpha) + v_j^{\delta,H}(\alpha)) = 1$ we invoke a result due to Laguerre (1883). The result extends Descartes' rule of signs, which provides a bound for the number of positive real roots of polynomials in δ, to the case of linear combinations of powers of δ. A corollary of Laguerre's result is that every finite linear combination of positive powers of δ which does not vanish everywhere has a finite number of solutions.[32]

the limit equilibrium payoffs for the corresponding sequence of discount factors vary accordingly. For the sequence of discount factors indexed by n given by $(1-1/n, 1-1/n)$ for odd n and $(1-1/n, (1-1/n)^2)$ for even n, the set of equilibrium payoffs has two accumulation points, $(1/2, 1/2)$ and $(2/3, 1/3)$. Similarly, for more complicated network structures, the limit equilibrium agreement network depends on the sequence of discount factors, and convergence does not always obtain.

[32]If all components of α are rational numbers we can avoid non-polynomial functions by using the substation $\delta \to \delta^c$, where c is the least common multiple of the denominators of $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Theorem 3$^\alpha$. *For every $G^*(\alpha)$-independent set M, the following bounds on limit equilibrium payoffs hold*

$$\min_{i \in M} v_i^*(\alpha) \leq \frac{\sum_{j \in L^{G^*(\alpha)}(M)} \alpha_j}{\sum_{i \in M} \alpha_i + \sum_{j \in L^{G^*(\alpha)}(M)} \alpha_j}$$
$$\max_{j \in L^{G^*(\alpha)}(M)} v_j^*(\alpha) \geq \frac{\sum_{i \in M} \alpha_i}{\sum_{i \in M} \alpha_i + \sum_{j \in L^{G^*(\alpha)}(M)} \alpha_j}.$$

Proof. We follow analogous steps to the proof of Theorem 3. The only innovation is that we rewrite the conclusion of Lemma 4 as

$$\frac{1-\delta^{\alpha_i}}{1-\delta} v_i^{*\delta}(\alpha) = \frac{1}{1-\delta} \sum_{\{j | ij \in G\}} \frac{1}{2e} \max(1 - \delta^{\alpha_i} v_i^{*\delta}(\alpha) - \delta^{\alpha_j} v_j^{*\delta}(\alpha), 0), \forall i \in N, \forall \delta \in (0,1).$$

Then

$$\sum_{j \in L} \frac{1-\delta^{\alpha_j}}{1-\delta} v_j^{*\delta}(\alpha) \geq \sum_{i \in M} \frac{1-\delta^{\alpha_i}}{1-\delta} v_i^{*\delta}(\alpha),$$

which after taking the limit $\delta \to 1$ becomes

$$\sum_{j \in L} \alpha_j v_j^*(\alpha) \geq \sum_{i \in M} \alpha_i v_i^*(\alpha).$$

The conclusion is reached as in the proof of Theorem 3. □

If we replace formula 5.1 in the algorithm $\mathcal{A}(G)$ with

$$r_s(\alpha) = \min_{M \subset N_s,\ M \in \mathcal{I}(G)} \frac{\sum_{j \in L^{G^*(\alpha)}(M)} \alpha_j}{\sum_{i \in M} \alpha_i},$$

and leave the definitions of the other variables unchanged, the new procedure delivers the limit equilibrium payoffs of $\Gamma^{\delta,\alpha}$ when $\delta \to 1$ as detailed in Theorem 4.[33]

9.2. **General matching technologies.** In the benchmark model of Section 2 it is assumed that every pair of connected players is equally likely to be matched to bargain at every round, and each of the two matched players is equally likely to be the proposer. Here we attempt to relax these assumptions. Let $(p(i \to j) > 0)_{ij \in G}$ be a probability distribution over the oriented links of G. Denote by $\Gamma^{\delta,\alpha}$ the bargaining game described in Section 2 with the moves by nature modified so that every period player i is chosen to make an offer to player j with probability $p(i \to j)$. Hence, each link ij is selected for bargaining with probability $p(i \to j) + p(j \to i)$, and conditional on the selection, i is the proposer with probability

[33]The extension of the proofs of Lemma 5 and Theorem 4 virtually consists in replacing everywhere the cardinality set operator $|\cdot|$ by the α-weight set operator $|\cdot|_\alpha$, defined by $|M|_\alpha = \sum_{i \in M} \alpha_i$ for every $M \subset N$.

$p(i \to j)/(p(i \to j) + p(j \to i))$. The uniform distribution p_u defining the benchmark game Γ^δ is given by $p_u(i \to j) = 1/(2e), \forall ij \in G$.

Fix p. We are interested in the asymptotic equilibrium behavior in $\Gamma^{\delta,p}$ as $\delta \to 1$. As in the previous subsection, Theorems 1 and 2 generalize with $\Gamma^\delta, v^{*\delta}, \underline{\delta}, G^{*\delta}, G^*, v^*, \ldots$ replaced by $\Gamma^{\delta,p}, v^{*\delta}(p), \underline{\delta}(p), G^{*\delta}(p), G^*(p), v^*(p), \ldots$

The observation that for every discount factor the equilibrium payoff of any player is equal to the expected present value of his stream of first mover advantage, which is the first ingredient for the proof of Theorem 3, generalizes to $\Gamma^{\delta,p}$. Next suppose that conditional on the selection of any link, each of the two players is equally likely to become the proposer, i.e., $p(i \to j) = p(j \to i)$ for all $ij \in G$. Then the second ingredient for the proof of Theorem 3 also generalizes to $\Gamma^{\delta,p}$. Any net first mover advantage that a player i gains from making an offer to a player j is mapped to an equal net first mover advantage that j gains from making an offer to i, and both gains are weighted by the probability $p(i \to j) = p(j \to i)$ in the expected payoffs of i and j. Therefore, the sum of the expected present values of the streams of first mover advantage enjoyed by all players in a mutually estranged set is not larger than the same expression evaluated for the corresponding partner set.

Proposition 4. *If $p(i \to j) = p(j \to i)$ for all $ij \in G$ then Theorems 3 and 4 extend without change. As $\delta \to 1$, the limit equilibrium payoffs of $\Gamma^{\delta,p}$ are identical to those of $\Gamma^{\delta,p_u} = \Gamma^\delta$.*

Remark 5. Suppose that G is the **complete network**,[34] and that $p(i \to j) = \pi_i \pi_j, \forall i \neq j \in N$ for some vector π describing relative matching frequencies ($\pi_i > 0, \forall i \in N$). In this setting, Proposition 4 implies that the limit equilibrium payoff of each player is 1/2, independently of π. While players with higher matching frequencies obtain larger equilibrium payoffs for any $\delta < 1$, all equilibrium payoffs converge to 1/2 as δ goes to 1. The intuition is that patient players can postpone agreement until matched to bargain with players that have low matching frequency (each match occurs with positive probability every period).

Another case where the results extend easily is the setting of buyer-seller networks where there exists $q > 0$ such that $p(j \to i) = qp(i \to j)$ for all $ij \in G, i \in B, j \in S$ (B and S denote the sets of buyers and sellers, respectively). In every pairwise matching, the seller is q times more likely than the buyer to be the proposer. Consider the sequence $(r_s, x_s, B_s, S_s, N_s, G_s)_s$

[34]The complete network includes every link ij with $i \neq j \in N$.

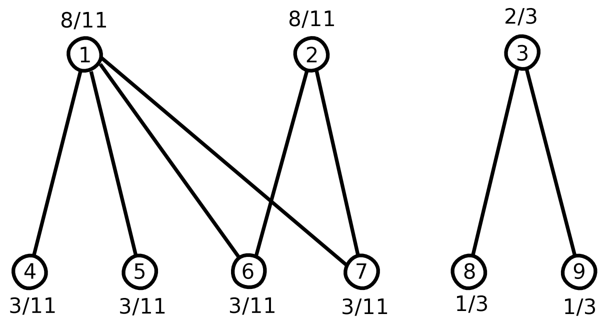

FIGURE 7. Network $G_3^*(p)$

generated by the algorithm $\mathcal{A}^{BS}(G)$, with the variable x_s redefined as $r_s/(q+r_s)$ for all s and the other variables left intact. The sequence delivers the limit equilibrium payoffs of $\Gamma^{\delta,p}$ when $\delta \to 1$ as detailed in Theorem 4.

The conclusions of Theorems 3 and 4 do not immediately extend to more general matching technologies. The following example illustrates some of the difficulties.

Example 4. Consider again the network G_3 with 9 players illustrated in Figure 3. Let p be the probability distribution of moves by nature induced as follows. Every link is selected with equal probability, and for each selection except $(2,6)$ and $(2,7)$, each of the two matched players is equally likely to be the proposer. If the link $(2,6)$ ($(2,7)$) is selected, then player 2 is twice as likely as player 6 (7) to be the proposer. Mathematically, $p(2 \to 6) = p(2 \to 7) = 1/18, p(6 \to 2) = p(7 \to 2) = 1/36$ and $p(i \to j) = 1/24$ for all other links ij in G_3. We similarly define the probability distribution p' to give player 2 asymmetric bargaining power in encounters with players 8 and 9, by $p'(2 \to 8) = p'(2 \to 9) = 1/18, p'(8 \to 2) = p'(9 \to 2) = 1/36$ and $p'(i \to j) = 1/24$ for all other links ij in G_3.

Consider first the game $\Gamma^{\delta,p}$. By arguments similar to those from Example 2, we obtain that the limit equilibrium agreement network is the subnetwork $G_3^*(p)$ illustrated in Figure 7. The limit equilibrium payoffs are $v_1^*(p) = v_2^*(p) = 8/11, v_3^*(p) = 2/3, v_4^*(p) = v_5^*(p) = v_6^*(p) = v_7^*(p) = 3/11$ and $v_8^*(p) = v_9^*(p) = 1/3$. The intuition is that player 2 can extort players 6 and 7 for more than $2/3$, since he enjoys increased bargaining power in pairwise interactions with each of these players. Players 6 and 7 have to reach agreements when matched to bargain with 1, since they would receive limit equilibrium payoffs of at most $1/5$ if they were monopolized by 2. Then player 1 will be able to take advantage of the weakness of 6 and 7, and also reach equally favorable agreements with 4 and 5. Player 1 can extort

4 and 5 because these two plyers do not have other bargaining partners. In the limit, since players 1 and 2 reach agreements on very favorable terms with 4, 5, 6 and 7, they are not attractive bargaining partners for 8 and 9. Players 8 and 9 have monopsony power over 3, and thus can secure limit equilibrium payoffs of 1/3. Hence, as players become patient, 8 and 9 do not have incentives to reach agreements with 1 or 2.

Consider next the game $\Gamma^{\delta,p'}$. The limit equilibrium agreement network is identical to G_3^* (consisting of the set of thick links in Fugure 3). The limit equilibrium payoffs are $v_1^* = v_2^* = v_3^* = 2/3$ and $v_4^* = v_5^* = v_6^* = v_7^* = v_8^* = v_9^* = 1/3$. Player 2 cannot use his stronger bargaining power in pairwise interactions with 8 and 9 to obtain a limit equilibrium payoff larger than 2/3. The intuition is that 8 and 9 can secure limit equilibrium payoffs of 1/3 in pairwise agreements with 3, since they constitute the only bargaining partners for 3. Hence 8 and 9 cannot be pressured to surrender more than 2/3 to player 2 in the limit, despite their relatively smaller chance of proposing when matched to bargain with 2. Equilibrium agreements do not arise across the links $(1, 8)$ and $(1, 9)$ when players are sufficiently patient for the reasons outlined in Example 2.

9.2.1. *More than one match per period.* The results are not sensitive to the assumption that only one pair of players is matched to bargain every period. Consider the following more general matching technology. Suppose that every period nature matches a set (possibly varying in cardinality) of disjoint pairs of linked players. All pairs matched in a given period bargain simultaneously. We assume that the distribution over matchings is stationary, and that each link is selected with positive probability. The preliminary results on essential equilibrium uniqueness and convergence as players become patient extend to this setting. Suppose further that for every matched pair, each of the two players is equally likely to be the proposer. Then, by an argument similar to the one of Proposition 4, the limit equilibrium payoffs of the bargaining game with the matching technology sketched out here are identical to those of the benchmark game.

10. Related Literature

One major study of decentralized trade is Rubinstein and Wolinsky (1985). The paper considers a market where each seller owns one unit of a homogeneous indivisible good, and each buyer demands one unit of the good. The values of buyers and sellers for the good are

1 and 0, respectively. At every round each buyer (seller) is matched to a new seller (buyer) with probability α_b (α_s). For every buyer-seller match, each of the two agents is selected with equal probability to make a price offer. If the offer is accepted then the two agents trade and leave the market. It is assumed that the market is at a steady state, where the numbers of buyers and sellers are constant over time. An important result is that as players become patient, the equilibrium payoffs of each buyer and each seller converge to $\alpha_b/(\alpha_s + \alpha_b)$ and $\alpha_s/(\alpha_s + \alpha_b)$, respectively.

Rubinstein and Wolinsky endogenize the variables α_b and α_s by means of the following matching technology. Suppose there are n_b buyers and n_s sellers. Each period m matchings are chosen with equal probability among the sets of m disjoint buyer-seller pairs. This framework is a special case of our model (see Subsection 9.2.1), whereby the network is a bipartite complete graph with n_b buyers and n_s sellers. If n_b and n_s are large,[35] the matching technology leads to $\alpha_b \approx m/n_b$ and $\alpha_s \approx m/n_s$. Then the limit equilibrium payoffs can be rewritten as $n_s/(n_b + n_s)$ and $n_b/(n_b + n_s)$, respectively. Hence, as players become patient, the unit surplus is split between buyers and sellers according to the *shortage ratio*, n_s/n_b, describing the relative strength of the group of buyers. Our analysis develops the idea that, in stationary environments, the shortage ratio of every mutually estranged group plays a key role in the prediction of bargaining power.

Gale (1987), Binmore and Herrero (1988a), and Rubinstein and Wolinsky (1990) further study the relationship between the equilibrium outcomes of various decentralized bargaining procedures and the competitive equilibrium price as the costs of search and delay become negligible. The findings are that the relationship is sensitive to the assumptions on the flow of agents entering the market over time, the amount of information available to agents, the matching technology, and the discount factor.[36] All buyers and respectively all sellers are treated anonymously by the various stochastic matching processes considered in this literature. The analogue of this modeling assumption in our setting is the special case of buyer-seller networks in which every buyer is connected to every seller. As argued above,

[35]Rubinstein and Wolinsky assume that every pair of matched agents continues to bargain until one of them is matched to a new partner. The large market condition implies that it is improbable that an agent will be re-matched with his current partner.

[36]Osborne and Rubinstein (1990) and Binmore et al. (1992) are excellent surveys of the literature on bargaining in markets.

for such networks our conclusions coincide with those of Rubinstein and Wolinsky (1985). In other networks some pairs of buyers and sellers are not connected and cannot engage in exchange. Since bargaining encounters are restricted by network connections, the competitive equilibrium analysis does not apply.

In Abreu and Manea (2008) we drop the stationarity assumption, and analyze the situation in which players that reach agreements are removed from the network without replacement. The bargaining protocol is identical to the one of the present paper. In that setting the production technology determines a maximum total surplus that may be generated cooperatively in every network. A family of equilibria is asymptotically efficient if the welfare they induce approaches the maximum total surplus as players become patient. To achieve efficiency some pairs of connected players need to refrain from reaching agreements in various subgames. For many networks, all Markov perfect equilibria are asymptotically inefficient. The main result is that asymptotically efficient subgame perfect equilibria exist for every network. Our equilibrium construction entails non-Markovian behavior as in the dynamic games literature. Players who resist the temptation of reaching inefficient agreements are rewarded by their neighbors, and players who do not conform to the rewarding procedure are punished via the threat of inefficient agreements that result in their isolation.

The latter paper also studies properties of Markov perfect equilibria (MPE). We establish existence of MPEs, and show that MPE payoffs are not necessarily unique. We provide a method to construct pure strategy MPEs for high discount factors based on conjectures about the set of links across which agreement may obtain in every subnetwork.

The two models differ in strategic complexity. In the present model bargaining opportunities are stationary over time. A player's decisions consist *solely* in determining who his most favorable bargaining partners are. In effect, each player solves a search problem with prizes endogenously and simultaneously determined by the network structure. In the model of Abreu and Manea (2008) a player's decisions *additionally* entail anticipating that passing up bargaining opportunities may lead to agreements involving other players which undermine or enhance his position in the network in future bargaining encounters. Technically, this means that we need to compute equilibrium payoffs for every subnetwork that may arise following a series of agreements. Clearly, the selection between the two models depends on the environment under investigation. In some markets bargaining opportunities are stationary, while in others bargaining opportunities decline over time.

Polanski (2007) studies a related model. The differences regard the matching technology and the steady state assumption. Polanski assumes that a maximum number of pairs of connected players are selected to bargain every period, and all players who reach agreement are removed from the network without replacement. Corominas-Bosch (2004) considers a model in which buyers and sellers alternate in making public offers that may be accepted by any of the responders connected to a specific proposer. As in the previous paper, the matching technology is chosen such that when there are multiple possibilities to match buyers and sellers (that is, there are multiple agents proposing or accepting identical prices) the maximum number of transactions take place. The efficient matching technologies of the preceding papers are fundamentally centralized. The market forces that would organize the matchings in order to maximize total surplus are not explicitly modeled by way of self-interested strategic behavior.

Centralized trading mechanisms may be employed in order to implement efficient matching outcomes. Kranton and Minehart (2001) study a model similar to the one of Corominas-Bosch (2004). The valuations of the buyers are heterogeneous, and sellers are non-strategic. Prices are determined by the ascending-bid auction mechanism designed by Demange, Gale and Sotomayor (1986). The unique equilibrium in weakly undominated strategies leads to an efficient allocation of the goods. The result makes possible the study of the relationship between overall economic welfare and the incentives of buyers to form the network when links are costly to maintain.

Calvo-Armengol (2001, 2003) introduces the following model of sequential bargaining on a network. In the first round a proposer is randomly selected, and a responder is randomly chosen among his neighbors in the network. Every round that ends in disagreement is followed by a new round where the disagreeing responder makes an offer to a randomly chosen neighbor. The game ends when the first agreement is obtained. The unique stationary subgame perfect equilibrium specifies offers and responses identical to those in the two-person Rubinstein (1982) game. Therefore, the network has no effect on the ex-post equilibrium division, and only influences payoffs via the probabilities that an agent is the proposer or the responder in the first round of the game. These probabilities are defined exogenously in the model. Furthermore, it is assumed that a proposer passes down the bargaining opportunity to the responder in case of disagreement and that the game ends as soon as one pair reaches an agreement.

Formal models of non-cooperative two-player bargaining were introduced in the pioneering work of Stahl (1972), Rubinstein (1982) and Binmore (1987). Pairwise stability is defined by Jackson and Wolinsky (1996); Jackson (2005) surveys the extensive related literature.

11. Conclusion

Networks are important in many economic and social interactions. In our setting networks represent patterns of trading opportunities. The network structure affects the set of feasible agreements, the division of surplus, and the relative bargaining strengths. Previous studies of surplus division in networks mainly focused on allocation rules defined by exogenous or cooperative, rather than non-cooperative, procedures. Some papers discussed in the literature review analyze non-cooperative division of surplus in networks under centralized mechanisms. Yet only a few papers explore the complex strategic issues that arise in bargaining over surplus in networks with decentralized, bilateral matching. The models of non-cooperative decentralized bargaining in networks of the present paper and of Abreu and Manea (2008) constitute initial endeavors in that exploration.

The model introduced here is well-behaved in that equilibria are essentially unique and converge as players become patient. The main result of the paper is the characterization of the limit equilibrium payoffs by iterative use of the finding that players with extreme limit equilibrium payoffs form an oligopoly subnetwork corresponding to the largest mutually estranged set that minimizes the shortage ratio. The result facilitates the analysis of equitable networks, stable networks, and non-discriminatory buyer-seller networks. The ideas of mutually estranged sets and minimal shortage ratios, along with induced oligopoly subnetworks, provide insights into the relative strengths of the positions in a network. The limit equilibrium payoffs deliver an index of bargaining power in networks.

Appendix A. Proofs

Proof of Lemma 1. Suppose $w_1 = \max(w_1, w_2, w_3, w_4)$. Then

$$|\max(w_1, w_2) - \max(w_3, w_4)| = w_1 - \max(w_3, w_4) \leq w_1 - w_3 \leq \max(|w_1 - w_3|, |w_2 - w_4|).$$

The proof is similar for the cases when w_2, w_3, or w_4 is equal to $\max(w_1, w_2, w_3, w_4)$. □

Proof of Lemma 2. Let σ be a steady state stationary equilibrium of Γ^δ. Under σ, for every $i \in N$, each player i_τ receives the same expected payoff, $\tilde{v}_i$, in any $\mathcal{H}(i_\tau)$ subgame. In equilibrium, it must be that each player j accepts any offer larger than $\delta\tilde{v}_j$, and rejects any offer smaller than $\delta\tilde{v}_j$. If $ij \in G$ and $\delta(\tilde{v}_i + \tilde{v}_j) < 1$, when i is selected to propose to j, he offers $\delta\tilde{v}_j$ and j accepts with probability 1. Similarly, if $ij \in G$ and $\delta(\tilde{v}_i + \tilde{v}_j) > 1$, when i is selected to propose to j, he makes an offer that j rejects with probability 1.

By the analysis above, the payoffs under σ solve the system

$$\tilde{v}_i = \frac{2e - e_i}{2e}\delta\tilde{v}_i + \frac{1}{2e}\sum_{\{j|ij\in G\}} \max(1 - \delta\tilde{v}_j, \delta\tilde{v}_i), \forall i \in N$$

(e and e_i are defined in Section 3).

Therefore, $\tilde{v} = (\tilde{v}_i)_{i\in N}$ is a fixed point of the function $f^\delta = (f_1^\delta, f_2^\delta, \ldots, f_n^\delta) : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$f_i^\delta(v) = \frac{2e - e_i}{2e}\delta v_i + \frac{1}{2e}\sum_{\{j|ij\in G\}} \max(1 - \delta v_j, \delta v_i).$$

Note that f^δ maps $[0,1]^n$ into itself.

We show that f^δ is a contraction with respect to the sup norm $||\cdot||_\infty$ on $\mathbb{R}^n$, defined by $||z||_\infty = \max_{i=\overline{1,n}} |z_i|$. Specifically,

$$||f^\delta(v) - f^\delta(u)||_\infty \leq \delta||v - u||_\infty, \forall v, u \in [0,1]^n.$$

We need to prove that for each i, $|f_i^\delta(v) - f_i^\delta(u)| \leq \delta||v - u||_\infty$.

By f^δ's definition,

$$\begin{aligned}
|f_i^\delta(v) - f_i^\delta(u)| &\leq \frac{2e - e_i}{2e}\delta|v_i - u_i| + \frac{1}{2e}\sum_{\{j|ij\in G\}} |\max(1 - \delta v_j, \delta v_i) - \max(1 - \delta u_j, \delta u_i)| \\
&\leq \frac{2e - e_i}{2e}\delta|v_i - u_i| + \frac{1}{2e}\sum_{\{j|ij\in G\}} \max(|1 - \delta v_j - (1 - \delta u_j)|, |\delta v_i - \delta u_i|) \\
&= \frac{2e - e_i}{2e}\delta|v_i - u_i| + \frac{1}{2e}\sum_{\{j|ij\in G\}} \delta\max(|v_j - u_j|, |v_i - u_i|) \\
&\leq \frac{2e - e_i}{2e}\delta||v - u||_\infty + \frac{1}{2e}\sum_{\{j|ij\in G\}} \delta||v - u||_\infty \\
&= \delta||v - u||_\infty,
\end{aligned}$$

where the second inequality follows from Lemma 1, and the others from algebraic manipulation and the definition of the sup norm.

Because f^δ is a contraction, it has exactly one fixed point, denoted $\tilde{v}^{*\delta}$. Since $f^\delta([0,1]^n) \subset [0,1]^n$, it follows that $\tilde{v}^{*\delta} \in [0,1]^n$. Steady state stationary equilibrium payoffs of Γ^δ need to be fixed points for f^δ, hence they are uniquely given by $\tilde{v}^{*\delta}$. □

Proof of Proposition 1. By definition, $ij \in G^{*\delta} \iff ij \in G \;\&\; \max(1-\delta v_j^{*\delta}, \delta v_i^{*\delta}) = 1-\delta v_j^{*\delta}$. Since $v^{*\delta}$ is a fixed point of f^δ, $v^{*\delta}$ solves the $n \times n$ linear system

$$v_i = \frac{2e - e_i^{G^{*\delta}}}{2e}\delta v_i + \frac{1}{2e}\sum_{\{j|ij\in G^{*\delta}\}}(1-\delta v_j), \forall i = \overline{1,n}.$$

For every $\delta \in (0,1)$ and every non-empty subnetwork H of G, consider more generally the $n \times n$ linear system

$$v_i = \frac{2e - e_i^{H}}{2e}\delta v_i + \frac{1}{2e}\sum_{\{j|ij\in H\}}(1-\delta v_j), \forall i = \overline{1,n} \tag{A.1}$$

(e and e_i^H are defined in Section 3). We argue below that the system A.1 has a unique solution $v^{\delta,H}$, and the solution belongs to $[0,1]^n$. In particular, the system A.1 is non-singular and $v^{*\delta} = v^{\delta,G^{*\delta}}$.

The simplest path to show uniqueness of the solution to A.1 is analytical rather than linear algebraic, by proving that the function $h^{\delta,H} : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$h_i^{\delta,H}(v) = \frac{2e - e_i^{H}}{2e}\delta v_i + \frac{1}{2e}\sum_{\{j|ij\in H\}}(1-\delta v_j), \forall i = \overline{1,n}$$

is a contraction with respect to the sup norm on $\mathbb{R}^n$. The proof is omitted as it is very similar (but simpler, since it does not involve Lemma 1) to the proof that f^δ is a contraction in Lemma 2.

All entries in the augmented matrix of the linear system A.1 are linear functions of δ. Then for each $i \in N$ the solution $v_i^{\delta,H}$ is given by Cramer's rule, as the ratio of two determinants that are polynomials in δ of degree at most n,

$$v_i^{\delta,H} = P_i^H(\delta)/Q_i^H(\delta). \tag{A.2}$$

$Q_i^H(\delta) \neq 0$ for all $\delta \in (0,1)$ and all non-empty subnetworks H of G as the corresponding system A.1 is non-singular.

Let $\bar{\Delta}$ be the set of δ for which there exist i, j, H with $\delta(v_i^{\delta,H} + v_j^{\delta,H}) = 1$. Fix i, j, H. The equation $\delta(v_i^{\delta,H} + v_j^{\delta,H}) = 1$ is equivalent to

$$1 = \delta(v_i^{\delta,H} + v_j^{\delta,H}) = \delta(P_i^H(\delta)/Q_i^H(\delta) + P_j^H(\delta)/Q_j^H(\delta)).$$

If the equation above has an infinite number of solutions δ, it follows that

$$\delta(P_i^H(\delta)Q_j^H(\delta) + P_j^H(\delta)Q_i^H(\delta)) = Q_i^H(\delta)Q_j^H(\delta)$$

is a polynomial identity, i.e., it holds for all δ. In particular, equality needs to hold in the two display equations above for $\delta = 1/3$, implying that $1/3(v_i^{1/3,H} + v_j^{1/3,H}) = 1$. Hence $v_i^{1/3,H} + v_j^{1/3,H} = 3$, which is a contradiction with $v^{1/3,H} \in [0,1]^n$.

Since for every triplet (i, j, H) the equation $\delta(v_i^{\delta,H} + v_j^{\delta,H}) = 1$ has a finite number of solutions δ, and the number of such triplets is finite, it follows that the set $\bar{\Delta}$ is finite. The equality $v^{*\delta} = v^{\delta,G^{*\delta}}$ implies that the set of δ for which there exist i, j s.t. $\delta(v_i^{*\delta} + v_j^{*\delta}) = 1$ is included in $\bar{\Delta}$. □

Proof of Proposition 2. Let $ij \in G$. If $ij \in G \setminus G^*$, then for all $\delta > \underline{\delta}$,

$$\delta(v_i^{*\delta} + v_j^{*\delta}) > 1. \tag{A.3}$$

If $ij \in G^*$, then for all $\delta > \underline{\delta}$,

$$v_i^{*\delta} = \frac{2e - e_i^{G^*}}{2e}\delta v_i^{*\delta} + \frac{1}{2e}\sum_{\{k|ik\in G^*\}}(1 - \delta v_k^{*\delta}) \geq \frac{2e-1}{2e}\delta v_i^{*\delta} + \frac{1}{2e}(1 - \delta v_j^{*\delta}), \tag{A.4}$$

since $1 - \delta v_k^{*\delta} \geq \delta v_i^{*\delta}$ for all $k \neq j$ such that $ik \in G^*$. Taking the limit as δ goes to 1 in either A.3 or A.4 we obtain that $v_i^* + v_j^* \geq 1$.

In conclusion, $v_i^* + v_j^* \geq 1, \forall ij \in G$. The other claims follow similarly. □

Proof of Lemma 3. Fix $\delta > \underline{\delta}$. If i had no link in G^*, then $v_i^{*\delta} = 0$. Hence $\delta(v_i^{*\delta} + v_j^{*\delta}) < 1$ for all $j \neq i$. Then by Theorem 1, $ij \in G^*$ for all j such that $ij \in G$. Since i has at least one link in G (see footnote 6), it follows that i has at least one link in G^*, a contradiction. □

Proof of Theorem 3. Let M be a mutually estranged set with partner set L. Fix $\delta > \underline{\delta}$, with $\underline{\delta}$ specified as in Theorem 2. Then in every equilibrium of Γ^δ, a pair of players connected in G reach agreement when matched to bargain if and only if they are connected in G^*.

By Lemma 4, for all i in M,

$$\begin{aligned} v_i^{*\delta} &= \frac{1}{1-\delta} \sum_{\{j|ij\in G\}} \frac{1}{2e} \max(1-\delta v_i^{*\delta} - \delta v_j^{*\delta}, 0) \\ &= \frac{1}{1-\delta} \sum_{\{j|ij\in G,\ j\in L\}} \frac{1}{2e} \max(1-\delta v_i^{*\delta} - \delta v_j^{*\delta}, 0), \end{aligned} \tag{A.5}$$

since i only has G^* links to players in L, so $\max(1-\delta v_i^{*\delta} - \delta v_j^{*\delta}, 0) = 0$ if $ij \in G, j \notin L$.

By Lemma 4, for all j in L,

$$\begin{aligned} v_j^{*\delta} &= \frac{1}{1-\delta} \sum_{\{k|kj\in G\}} \frac{1}{2e} \max(1-\delta v_k^{*\delta} - \delta v_j^{*\delta}, 0) \\ &\geq \frac{1}{1-\delta} \sum_{\{i|ij\in G,\ i\in M\}} \frac{1}{2e} \max(1-\delta v_i^{*\delta} - \delta v_j^{*\delta}, 0). \end{aligned} \tag{A.6}$$

Adding up the equalities A.5 across all $i \in M$ and the inequalities A.6 across all $j \in L$ we obtain

$$\begin{aligned} \sum_{i\in M} v_i^{*\delta} &= \frac{1}{1-\delta} \sum_{\{(i,j)|ij\in G,\ i\in M,\ j\in L\}} \frac{1}{2e} \max(1-\delta v_i^{*\delta} - \delta v_j^{*\delta}, 0) \\ \sum_{j\in L} v_j^{*\delta} &\geq \frac{1}{1-\delta} \sum_{\{(i,j)|ij\in G,\ i\in M,\ j\in L\}} \frac{1}{2e} \max(1-\delta v_i^{*\delta} - \delta v_j^{*\delta}, 0). \end{aligned}$$

Therefore,

$$\sum_{j\in L} v_j^{*\delta} \geq \sum_{i\in M} v_i^{*\delta}, \tag{A.7}$$

which after taking the limit $\delta \to 1$ becomes

$$\sum_{j\in L} v_j^{*} \geq \sum_{i\in M} v_i^{*}.$$

We can manipulate the latter inequality to obtain that

$$|L| \max_{j\in L} v_j^* \geq |M| \min_{i\in M} v_i^*.$$

Player $\underline{i} \in \arg\min_{i\in M} v_i^*$ is connected in G^* to a player $\tilde{j} \in L$ (Lemma 3), hence by Proposition 2, $v_{\underline{i}}^* + v_{\tilde{j}}^* = 1$. Thus $\min_{i\in M} v_i^* = 1 - v_{\tilde{j}}^* \geq 1 - \max_{j\in L} v_j^*$.

Also, any $\overline{j} \in \arg\max_{j\in L} v_j^*$ is connected in G^* to a player $\tilde{i} \in M$, and $v_{\tilde{i}}^* + v_{\overline{j}}^* = 1$ by Proposition 2. Hence $\max_{j\in L} v_j^* = 1 - v_{\tilde{i}}^* \leq 1 - \min_{i\in M} v_i^*$.

We proved that $\min_{i\in M} v_i^* = 1 - \max_{j\in L} v_j^*$. It follows that

$$|L| \max_{j\in L} v_j^* \geq |M|(1 - \max_{j\in L} v_j^*),$$

which is equivalent to

$$\max_{j\in L} v_j^* \geq \frac{|M|}{|M| + |L|}.$$

Moreover,

$$\min_{i\in M} v_i^* = 1 - \max_{j\in L} v_j^* \leq 1 - \frac{|M|}{|M| + |L|} = \frac{|L|}{|M| + |L|}.$$

□

Proof of Lemma 5. Suppose that $r := \min_{M\in\mathcal{I}(H)} |L^H(M)|/|M| < 1$, and let M', M'' be two H-independent sets achieving the minimum. Decompose the set M' as the union of the sets $A_2 = M' \cap M''$, $A_1 = (M' \setminus M'') \setminus L^H(M'')$, that is, the set of players in $M' \setminus M''$ who do not have any H links to M'', and $A_4 = (M' \setminus M'') \cap L^H(M'')$, that is, the set of players in $M' \setminus M''$ who have H links to M''. Similarly, decompose the set M'' as the union of the sets A_2, $A_3 = (M'' \setminus M') \setminus L^H(M')$ and $A_5 = (M'' \setminus M') \cap L^H(M')$. Let $B_2 = L^H(A_2), B_1 = L^H(A_1) \setminus B_2, B_3 = L^H(A_3) \setminus B_2$. Denote $|A_i| = a_i, |B_j| = b_j$ for $i = \overline{1,5}, j = \overline{1,3}$.

Since M'' is H-independent, there are no H links between A_5 and A_2. Also, there are no H links between A_5 and A_1 because $A_1 \cap L^H(M'') = \emptyset$. Then, as $L^H(M') \supset A_5$, it must be that $L^H(A_4) \supset A_5$. Similarly, $L^H(A_5) \supset A_4$. Therefore,[37]

$$\begin{aligned}
L^H(A_1 \cup A_2 \cup A_3) &= B_1 \cup B_2 \cup B_3 \\
L^H(M') = L^H(A_1 \cup A_2 \cup A_4) &\supset B_1 \cup B_2 \cup A_5 \\
L^H(M'') = L^H(A_2 \cup A_3 \cup A_5) &\supset B_2 \cup B_3 \cup A_4 \\
L^H(A_2) &= B_2.
\end{aligned}$$

Since there are no H links between $A_1 \cup A_2$ and M'', it follows that $(B_1 \cup B_2) \cap A_5 \subset (B_1 \cup B_2) \cap M'' = \emptyset$. Analogously, $(B_2 \cup B_3) \cap A_4 = \emptyset$. By definition, $B_1 \cap B_2 = \emptyset$ and $B_2 \cap B_3 = \emptyset$. It follows that the triplets (B_1, B_2, A_5) and (B_2, B_3, A_4) consist of pairwise

[37]The middle two expressions may be strict inclusions as players in A_4 (A_5) may have H links to players not in $B_1 \cup B_2 \cup A_5$ ($B_2 \cup B_3 \cup A_4$).

disjoint sets, hence $|B_1 \cup B_2 \cup A_5| = b_1 + b_2 + a_5$ and $|B_2 \cup B_3 \cup A_4| = b_2 + b_3 + a_4$. The intersection of B_1 and B_3 may be non-empty, hence $|B_1 \cup B_2 \cup B_3| \leq b_1 + b_2 + b_3$.

The definitions of r, M', M'', and the arguments above above imply[38]

$$\frac{b_1 + b_2 + b_3}{a_1 + a_2 + a_3} \geq \frac{|L^H(A_1 \cup A_2 \cup A_3)|}{|A_1 \cup A_2 \cup A_3|} \geq r \tag{A.8}$$

$$r = \frac{|L^H(M')|}{|M'|} \geq \frac{b_1 + b_2 + a_5}{a_1 + a_2 + a_4} \tag{A.9}$$

$$r = \frac{|L^H(M'')|}{|M''|} \geq \frac{b_2 + b_3 + a_4}{a_2 + a_3 + a_5} \tag{A.10}$$

$$\frac{b_2}{a_2} = \frac{|L^H(A_2)|}{|A_2|} \geq r, \tag{A.11}$$

which can be rewritten as

$$b_1 + b_2 + b_3 \geq ra_1 + ra_2 + ra_3 \tag{A.12}$$

$$ra_1 + ra_2 + ra_4 \geq b_1 + b_2 + a_5 \tag{A.13}$$

$$ra_2 + ra_3 + ra_5 \geq b_2 + b_3 + a_4 \tag{A.14}$$

$$b_2 \geq ra_2. \tag{A.15}$$

Adding up all the inequalities above and canceling terms we obtain that

$$(r-1)(a_4 + a_5) \geq 0. \tag{A.16}$$

Since $r < 1$, it follows that $a_4 + a_5 = 0$, hence there are no H links between M' and M'', so the set $M' \cup M''$ is H-independent. Moreover, $A_4 = A_5 = \emptyset$, and thus $M' \cup M'' = A_1 \cup A_2 \cup A_3$.

As the sum of all weak inequalities A.12-A.15 leads to an equality, it follows that A.8-A.15 hold with equality. In particular,

$$\frac{b_1 + b_2 + b_3}{a_1 + a_2 + a_3} = \frac{|L^H(A_1 \cup A_2 \cup A_3)|}{|A_1 \cup A_2 \cup A_3|} = r.$$

Therefore,

$$\frac{|L^H(M' \cup M'')|}{|M' \cup M''|} = r,$$

which finishes the proof. □

[38]The case $a_1 + a_2 + a_3 = 0$ is not possible, as it would lead to $L^H(M') \supset M''$ and $L^H(M'') \supset M'$, which can hold only if $r \geq 1$. If $a_2 = 0$, the bottom inequality becomes irrelevant for the proof.

Proof of Proposition 3. It is sufficient to show that $(r_s)_s$ is strictly increasing. We proceed by contradiction. Suppose that $r_s \leq r_{s-1}$. Then it must be that $1 < s < \overline{s}$.

By Lemma 6, $L^{G_{s-1}}(M_{s-1} \cup M_s) = L_{s-1} \cup L_s$ and $M_{s-1} \cup M_s$ is a G-independent set. Since

$$\frac{|L_{s-1}|}{|M_{s-1}|} = r_{s-1} \text{ and } \frac{|L_s|}{|M_s|} = r_s \leq r_{s-1},$$

it follows that

$$\frac{|L^{G_{s-1}}(M_{s-1} \cup M_s)|}{|M_{s-1} \cup M_s|} = \frac{|L_{s-1}| + |L_s|}{|M_{s-1}| + |M_s|} \leq r_{s-1}.$$

Therefore,

$$M_{s-1} \cup M_s \in \arg \min_{M \subset N_{s-1},\ M \in \mathcal{I}(G)} \frac{|L^{G_{s-1}}(M)|}{|M|},$$

a contradiction with M_{s-1} being the union of all the minimizers of the expression above. □

Proof of Theorem 4, case $s = \overline{s}$. This case is only relevant when $N_{\overline{s}} \neq \emptyset$, which is assumed in the claims below. Note that $r_{\overline{s}} \geq 1$.

Claim 4.5. $v_k^* \geq 1/2, \forall k \in N_{\overline{s}}$

Again, let $\underline{x}_{\overline{s}} = \min_{i \in N_{\overline{s}}} v_i^*$, $\underline{M}_{\overline{s}} = \arg\min_{i \in N_{\overline{s}}} v_i^*$ and $\underline{L}_{\overline{s}} = L^{G_{\overline{s}}}(\underline{M}_{\overline{s}})$. We show that $\underline{x}_{\overline{s}} \geq 1/2$ by contradiction. Suppose that $\underline{x}_{\overline{s}} < 1/2$.

By arguments parallel to those in Claim 4.2, under the assumption that $\underline{x}_{\overline{s}} < 1/2$, $L^{G^*}(\underline{L}_{\overline{s}}) \subset \underline{M}_{\overline{s}}$ and $\underline{L}_{\overline{s}}$ is G^*-independent. Theorem 3 implies that

$$\underline{x}_{\overline{s}} = \max_{i \in L^{G^*}(\underline{L}_{\overline{s}})} v_i^* \geq \frac{|\underline{L}_{\overline{s}}|}{|\underline{M}_{\overline{s}}| + |\underline{L}_{\overline{s}}|}.$$

Since $\underline{x}_{\overline{s}} < 1/2$ and $\underline{L}_{\overline{s}} = L^{G_{\overline{s}}}(\underline{M}_{\overline{s}})$, we obtain that

$$1 > \frac{|L^{G_{\overline{s}}}(\underline{M}_{\overline{s}})|}{|\underline{M}_{\overline{s}}|},$$

which is a contradiction with $r_{\overline{s}} \geq 1$.

Claim 4.6. $v_k^* \leq 1/2, \forall k \in N_{\overline{s}}$

Fix $k \in N_{\overline{s}}$. By Claim 4.5, $v_k^* \geq 1/2$. One consequence of Lemma 6 is that k has no G links to players in $M_1 \cup M_2 \cup \ldots \cup M_{\overline{s}-1}$. By Proposition 2, as $v_k^* \geq 1/2$, there are no G^* links from k to players $j \in L_1 \cup L_2 \cup \ldots \cup L_{\overline{s}-1}$, since for these players $v_j^* \geq 1 - x_{\overline{s}-1} > 1/2$. Therefore, k may only have G^* links to players in $N_{\overline{s}}$. But Claim 4.5 showed that the limit equilibrium payoff of every player in $N_{\overline{s}}$ is at least 1/2. Then Proposition 2 and Lemma 3 imply that $v_k^* \leq 1/2$.

Claims 4.5 and 4.6 show that $v_k^* = 1/2$ for all $k \in N_{\overline{s}}$. □

Proof of Theorem 6. (i) Let $\tilde{G}$ be a network with $ij \in \tilde{G}$, and let $G = \tilde{G} - ij$ be the network obtained by deleting the link ij from $\tilde{G}$. Denote by $\mathcal{A}(G) = (r_s, x_s, M_s, L_s, N_s, G_s)_{s=1,2,\ldots,\overline{s}}$ and $\mathcal{A}(\tilde{G}) = (\tilde{r}_{\tilde{s}}, \tilde{M}_{\tilde{s}}, \tilde{L}_{\tilde{s}}, \tilde{N}_{\tilde{s}}, \tilde{G}_{\tilde{s}})_{\tilde{s}=1,2,\ldots,\tilde{\overline{s}}}$ the outcomes of the algorithm for computing the limit equilibrium payoffs for the bargaining games on the networks G and $\tilde{G}$, respectively. Let $s(k)$ and $\tilde{s}(k)$ denote the steps at which player k is removed in the algorithms $\mathcal{A}(G)$ and respectively $\mathcal{A}(\tilde{G})$, i.e., $s(k) = \max\{s | k \in N_s\}, \tilde{s}(k) = \max\{\tilde{s} | k \in \tilde{N}_{\tilde{s}}\}$.

Without loss of generality, we may assume that $s(i) \leq s(j)$ and set out to prove both of the following inequalities,

$$v_i^*(\tilde{G}) \geq v_i^*(G) \text{ and } v_j^*(\tilde{G}) \geq v_j^*(G).$$

It can be easily shown that if $i \in L_{s(i)}$ or $s(i) = \overline{s}$ then $\mathcal{A}(G)$ and $\mathcal{A}(\tilde{G})$ lead to identical outcomes. Therefore, we may assume that $i \in M_{s(i)}$ and $s(i) < \overline{s}$. In particular, $r_{s(i)} < 1$.

Note that the outcomes of the algorithms $\mathcal{A}(G)$ and $\mathcal{A}(\tilde{G})$ are identical for steps $1, \ldots, s(i)-1$ and $\tilde{r}_{s(i)} \geq r_{s(i)}$. Since $i \in M_{s(i)}, r_{s(i)} < 1$ and $\tilde{r}_{\tilde{s}} \geq r_{s(i)}$ for $\tilde{s} \geq s(i)$, it must be that $v_k^*(\tilde{G}) \geq r_{s(i)}/(1 + r_{s(i)}) = v_i^*(G)$ for all $k \in N_{s(i)} = \tilde{N}_{s(i)}$. Hence $v_i^*(\tilde{G}) \geq v_i^*(G)$.

We next show that $v_j^*(\tilde{G}) \geq v_j^*(G)$. There are three cases to consider: $j \in L_{s(j)}, j \in M_{s(j)}$, and $s(j) = \overline{s}$. We only solve the former case; the other two can be handled by similar methods.

Henceforth we focus on the case $i \in M_{s(i)}, j \in L_{s(j)}$. Then $s(j) < \overline{s}$ and $r_{s(j)} < 1$. The following lemma will be used repeatedly.

Lemma 7. *Suppose that $(r_s, x_s, M_s, L_s, N_s, G_s)_{s=1,2,\ldots,\overline{s}}$ is the outcome of the algorithm $\mathcal{A}(G)$. For any $s < \overline{s}$, and any non-empty $L' \subset L_s$,*

$$\frac{|L'|}{|L^{G_s}(L') \cap M_s|} \leq r_s.$$

Proof. Let $M' = L^{G_s}(L') \cap M_s$. Then $L_s = L^{G_s}(M_s)$ and $L^{G_s}(L') \cap (M_s \setminus M') = \emptyset$ imply that $L^{G_s}(M_s \setminus M') \subset L_s \setminus L'$. Since

$$r_s = \min_{M \subset N_s,\ M \in \mathcal{I}(G)} |L^{G_s}(M)|/|M|,$$

it follows that $|L_s \setminus L'|/|M_s \setminus M'| \geq r_s = |L_s|/|M_s|$. Hence $|L'|/|M'| \leq |L_s|/|M_s| = r_s$. □

For $\tilde{s} = s(i)-1, s(i), \ldots, \tilde{s}(j)$ we show that

(1) $\tilde{r}_{\tilde{s}} \leq r_{s(j)}$
(2) $\tilde{M}_{\tilde{s}} \cap (\cup_{s=s(i)}^{s(j)} L_s) = \emptyset$
(3) $\tilde{L}_{\tilde{s}} \cap (\cup_{s=s(i)}^{s(j)} M_s) = \emptyset$

by induction on $\tilde{s}$. The induction base case, $\tilde{s} = s(i)-1$, follows trivially. Suppose we proved the three assertions for all lower values, and we proceed to proving them for $\tilde{s}$. Each of the parts below establishes the corresponding assertion.

Part 6.1. We prove the first part of the induction step, $\tilde{r}_{\tilde{s}} \leq r_{s(j)}$. Let $\tilde{M} = \tilde{M}_{s(i)} \cup \ldots \cup \tilde{M}_{\tilde{s}-1}$ and $\tilde{L} = L^{\tilde{G}_{s(i)}}(\tilde{M}) = \tilde{L}_{s(i)} \cup \ldots \cup \tilde{L}_{\tilde{s}-1}$. Note that $\tilde{L} = L^{\tilde{G}_{s(i)}}(\tilde{M}) = L^{G_{s(i)}}(\tilde{M})$ since $i, j \notin \tilde{M}$ (if $i \in \tilde{M}$ then $j \in \tilde{L}$, a contradiction with $\tilde{s}-1 < \tilde{s}(j)$; $j \in \tilde{M}$ leads to a similar contradiction).

Note that $\cup_{s=s(i)}^{s(j)} M_s \setminus \tilde{M}$ is non-empty as it contains i, and is contained in $\tilde{N}_{\tilde{s}}$ since by the induction hypothesis, $\tilde{L}_{\tilde{s}'} \cap (\cup_{s=s(i)}^{s(j)} M_s) = \emptyset$ for $\tilde{s}' < \tilde{s}$. Thus it is sufficient to prove that

$$\frac{|L^{\tilde{G}_{\tilde{s}}}(\cup_{s=s(i)}^{s(j)} M_s \setminus \tilde{M})|}{|\cup_{s=s(i)}^{s(j)} M_s \setminus \tilde{M}|} \leq r_{s(j)}.$$

Indeed, $\cup_{s=s(i)}^{s(j)} M_s \setminus \tilde{M}$ is G-independent (Lemma 6), and also $\tilde{G}$-independent ($\tilde{G} = G + ij, i \in M_{s(i)}, j \in L_{s(j)}$), so the inequality above implies that $\tilde{r}_{\tilde{s}} \leq r_{s(j)}$.

Fix $s \in \overline{s(i), s(j)}$. Let $L' = L_s \setminus L^{G_s}(M_s \cap \tilde{M})$. Note that $L^{G_s}(L') \cap M_s \subset M_s \setminus \tilde{M}$. Lemma 7 applied to step s of $\mathcal{A}(G)$ with L' defined above implies that[39]

$$\frac{|L_s| - |L^{G_s}(M_s \cap \tilde{M})|}{|M_s \setminus \tilde{M}|} \leq r_s.$$

As $L^{G_s}(M_s \cap \tilde{M}) \subset L^{G_{s(i)}}(\tilde{M}) \cap L_s$, it follows that

$$\frac{|L_s| - |L^{G_{s(i)}}(\tilde{M}) \cap L_s|}{|M_s \setminus \tilde{M}|} \leq r_s.$$

Since $r_s \leq r_{s(j)}$ for all $s \in \overline{s(i), s(j)}$, the set of inequalities above imply that

$$\frac{\sum_{s=s(i)}^{s(j)} (|L_s| - |L^{G_{s(i)}}(\tilde{M}) \cap L_s|)}{\sum_{s=s(i)}^{s(j)} |M_s \setminus \tilde{M}|} \leq r_{s(j)},$$

[39]The argument is only necessary and relevant when $M_s \cap \tilde{M} \neq \emptyset, M_s$.

or equivalently,

$$\frac{|\cup_{s=s(i)}^{s(j)} L_s| - |L^{G_{s(i)}}(\tilde{M}) \cap (\cup_{s=s(i)}^{s(j)} L_s)|}{|\cup_{s=s(i)}^{s(j)} M_s \setminus \tilde{M}|} \leq r_{s(j)}.$$

The letter inequality can be rewritten as

$$\frac{|\cup_{s=s(i)}^{s(j)} L_s \setminus L^{G_{s(i)}}(\tilde{M})|}{|\cup_{s=s(i)}^{s(j)} M_s \setminus \tilde{M}|} \leq r_{s(j)}.$$

But $L^{\tilde{G}_{\tilde{s}}}(\cup_{s=s(i)}^{s(j)} M_s \setminus \tilde{M}) \subset L^{\tilde{G}_{s(i)}}(\cup_{s=s(i)}^{s(j)} M_s) = L^{G_{s(i)}}(\cup_{s=s(i)}^{s(j)} M_s) = \cup_{s=s(i)}^{s(j)} L_s$ (the first equality follows from $\tilde{G} = G + ij, i \in M_{s(i)}, j \in L_{s(j)}$), and $\tilde{G}_{\tilde{s}}$ does not contain any players in $\tilde{L} = L^{G_{s(i)}}(\tilde{M})$. Then $L^{\tilde{G}_{\tilde{s}}}(\cup_{s=s(i)}^{s(j)} M_s \setminus \tilde{M}) \subset \cup_{s=s(i)}^{s(j)} L_s \setminus L^{G_{s(i)}}(\tilde{M})$, and the inequality above implies that

$$\frac{|L^{\tilde{G}_{\tilde{s}}}(\cup_{s=s(i)}^{s(j)} M_s \setminus \tilde{M})|}{|\cup_{s=s(i)}^{s(j)} M_s \setminus \tilde{M}|} \leq r_{s(j)},$$

as desired.

Part 6.2. We prove the second part of the induction step, $\tilde{M}_{\tilde{s}} \cap (\cup_{s=s(i)}^{s(j)} L_s) = \emptyset$, by contradiction. Suppose that $\tilde{M}_{\tilde{s}} \cap (\cup_{s=s(i)}^{s(j)} L_s) \neq \emptyset$, and let s_0 be the smallest index $s \in \overline{s(i), s(j)}$ for which $\tilde{M}_{\tilde{s}} \cap L_s \neq \emptyset$. Define $B = \tilde{M}_{\tilde{s}} \cap L_{s_0}$ and $A = L^{G_{s_0}}(B) \cap M_{s_0}$.

We argue that $A \subset \tilde{N}_{\tilde{s}}$. Fix $k \in A$. Player k has a G-link to a player $l \in B$. If k is removed at step $\tilde{s}' < \tilde{s}$ in the algorithm $\mathcal{A}(\tilde{G})$ then $k \in \tilde{M}_{\tilde{s}'}$ by the induction hypothesis ($\tilde{L}_{\tilde{s}'} \cap (\cup_{s=s(i)}^{s(j)} M_s) = \emptyset$). Then $l \in \tilde{L}_{\tilde{s}'}$ or $l \notin \tilde{N}_{\tilde{s}'}$, contradicting that $l \in \tilde{M}_{\tilde{s}}$. Therefore, $k \in \tilde{N}_{\tilde{s}}$.

Note that $L^{\tilde{G}_{\tilde{s}}}(A) \cap \tilde{M}_{\tilde{s}} \subset B \cup \{j\}$ since players in $A \subset M_{s_0}$ may only have G-links to players in $L_1 \cup L_2 \cup \ldots \cup L_{s_0}$ (Lemma 6), and $\tilde{M}_{\tilde{s}} \cap (L_1 \cup L_2 \cup \ldots \cup L_{s_0}) = B$ by the definition of s_0. If $i \in A$ then we could have $j \in L^{\tilde{G}_{\tilde{s}}}(A) \cap \tilde{M}_{\tilde{s}}$. Lemma 7 applied for step $\tilde{s}$ of $\mathcal{A}(\tilde{G})$ with $L' = A$ and Part 6.1 imply that

$$\frac{|A|}{|B| + 1} \leq \tilde{r}_{\tilde{s}} \leq r_{s(j)} < 1.$$

Hence $|A| < |B| + 1$, or $|A| \leq |B|$.

Since $A = L^{G_{s_0}}(B) \cap M_{s_0}$, Lemma 7 applied to step s_0 of $\mathcal{A}(G)$ with $L' = B$ implies that

$$\frac{|B|}{|A|} \leq r_{s_0} \leq r_{s(j)} < 1.$$

Hence $|A| > |B|$, a contradiction with $|A| \leq |B|$. Therefore $\tilde{M}_{\tilde{s}} \cap (\cup_{s=s(i)}^{s(j)} L_s) = \emptyset$.

Part 6.3. To establish the third part of the induction hypothesis, $\tilde{L}_{\tilde{s}} \cap (\cup_{s=s(i)}^{s(j)} M_s) = \emptyset$, we proceed by contradiction. Suppose that $k \in \tilde{L}_{\tilde{s}} \cap (\cup_{s=s(i)}^{s(j)} M_s)$. It should be that k has a $\tilde{G}_{\tilde{s}}$ link to a player $l \in \tilde{M}_{\tilde{s}}$. By Lemma 6, since $\tilde{G}_{\tilde{s}}$ is a subnetwork of $G_{s(i)}$, $k \in \cup_{s=s(i)}^{s(j)} M_s$ may only have $\tilde{G}_{\tilde{s}}$ links to players in $\cup_{s=s(i)}^{s(j)} L_s$, so $l \in \cup_{s=s(i)}^{s(j)} L_s$. Therefore, $l \in \tilde{M}_{\tilde{s}} \cap (\cup_{s=s(i)}^{s(j)} L_s)$, a contradiction with Part 6.2.

In particular, for $\tilde{s} = \tilde{s}(j)$ the induction hypothesis implies that $j \in \tilde{L}_{\tilde{s}(j)}$ and $\tilde{r}_{\tilde{s}(j)} \leq r_{s(j)}$. Then $v_j^*(\tilde{G}) = 1/(1 + \tilde{r}_{\tilde{s}(j)}) \geq 1/(1 + r_{s(j)}) = v_j^*(G)$.

(ii) To prove the "if" part of the statement, let $\tilde{G}$ be an equitable network. Part (i) shows that $\tilde{G}$ is unilaterally stable with respect to $(v_i^*(G))_{i \in N, G \in \mathcal{G}}$. Note that by Theorem 5 when a link is added to an equitable network another equitable network obtains. Hence $\tilde{G} + ij$ is equitable, and $v_i^*(\tilde{G} + ij) = v_i^*(\tilde{G}) = 1/2$ for all $i \neq j \in N$. Therefore, $\tilde{G}$ is pairwise stable with respect to $(v_i^*(G))_{i \in N, G \in \mathcal{G}}$.

To prove the "only if" part of the statement, let $\tilde{G}$ be a network that is pairwise stable with respect to $(v_i^*(G))_{i \in N, G \in \mathcal{G}}$. Suppose that $\tilde{G}$ is not equitable. Let $(\tilde{r}_{\tilde{s}}, \tilde{M}_{\tilde{s}}, \tilde{L}_{\tilde{s}}, \tilde{N}_{\tilde{s}}, \tilde{G}_{\tilde{s}})_{\tilde{s}}$ denote the outcome of the algorithm $\mathcal{A}(\tilde{G})$. Then there exist $i, j \in \tilde{M}_1$ such that $v_i^*(\tilde{G}) = v_j^*(\tilde{G}) < 1/2$ ($|\tilde{M}_1| \geq 2$). The limit equilibrium payoffs of players i and j in the game on the network $\tilde{G} + ij$ satisfy $v_i^*(\tilde{G} + ij) \geq v_i^*(\tilde{G})$ and $v_j^*(\tilde{G} + ij) \geq v_j^*(\tilde{G})$ by part (i) of the theorem. By Proposition 2, $v_i^*(\tilde{G} + ij) + v_j^*(\tilde{G} + ij) \geq 1$. Hence, $v_i^*(\tilde{G} + ij) + v_j^*(\tilde{G} + ij) > v_i^*(\tilde{G}) + v_j^*(\tilde{G})$, which together with $v_i^*(\tilde{G} + ij) \geq v_i^*(\tilde{G})$ and $v_j^*(\tilde{G} + ij) \geq v_j^*(\tilde{G})$, leads to a violation of the pairwise stability of $\tilde{G}$.[40] The contradiction proves that $\tilde{G}$ is equitable. □

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[40] It can be shown that the latter two inequalities hold strictly, which is necessary for the proof of Corollary 2.

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