Dynamic Recontracting Processes
with Multiple Indivisible Goods*

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Abstract

We consider multiple-type housing markets with strict preferences. To capture the dynamic aspect of trades in such markets, we study a dynamic re-contracting process as introduced by Serrano and Volij (2005). First, we analyze the set of recurrent classes of this process as a (non-empty) solution concept. We show that each core allocation always constitutes a singleton recurrent class and provide examples of non-singleton recurrent classes consisting of blocking-cycles of individually rational allocations. Contrary to Serrano and Volij (2005), and to many standard applications of dynamic Markov processes, we show that for multiple-type housing markets stochastic stability never serves as a selection device among recurrent classes.

Next, we propose a method to explicitly compute the limit invariant distribution of the dynamic re-contracting process. This method exploits the interplay of coalitional stability and accessibility that determines the probability distribution over final allocations. We provide various examples to demonstrate how the limit invariant distribution discriminates among stochastically stable allocations: surprisingly sometimes core allocations are less likely to be final allocations of the dynamic process than cycles composed of non-core allocations.

Keywords: coalitional accessibility, core, indivisible goods, limit invariant distribution, stochastic stability.
JEL classification: D63, D70

1 Introduction

Dynamic re-contracting processes Consider Shapley and Scarf’s (1974) model of housing markets. One of the most important solution concepts for such markets is the core. An allocation \( x \) is in the core if there does not exist a coalition that can improve upon \( x \) using its own endowments: \( x \) cannot be blocked. The core of a housing market satisfies some remarkable properties. Most prominently, the core consists of a unique allocation that also turns out to be the unique Walrasian allocation of the market (Roth and Postlewaite, 1977). However, while blocking of allocations captures a dynamic aspect of trades, the core itself is a static concept. In other words, if agents block, they cause a transition from one state of the

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1In Shapley-Scarf housing markets, each agent is endowed with a house, has strict preferences over the set of houses in the market, and wishes to consume exactly one house.

2Note that the core we introduce here is sometimes referred to as the strong (or strict) core. The notion of blocking associated with the strong core is weak blocking. That is, when a coalition \( S \) blocks \( x \), no member of \( S \) is worse-off and at least one of its member is better-off.
world (an allocation) to another state of the world (an allocation where the members of the blocking coalition are better off). So the core incorporates robustness against these potential transitions in a model that does not accommodate the possibility of transition.

Our aim here is to better understand the dynamics of trade. In particular we are interested in its resulting allocations – inside as well as outside the core. To this end, we study the following *dynamic recontracting process*. The dynamic recontracting process starts with the agents’ endowment vector as the initial allocation for trade. Note that throughout the dynamic recontracting process we do not redefine endowments (property rights are not exchanged). Next, at any stage of the dynamic recontracting process agents can recontract upon the allocation $x$ that resulted from previous trades. A coalition is randomly selected and is allowed to recontract over $x$ if it can block allocation $x$ using the coalition’s endowments. Based on the blocking the coalition uses to recontract, the new allocation is obtained as follows: agents in the coalition reallocate their endowments according to the blocking. If this reallocation is feasible because no agent outside the coalition was consuming the endowment of an agent inside the coalition, then agents outside the coalition stick to their assignment at allocation $x$. If the coalition’s recontracting is not feasible, then agents outside the coalition are send to their endowments. Thus, at each period a coalition is randomly selected and has the power to make the process transit from the prevailing allocation to another one. This determines a Markov process on the space of allocations. In the long-run such a Markov-process will always end-up in one of its recurrent classes: a set of allocations that once reached by the process will never be abandoned.

Next, we allow that agents make mistakes when they recontract; we “perturb” the dynamic recontracting process. For a dynamic recontracting process this means that in every period, each agent with a small probability $\epsilon$ agrees on a reallocation that makes him worse off. In such a perturbed dynamic recontracting process any allocation can be reached from any other allocation from one period to another (with sufficiently many mistakes by the agents involved). Hence, a perturbed dynamic recontracting process has only one recurrent class – the entire allocation space – and the probability distribution over allocations induced by the perturbed dynamic recontracting process converges (in the long-run and for small $\epsilon$) to the so-called *limit invariant distribution*, which is uniquely determined. The support of this distribution – the set of *stochastically stable allocations* – is the set of allocations to which the perturbed dynamic recontracting process will converge to with strictly positive probability. Hence, stochastically stable allocations can be regarded as the potentially *final allocations* of the dynamic recontracting process. Furthermore, the limit invariant distribution is a probability distribution over these candidates for final allocations (the stochastically stable allocations) of the process.

Note that similarly as the core, recurrent classes (of the unperturbed dynamic recontracting process) are static. A set of allocations is a recurrent class if it exhibits the following *stability*: no allocation outside the recurrent class blocks an allocation in the recurrent class. In contrast, the limit invariant distribution is a inherently dynamic concept as it also captures the *accessibility* of an allocation. To be more specific, the easier it is for the dynamic recontracting process to reach an allocation - and, the more difficult to exit - the larger is the probability that the dynamic reconcontracting process converges to the respective allocation in the long run. Hence, the limit invariant distribution is a probability distribution over final allocations that combines a core-like stability concept (each stochastically stable allocation

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3We discuss basic notions and terminology of Markov processes in Appendix B.
is an element of a recurrent class) with a notion of accessibility incorporated by the dynamic recontracting process.

One of our objectives is to introduce a computational method that elicits the set of stochastically stable allocations and the limit invariant distribution. In particular, we are interested in the relation between the set of final allocations of dynamic recontracting processes (and the respective probability distributions) and the core. This is the main reason why we have modeled the dynamic recontracting process using recontracting based on core blocking and without the transfer of property rights (once property rights are transferred throughout the process, it is obvious that the core of the initial market will not play any specific role). Given that we model the dynamic recontracting process as closely as possible to the implicit dynamic elements incorporated into the core, we ask the following questions: Are core allocations necessarily elements of recurrent classes or stochastically stable? Does the process converge to every core allocation with the same probability? Can the process converge to non-core allocations? And, how do all these solution concepts relate to Walrasian allocation(s)?

Relation to the Literature  Pioneering work on dynamic recontracting processes for exchange economies has been conducted by Feldman (1974) and Green (1974). They provide conditions for which a dynamic recontracting process converges to the core and thereby formalize Edgeworth’s initial intuition that the final allocation of an exchange economy can be reached through dynamic recontracting. In a recent contribution, Serrano and Volij (2005) use dynamic recontracting processes to analyze Shapley-Scarf (Shapley and Scarf, 1974) housing markets. Using a Markov process identical to the one described above, Serrano and Volij (2005) show that the unique core allocation is the unique recurrent class (and, hence, the unique stochastically stable allocation) of the dynamic recontracting process. This “equivalence” result between the core and the set of final allocations of the dynamic recontracting process is driven by the global dominance property of the core for Shapley-Scarf housing markets.\footnote{Roth and Postlewaite (1977) demonstrate that in a Shapley-Scarf housing market each allocation outside the core can be blocked by the unique core allocation.} Next, Serrano and Volij (2005) extend the classical housing market model by allowing for indifferences in the agents’ preferences. For this model they characterize the set of recurrent classes and stochastically stable allocations. In particular, they show that every allocation in the core forms a singleton recurrent class of the dynamic recontracting process. However, not every core allocation is stochastically stable. The authors provide examples where (i) The set of stochastically stable allocations coincides with the set of core allocations, (ii) Requiring stochastic stability selects certain core allocations that are not necessarily Walrasian, and (iii) The set of stochastically stable allocations overlaps with the set of core allocations but also contains a cycle of non-core allocations. Serrano and Volij’s (2005) results suggest that the set of stochastically stable allocations is a dynamic solution concept that relates to the core in a non-trivial way. In particular, they demonstrate how stochastic stability serves as a selection device among recurrent classes.

Our Contribution  We consider a different extension of Shapley-Scarf housing markets than Serrano and Volij (2005). We keep preferences strict, but extend the analysis to multiple-type housing markets (Konishi et al., 2001; Moulin, 1995). Hence, we endow each agent with one commodity of each type (\textit{e.g.}, houses and cars) and analyze simultaneous trade in all these types. Konishi et al. (2001) show that the core of such an economy may well be empty
or multi-valued. For the case of only one type the multiple-type housing market model is identical to Serrano and Volij’s (2005) benchmark model with strict preferences.

Similar to Serrano and Volij (2005), we show that each allocation in the core forms a singleton recurrent classes of the dynamic recontracting process while there are possibly non-singleton recurrent classes consisting of blocking cycles. In contrast to Serrano and Volij’s findings – and to many standard applications of Markov processes (see literature on equilibrium selection in non-cooperative games as proposed by Kandori et al., 1993; Young, 1993) – we show that stochastic stability never serves as a selection device among recurrent classes. There is no allocation in a recurrent class of the process that fails to be stochastically stable. However, this does not imply that each stochastically stable allocation (each allocation in the support of the limit invariant distribution) has the same probability mass (i.e., is the final allocation of the process with the same probability).

Starting with a result by Freidlin and Wentzell (2004), we develop a method to compute the limit invariant distribution of a dynamic recontracting process. As the limit invariant distribution is the unique probability distribution over allocations to which the recontracting process is converging to in the long-run and for small probability of mistakes, its use is to discriminate between stochastically stable allocations as contenders for the final allocations of the economy. This discrimination hinges on one crucial conceptual difference between what determines the stochastic stability of an allocation and what determines the probability that this allocation will be the final allocation of the process (i.e., the respective component in the limit invariant distribution). While stochastic stability solely depends on the minimum number of mistakes needed to exit and enter a certain allocation, a component of the limit invariant distribution is also determined by the size of the transition probabilities of the original (unperturbed) dynamic recontracting process. This additional dependence on the underlying dynamic recontracting process will prove to allow for a finer characterization of the final allocations of the dynamic recontracting process.

We illustrate the computational techniques and their interpretation with several examples. In particular we show that some core and Walrasian allocations may be the least likely of all possible final allocations. On the other hand, blocking-cycles may emerge as powerful contenders for final allocations even if the core is non-empty. Hence, following the long-run predictions of a dynamic recontracting process might lead us to non-trivial relations between core allocations, Walrasian allocations, and blocking-cycles.

Moreover, we regard our analysis of dynamic recontracting processes as an instructive illustration of our method to compute limit invariant distributions. This method should be of added-value in most applications of Markov processes (such as methods of equilibrium selection in non-cooperative games (Kandori et al., 1993; Young, 1993) or models of network formation (Jackson and Watts, 2002).

**Organization of the Paper** The remainder of the paper is organized as follows. In Section 2 we define multiple-type housing markets. In Section 3 we discuss some basic results for multiple-type housing markets and introduce examples. We introduce the dynamic recontracting process and characterize its recurrent classes in Section 4. Next, we introduce the perturbed process in Section 5 and then derive its set of stochastically stable allocations and the limit invariant distribution. Finally, in Section 6, we conclude with some remarks on the robustness of our findings and on the relation between the different solution concepts.

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5 We show that the same holds for the set of Walrasian allocations.
2 Multiple-Type Housing Markets

Let \( N = \{1, \ldots, n\} \), \( n \geq 2 \), be the set of agents, which we sometimes call the grand coalition. There exist \( \ell \geq 1 \) types of indivisible objects. The set of object types is denoted by \( L = \{1, \ldots, \ell\} \) and each agent \( i \in N \) is endowed with one object of each type \( \ell \in L \), denoted by \( i \). Thus, \( N \) also denotes the set of objects of each type.

**Allocations** An allocation is an assignment of objects such that each agent receives exactly one object of each type, i.e., an allocation is a matrix \( x = (x_i(\ell))_{i \in N, \ell \in L} \in N^{N \times L} \) such that

(i) For each \( i \in N \) and each \( \ell \in L \), \( x_i(\ell) \in N \) denotes the object of type \( \ell \) that agent \( i \) consumes, e.g., if \( x_i(\ell) = j \), then agent \( i \) receives agent \( j \)'s endowment of type \( \ell \), and

(ii) no object of any type is assigned to more than one agent, i.e., for each \( \ell \in L \), \( \cup_{i \in N} \{x_i(\ell)\} = N \).

Let \( X \) denote the set of allocations. Given \( x \in X \) and \( \ell \in L \), \( x(\ell) = (x_1(\ell), \ldots, x_n(\ell)) \) denotes the allocation of type-\( \ell \) objects. Given \( x \in X \) and \( i \in N \), \( x_i = (x_i(1), \ldots, x_i(\ell)) \) denotes the list of objects that agent \( i \) receives at allocation \( x \). We call \( x_i \) agent \( i \)'s bundle. Note that the set of bundles for each \( i \in N \) can be denoted by \( N^L \). We denote each agent \( i \)'s endowment by \( e_i = (i, \ldots, i) \in N^L \). Similarly, for any coalition \( S \subseteq N \), we denote coalition \( S \)'s endowment by \( e_S = (e_i)_{i \in S} \).

**Markets** Each agent \( i \in N \) has complete, transitive, and strict preferences \( R_i \) over bundles, i.e., \( R_i \) is a linear order over \( N^L \). We denote the strict part of \( R_i \) by \( P_i \). Thus, for bundles \( x_i, y_i \in N^L \), \( x_i R_i y_i \) implies \([x_i \neq y_i \text{ and } x_i P_i y_i]\) or \([x_i = y_i]\). By \( \mathcal{R} \) we denote the set of preferences over \( N^L \). By \( \mathcal{R}^N = \times_{i \in N} \mathcal{R} \) we denote the set of (preference) profiles. Since the set of agents and their endowments remain fixed throughout, \( \mathcal{R}^N \) also denotes the set of multiple-type housing markets. For \( \ell = 1 \), our model coincides with the classical Shapley and Scarf (1974) housing market model.\(^6\)

**Individual Rationality** An allocation \( x \) is individually rational for \( R \in \mathcal{R}^N \) if for each \( i \in N \), \( x_i R_i e_i \). Let \( IR(R) \) be the set of individually rational allocations for \( R \in \mathcal{R}^N \).

To introduce the standard (cooperative) solution concepts for multiple-type housing markets we need some additional notation. The set of all feasible reallocations of objects among the members of coalition \( S \subseteq N \) is denoted by,

\[
X_S = \{(x_i(\ell))_{i \in S, \ell \in L} \in N^{S \times L} \mid \text{for each } \ell \in L, \cup_{i \in S} \{x_i(\ell)\} = S\}.
\]

Let \( y \in X \) and \( S \subseteq N \). Then, by \( y_S = (y_i)_{i \in S} \) we denote the restriction of allocation \( y \) to coalition \( S \). For notational convenience we will also use \( X_{-S} = X_{N \setminus S} \) and \( y_{-S} = y_{N \setminus S} \).

If for \( R \in \mathcal{R}^N \), \( x, y \in X \), and \( S \subseteq N \), (i) \( y_S \in X_S \), (ii) for each \( i \in S \), \( y_i R_i x_i \), and (iii) for some \( j \in S \), \( y_j P_j x_j \), then \( y \) \( S \)-blocks \( x \).

\(^6\)Note that instead of considering the whole domain of linear orders \( \mathcal{R} \) as our reference domain, we could restrict the domain to the domain of separable preferences \( \mathcal{R}^s \) (see Klaus, 2005) or to the domain of additively separable preferences \( \mathcal{R}^{as} \) (see Konishi et al., 2001). However, separability plays no role in our analysis.
The Core
An allocation is in the (strong) core if no coalition of agents can improve their welfare by reallocating their endowments among themselves, i.e., an allocation \( x \in X \) is a core allocation for \( R \in \mathcal{R} \) if there exists no coalition \( S \subseteq N \) and no \( y_S \in X_S \) such that \( y_S \)-blocks \( x \). Let \( \text{Core}(R) \) be the set of core allocations for \( R \in \mathcal{R} \).

Walrasian Allocations
Define a price system by \( p \equiv (p_\ell)_{\ell \in L} \in \mathbb{R}_+^L \) such that for all \( \ell \in L \), \( p_\ell = (p_\ell(1), \ldots, p_\ell(n)) \in \mathbb{R}_+^n \). An allocation \( x \) is a Walrasian allocation for \( R \in \mathcal{R} \) if there exists a price system \( p \in \mathbb{R}_+^L \setminus \{0\} \) such that for each \( i \in N \), \( x \) is a best affordable bundle, i.e., (i) \( \sum_{\ell \in L} p_\ell(i) \geq \sum_{\ell \in L} p_\ell(x_i(\ell)) \) and (ii) if \( y_i P x_i \), then \( \sum_{\ell \in L} p_\ell(y_i(\ell)) \sum_{\ell \in L} p_\ell(i) \). Note that the budget inequality (i) can be replaced by a budget equality: this can be easily checked by adding (i) up over all agents \( i \in N \) and applying \( \sum_{i \in N} \left( \sum_{\ell \in L} p_\ell(i) \right) = \sum_{i \in N} \left( \sum_{\ell \in L} p_\ell(x_i(\ell)) \right) \). Let \( W(R) \) be the set of Walrasian allocations for \( R \in \mathcal{R} \).

3 Multiple-Type Housing Markets: Basic Results & Examples
First, we summarize some results for the benchmark case of one object type housing markets.

Remark 1. The Benchmark Case: Housing Markets with Strict Preferences
For any housing market with one object type (and strict preferences), Shapley and Scarf (1974) showed that a core allocation always exists. Roth and Postlewaite (1977) proved that the set of core allocations for any housing market with one object type equals the set of Walrasian equilibria and is a singleton. Using the so-called top-trading algorithm (due to David Gale, see Shapley and Scarf, 1974) one can easily calculate the unique core allocation for any housing market with one object type. Furthermore, the core is externally stable, i.e., for any non-core allocation \( x \) there exists a coalition \( S \) such that the core allocation \( S \)-blocks \( x \) (Roth and Postlewaite, 1977, Lemma 1). Serrano and Volij (2005) refer to this particular feature of the core as global dominance.

As soon as we either relax the assumption of strict preferences or increase the number of object types, existence, single-valuedness, and the global dominance property of the core do not necessarily hold anymore. For markets with \( \bar{\ell} \geq 2 \), Konishi et al. (2001) show that the core may be empty or multi-valued – even for additively separable preferences. Moreover (Konishi et al., 2001, Proposition 3.1), for each \( R \in \mathcal{R} \), \( W(R) \subseteq \text{Core}(R) \).

We next introduce several examples that we will analyze in the sequel. All our examples are multiple-type housing market with two object types and three agents and will be denoted by the agents’ preferences.

Example 1. An Empty Core
Consider \( R \) such that
\[
(3, 1) P_1 (1, 2) P_1 (1, 1) P_1 \text{ anything},
(2, 1) P_2 (3, 2) P_2 (2, 2) P_2 \text{ anything},
(2, 3) P_3 (1, 3) P_3 (3, 3) P_3 \text{ anything}.
\]

\footnote{The exception is Example 5, which we comment on in Remark 2.}
The set of individually rational allocations equals $IR(R) = \{x^1, x^2, x^3, x^4\}$ with

\[
x^1 = \{(1,1), (2,2), (3,3)\}, \quad x^2 = \{(1,1), (3,2), (2,3)\},
\]
\[
x^3 = \{(1,2), (2,1), (3,3)\}, \quad x^4 = \{(3,1), (2,2), (1,3)\}.
\]

Clearly, $x^2$ \{2,3\}-blocks $x^1$, $x^3$ \{1,2\}-blocks $x^2$, $x^4$ \{1,3\}-blocks $x^3$, and $x^2$ \{2,3\}-blocks $x^4$. Hence, the core and the set of Walrasian allocations coincide and are empty.

We relegate the computation of the core and the set of Walrasian allocations for all remaining examples to Appendix A.

**Example 2. The Unique Walrasian Allocation Equals the Core Allocation**

Consider $R$ such that

\[
(3,1) P_1 (2,2) P_1 (1,2) P_1 (1,1) P_1 \text{ anything},
\]
\[
(2,1) P_2 (3,3) P_2 (3,2) P_2 (2,2) P_2 \text{ anything},
\]
\[
(2,3) P_3 (1,1) P_3 (1,3) P_3 (3,3) P_3 \text{ anything}.
\]

The set of individually rational allocations equals $IR(R) = \{x^1, x^2, x^3, x^4, x^5\}$ with

\[
x^1 = \{(1,1), (2,2), (3,3)\}, \quad x^2 = \{(1,1), (3,2), (2,3)\},
\]
\[
x^3 = \{(1,2), (2,1), (3,3)\}, \quad x^4 = \{(3,1), (2,2), (1,3)\},
\]
\[
x^5 = \{(2,2), (3,3), (1,1)\}.
\]

The core and the set of Walrasian allocations coincide and equal $Core(R) = W(R) = \{x^5\}$.  

Konishi et al. (2001, Example 3.3) show that, even for additively separable preferences, the core may be multi-valued. The next example has multiple core allocations, of which only one is Walrasian. Since in our context separability does not play a role, we introduce alternative examples that are easier to analyze; for instance, our next multiple-type housing market has four individually rational allocations while Konishi et al.’s (2001) corresponding example has eleven individually rational allocations.

**Example 3. Multiple Core Allocations and Unique Walrasian Allocation**

Consider $R$ such that

\[
(1,2) P_1 (3,3) P_1 (2,3) P_1 (1,1) P_1 \text{ anything},
\]
\[
(1,3) P_2 (1,2) P_2 (3,3) P_2 (2,2) P_2 \text{ anything},
\]
\[
(3,1) P_3 (2,1) P_3 (3,3) P_3 \text{ anything}.
\]

The set of individually rational allocations equals $IR(R) = \{x^1, x^2, x^3, x^4\}$ with

\[
x^1 = \{(1,1), (2,2), (3,3)\}, \quad x^2 = \{(2,3), (1,2), (3,1)\},
\]
\[
x^3 = \{(1,2), (3,3), (2,1)\}, \quad x^4 = \{(3,3), (1,2), (2,1)\}.
\]

The set of Walrasian allocations $W(R) = \{x^3\}$ is a strict subset of $Core(R) = \{x^2, x^3, x^4\}$.  


Next we illustrate that the set of Walrasian allocations may contain multiple allocations.

**Example 4. Multiple Walrasian Allocations**
Consider $R$ such that

$$(1, 2) P_1 (3, 3) P_1 (2, 3) P_1 (1, 1) P_1 \text{ anything},$$

$$(1, 3) P_2 (1, 2) P_2 (3, 3) P_2 (2, 2) P_2 \text{ anything},$$

$$(2, 1) P_3 (3, 1) P_3 (3, 3) P_3 \text{ anything}.$$

The set of individually rational allocations equals $IR(R) = \{x^1, x^2, x^3, x^4\}$ with

$x^1 = \{(1, 1), (2, 2), (3, 3)\}, \quad x^2 = \{(2, 3), (1, 2), (3, 1)\},$

$x^3 = \{(3, 3), (1, 2), (2, 1)\}, \quad x^4 = \{(1, 2), (3, 3), (2, 1)\}.$

The core equals the set of Walrasian allocations $W(R) = Core(R) = \{x^3, x^4\}.$

In our last example the set of Walrasian allocations is empty while the core is nonempty.

**Example 5. No Walrasian Allocations and a Multi-Valued Core**
Consider $R$ such that

$$(1, 2) P_1 (3, 3) P_1 (2, 3) P_1 (1, 1) P_1 \text{ anything},$$

$$(3, 2) P_2 (1, 2) P_2 (3, 3) P_2 (2, 2) P_2 \text{ anything},$$

$$(1, 3) P_3 (3, 1) P_3 (2, 3) P_3 (2, 1) P_3 (3, 3) P_3 \text{ anything}.$$

The set of individually rational allocations equals $IR(R) = \{x^1, x^2, x^3, x^4, x^5\}$ with

$x^1 = \{(1, 1), (2, 2), (3, 3)\}, \quad x^2 = \{(1, 1), (3, 2), (2, 3)\},$

$x^3 = \{(2, 3), (1, 2), (3, 1)\}, \quad x^4 = \{(3, 3), (1, 2), (2, 1)\},$

$x^5 = \{(1, 2), (3, 3), (2, 1)\}.$

The core equals $Core(R) = \{x^2, x^3\}$ and the set of Walrasian allocations is empty.

The last two examples illustrate two features of the relationship between the core and the set of Walrasian allocations not yet recognized in the literature.

**Remark 2. New Insights on Walrasian Allocations through Examples 4 and 5**
Konishi et al. (2001) prove that the set of Walrasian equilibria is a subset of the core and that it might be empty if the core is also empty. With Example 5 we show that the set of Walrasian allocations might even be empty if the core is nonempty. Second, with Example 4 we provide a multiple-type housing market where the set of Walrasian allocations contains more than one allocation.

Moreover, the examples collected in this section demonstrate that the core, or the set of Walrasian allocations, are not necessarily satisfactory solution concepts for multiple-type housing markets.
Remark 3. Walrasian and/or Core Allocations as Solutions?
First, note that both standard static solution sets, the set of core allocations and the set of Walrasian allocations, can be empty\(^8\). Then, we do not have any (static) predictions to offer as to what will happen in the market. Will agents keep their endowments or will they trade? Second, the set of core allocations and the set of Walrasian allocations can be multi-valued. Then, again we cannot make (static) predictions which, if any, of the possible allocations in the core will result from trade. As discussed in the Introduction, we will explicitly model the dynamic aspect of the core through a dynamic recontracting process. By doing so, we hopefully can also address the role of core and Walrasian allocations as solutions of a dynamic trading process. △

Next, we model the dynamic recontracting process and characterize market outcomes via the respective set of recurrent classes (Section 4), stochastic stability (Section 5.1), and – last but not least – the limit invariant distribution (Section 5.2).

4 Unperturbed Dynamic Recontracting Processes

For each \( R \in \mathcal{R}^N \), the dynamic recontracting is modeled by a Markov Process \((X, M(R))\).\(^3\) The state space is given by the set of allocations \( X \) and \( M(R) \) is a transition matrix that describes the following dynamics. In each period \( t \), the process is at an allocation \( x(t) \in X \) and a coalition \( S \) of agents is randomly selected. The system moves from one allocation to another when the agents in \( S \) recontract among themselves, i.e., agree upon a redistribution of their endowments. Agents never make mistakes in the sense that they only recontract if they benefit from doing so by means of blocking. As we will allow for mistakes later on, we refer to the dynamic recontracting process discussed in this section as unperturbed. The following three assumptions are satisfied at each period.

**Assumption 1. Opportunities to Recontract**
Each coalition \( S \subseteq N \) is chosen with positive probability to recontract among themselves. A coalition that has the opportunity to recontract is an active coalition.

This rather mild assumption covers most of the common models of coalition formation. In particular, the probability with which a certain coalition has the opportunity to recontract can depend on the size of the coalition, the allocation in period \( t \), or the identities of the agents involved as long as the corresponding probability distribution has full support in the set of coalitions.

For each \( S \subseteq N \) and each \( x \in X \), let \( B_S(x) = \{ y_S \in X_S \mid \text{for each } i \in S, \ y_i R_i x_i \text{ and for some } j \in S, \ y_j P_j x_j \} \) be the set of blockings of \( S \) for \( x \).

**Assumption 2. Recontracting Behavior**
Given \( x(t) \), an active coalition \( S \subseteq N \) recontracts if \( B_S(x(t)) \neq \emptyset \). If \( |B_S(x(t))| > 1 \), then each \( y_S \in B_S(x(t)) \) is chosen with positive probability.

Agents are myopic in the sense that they agree upon a reallocation of their endowments if it is weakly improving in the subsequent period. Hence, they ignore the prospect of future reallocations along the path of the recontracting process.

\(^8\)The set of Walrasian allocations might even be empty while the set of core allocations is nonempty
The assumption on which weakly improving reallocation is chosen is again mild. It is only important that any such reallocation is chosen with positive probability. The probability itself can depend on the identities or even the preferences of the agents in the active coalition.

We now fix the allocation for agents that did not participate in the recontracting.

**Assumption 3. Allocations Resulting from Recontracting**

Let $S$ be the active coalition at period $t$. If $B_S(x(t)) = \emptyset$, then $x(t+1) = x(t)$. Otherwise, there exists $y_S \in B_S(x(t))$ agreed upon by $S$ and $x(t+1) = (y_S, x_S(t))$ if $x_S(t) \in X_S$ and $x(t+1) = (y_S, e_S)$ otherwise.\(^9\)

A Markov process $(X, M(R))$ that satisfies Assumptions 1, 2, and 3 is an *unperturbed dynamic recontracting process* or *u.d.r. process* for short. Serrano and Volij (2005, Section 7) consider this specification of the u.d.r. process for Shapley-Scarf economies $(\ell = 1).\(^{10}\)

An important characteristic of the u.d.r. process is its set of recurrent classes. A set $A \subseteq X$ is a *recurrent class* if it is a minimal set of allocations that once entered throughout the u.d.r. process is never abandoned, i.e., for each $x \in A$ and each $x' \notin A$, $M(x, x') = 0$.

Let $R \in \mathcal{R}^N$. Then, we denote the set of recurrent classes by $\mathcal{RC}(R) = \{A \subseteq X \mid$ for each $x \in A$ and each $x' \notin A, M(x, x') = 0\}$. Notice that $\mathcal{RC}(R)$ is a set of sets of allocations. It is convenient to also denote the set of allocations contained in a recurrent class by $RC(R) = \{x \in X \mid$ there exists $A \in \mathcal{RC}(R)$ with $x \in A\}$.

**Theorem 1. Recurrent Classes**

Let $(X, M(R))$ be an u.d.r. process. Then, the following holds.

1. $RC(R) \neq \emptyset$.
2. A recurrent class is a singleton if and only if it is a core allocation.
3. Each element of a recurrent class is an individually rational allocation.
4. $IR(R) \supseteq RC(R) \supseteq \text{Core}(R) \supseteq W(R)$.

*Proof.* Statement (i) follows from the finiteness of the Markov process (the finiteness of $X$).

Statement (ii) is a direct implication of the definition of the u.d.r. process and the properties of the core.

To prove Statement (iii), suppose $x(t) \notin IR(R)$ and $\{x(t)\} \in \mathcal{RC}(R)$. Recall that Assumption 1 implies that there is a positive probability that any singleton coalition $\{i\}$ is allowed to recontract. If $x(t)$ is not individually rational for agent $i$, then as soon as $\{i\}$ is chosen as active coalition, agent $i$ will recontract with himself and receive his endowment. Hence, $\{x(t)\}$ cannot form a singleton recurrent class. Moreover, $x(t)$ cannot be an element of a non-singleton recurrent class as agent $i$ would have had to agree on $x(t)$, as a member of an active coalition in a previous period.

Statement (iv): By (iii) each element of a recurrent class has to be individually rational. It is also clear that not every individually rational allocation has to be member of a recurrent class. This implies $IR(R) \supseteq RC(R)$. By (ii) each allocation in the core is in $RC(R)$.

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\(^9\)This assumption on the allocation of the complement of $S$ turns out to be important in Section 5, where we discuss its impact in detail.

\(^{10}\)We assume that a coalition can recontract using weak blocking. Alternatively, we could model recontracting by strong blocking. We discuss how this modification affects our results in the Conclusions (Section 6).
Hence, \( RC(R) \supseteq Core(R) \).\(^{11}\) Finally, \( Core(R) \supseteq W(R) \) follows from Konishi et al. (2001, Proposition 3.1).\(^{12}\)

Theorem 1 shows that the set of recurrent classes of an u.d.r. process deserves attention as a solution for multiple-type housing markets: for any multiple-type housing market \( R \in \mathcal{R}^N \), \( RC(R) \) is non-empty, consists only of individually rational allocations, and contains all core allocations. To demonstrate (i) the non-emptiness of \( RC(R) \), (ii) the coexistence of singleton recurrent classes (i.e., allocations in the core) and non-singleton recurrent classes, and (iii) the absence of selection between core-allocations, we continue with three of the examples introduced in Section 3.\(^{13}\)

In Example 1 we observe a multi-valued (non-core) recurrent class.

**Example 1 (continued). Empty Core and Non-Singleton Recurrent Class**

Recall that \( Core(R) = \emptyset \). By Theorem 1 (iii) only allocations in \( IR(R) = \{x^1, x^2, x^3, x^4\} \) are candidates to be elements of a recurrent class. Since \( x^1 \) can be blocked by any other allocation in \( IR(R) \) it can never be an element of a recurrent class. Hence, we are left with the allocations \( x^2, x^3, \) and \( x^4 \). Recall that we have a blocking cycle where \( x^3 \{1, 2\}-blocks x^2, x^4 \{1, 3\}-blocks x^3, \) and \( x^2 \{2, 3\}-blocks x^4 \). Hence, none of the individually rational allocations \( x^2, x^3, \) and \( x^4 \) can form a singleton recurrent class. Furthermore, each of the allocations in \( \{x^2, x^3, x^4\} \) can be reached from one another through (a sequence of) blocking(s) while once an allocation in \( \{x^2, x^3, x^4\} \) is reached no outside allocation can block. Therefore, \( \{x^2, x^3, x^4\} \) constitutes the only recurrent class. Hence, \( RC(R) = \{\{x^2, x^3, x^4\}\} \) and \( RC(R) = \{x^2, x^3, x^4\} \). \( \diamond \)

In Example 2 we have two recurrent classes (one of them equals the core and contains the unique Walrasian allocation, the other contains three non-core allocations).

**Example 2 (continued). The Set of Recurrent Classes Exceeds the Core**

Recall that \( Core(R) = \{x^5\} \). By Theorem 1 (iii) only individually rational allocations in \( IR(R) = \{x^1, x^2, x^3, x^4, x^5\} \) are candidates to be elements of a recurrent class. By Theorem 1 (ii), \( Core(R) = \{x^5\} \) is the only singleton recurrent class. Next, recall that the only difference between Examples 1 and 2 is that – loosely speaking – we added allocation \( x^5 \) to the agents’ preferences such that \( x^5 \) is now not only individually rational, but also the unique core allocation. However, none of the allocations \( x^2, x^3, \) or \( x^4 \) can be blocked by \( x^5 \) (or \( x^1 \)) while there is again the blocking cycle where \( x^3 \{1, 2\}-blocks x^2, x^4 \{1, 3\}-blocks x^3, \) and \( x^2 \{2, 3\}-blocks x^4 \). Hence, \( \{x^2, x^3, x^4\} \) forms (as in Example 1) the only non-singleton recurrent class. Hence, \( RC(R) = \{\{x^2, x^3, x^4\}, \{x^5\}\} \) and \( RC(R) = \{x^2, x^3, x^4, x^5\} \). \( \diamond \)

We conclude with a multiple-type housing market where the set of recurrent classes coincides with the set (of singleton sets) of core-allocations.

**Example 3 (continued). The Core Equals the Set of Recurrent Classes**

Recall that \( Core(R) = \{x^2, x^3, x^4\} \). By Theorem 1 (iii) only allocations in \( IR(R) = \{x^1, x^2, x^3, x^4\} \) are candidates to be elements of a recurrent class. Since \( x^1 \) can be blocked by any other allocation in \( IR(R) \) it can never be an element of a recurrent class. By Theorem 1 (ii), \( \{x^2\}, \{x^3\} \) and \( \{x^4\} \) are the only recurrent classes. Hence, \( RC(R) = \{\{x^2\}, \{x^3\}, \{x^4\}\} \) and \( RC(R) = Core(R) \). \( \diamond \)

\(^{11}\)Example 2 is a multiple-type housing market with \( IR(R) \supseteq RC(R) \supseteq Core(R) \).

\(^{12}\)Example 3 is a multiple-type housing market with \( Core(R) \supseteq W(R) \).

\(^{13}\)The recurrent classes of the other examples are determined spelled in Appendix A.
To summarize, even in the case of an empty core, \( RC(R) \) offers a prediction for the outcome of a multiple-type housing market (see Example 1). This, however, is achieved at the expense of a weakly larger set of final outcomes whenever the core is non-empty (see Example 2). This closely resembles the situation in the literature on evolutionary selection of Nash equilibria in, e.g., coordination games (Young, 1993): while every Nash equilibrium of the coordination game is also a singleton recurrent class of the unperturbed learning process, the set of recurrent classes typically exceeds the set of Nash equilibria.

5 Perturbed Dynamic Recontracting Processes

We will now – similarly as in the benchmark model of Serrano and Volij (2005) – perturb the dynamic recontracting process by allowing agents to make mistakes. This is a realistic modeling assumption in a dynamic recontracting model where agents trade over time; after all people tend to make mistakes once in a while. We follow the standard approach to Markov processes by assuming that at any given period, any agent in an active coalition can make a mistake when recontracting with probability \( \epsilon > 0 \) – making a mistake here means that an agent agrees on a reallocation that makes him worse off. We restrict perturbations to mistakes where agents agree individually rational allocations that make them worse off.\(^{14}\) One could argue that any active coalition knows that an individually irrational block is never sustainable because an agent who receives a bundle which is worse than his endowment later (with positive probability) will recontract with himself and then has the good senses to improve his bundle by enforcing his endowment.

From now on, and without loss of generality, we assume that the state space for any \( R \in \mathcal{R} \) equals the set of individually rational allocations, i.e., \( X = IR(R) \).

Assumption 4. The probability with which a member of an active coalition \( i \in S \) agrees on a reallocation \( y_{iS} \) with \( x_i(t) P_i y_i R_i e_i \) equals \( \epsilon > 0 \).

In particular, the probability of a mistake does not depend on the allocation or the agents’ identities. We call a dynamic recontracting process that satisfies Assumptions 1 – 4 a perturbed dynamic recontracting process, a p.d.r. process for short, and denote it by \((X, M'(R))\). The p.d.r. process is clearly ergodic: as mistakes induce (indirect) transitions between any two individually rational allocations, and non-individually rational allocations remain transient, its unique recurrent class is \( IR(R) \). Hence, the u.d.r. process exhibits a unique invariant distribution \( \mu_e \) with support \( IR(R) \) that displays the long-run probability distribution over allocations. Moreover, the perturbation is regular on \( IR(R) \), i.e., transition probabilities between any two individually rational allocations are non-zero and polynomials in \( \epsilon \). As every transition from an allocation \( x' \) to an allocation \( x'' \) can involve a maximum of \( n \) mistakes (by every agent in \( N \)) we can denote \( M'(x', x'') \equiv \sum_{k=0,...,n} m_k(x', x'') \epsilon^k \) where \( m_k(x', x'') \) captures the probability that a certain (set of) coalition(s) form(s) at allocation \( x' \) and agrees upon allocation \( x'' \) with \( k \) agents making a mistake. Note in particular that \( m_0(x', x'') = M(x', x'') \) – the intercept of the polynomial – is the respective entry in the transition matrix of the u.d.r. process.

\(^{14}\)The consequences of also allowing “individually irrational mistakes” would be the following. First, we would observe a significant enlargement of the state space that has to be taken into account in our later analysis (Subsection 5.2). Second, in all examples preferences over all bundles (also the individually irrational ones) would have to be specified and taken into account. Third, none of the general results described in Theorem 2 and 3 would continue to hold.
5.1 Stochastic Stability

Young (1993, Theorem 4 (i)) has shown that the limit invariant distribution \( \mu^* \equiv \lim_{\varepsilon \to 0} \mu_\varepsilon \) of a perturbed Markov process \((X, M')\) exists and that it is an invariant distribution of the u.d.r. process \((X, M)\) if the perturbation is regular, i.e., if Assumption 4 holds. The support of every invariant distribution of the unperturbed process is a (non-empty) collection of recurrent classes.\(^{15}\) Hence, not every recurrent class of the u.d.r. process has to be in the support of \( \mu^* \), but the support is a collection of recurrent classes. Allocations in the support of \( \mu^* \) are called stochastically stable. We denote the support of \( \mu^* \) of \( R \) by \( SRC(R) \) and refer to it as the set of stochastically stable allocations. For further reference we summarize as follows.

**Lemma 1. Young (1993, Theorem 4 (i))**

Let \((X, M(R))\) be an u.d.r. process. Then, \( SRC(R) \neq \emptyset \) and \( SRC(R) \subseteq RC(R) \).

Next, for each \( R \in \mathcal{R} \), we introduce a general methodology to determine the set of stochastically stable allocations \( SRC(R) \) and apply it to our examples.

**x-Trees** Consider the set of directed graphs that have vertex set \( X \). Then, any directed graph is defined by its set of directed edges. We denote a directed edge from \( x' \) to \( x'' \) by \([x', x'']\) and interpret it as \( x'' \) is the outcome of recontracting that started from \( x' \). Note that the irreducibility of the p.d.r. process implies that for any directed edge \([x', x'']\) we have that \( M^\varepsilon(R)(x', x'') > 0 \).

To characterize the stochastically stable allocations of a u.d.r. process \((X, M(R))\), for every \( x \in X \) we consider the set of (directed) spanning trees or \( x \)-trees \( T_x \). An \( x \)-tree \( T_x \) is a directed graph such that for every \( y \in X \) with \( x \neq y \) there is exactly one (cycle-free) sequence of edges (a directed path) from \( y \) to \( x \). Denote by \( T_x \) the set of all \( x \)-trees. As the p.d.r. process is irreducible, \( T_x \) is non-empty for all \( x \in X \).

**Stochastic Potential** Let \([x', x'']\) be an edge in an \( x \)-tree \( T_x \in T_x \). The edge-resistance \( r(x', x'') \) is the minimum number of mistakes needed to get directly from \( x' \) to \( x'' \), i.e., the minimal number of agents that are worse off through recontracting when actively participating in a blocking of allocation \( x' \) that results in allocation \( x'' \). Formally, \( r(x', x'') = \min\{|r \geq 0 | \infty > \lim_{\varepsilon \to 0} e^{-\varepsilon} M^\varepsilon(x', x'') > 0\} \). Finally, the stochastic potential of \( x \in X \), denoted by \( \gamma(x) \), is the minimal sum of edge-resistances over all \( x \)-trees, i.e., \( \gamma(x) = \min_{T_x \in T_x} \sum_{[x', x''] \in T_x} r(x', x'') \). We refer to an \( x \)-tree \( T_x \) that minimizes \( \sum_{[x', x''] \in T_x} r(x', x'') \) as a least resistance tree.

**Stochastic Stability** An allocation \( x \) is stochastically stable if and only if it minimizes the number of mistakes needed to reach the allocation through the p.d.r. process, i.e., if the allocation \( x \) minimizes \( \gamma(x) \). The set of stochastically stable allocations \( SRC(R) \) can be characterized as follows.

**Lemma 2. Young (1993, Theorem 4 (ii))**

Let \((X, M(R))\) be an u.d.r. process. Then, allocation \( x \in X \) is stochastically stable if and only if for each \( y \in X \), \( \gamma(x) \leq \gamma(y) \).

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\(^{15}\) Every recurrent class corresponds to exactly one invariant distribution and the set of all invariant distributions is the convex hull of the invariant distribution for all its recurrent classes (Appendix B).
Interpretation and Basic Implications  The set of recurrent classes of a dynamic recontracting process are internally because no allocation in a recurrent class can be blocked by an “outside allocation”. In the case of singleton recurrent classes this internal stability requirement coincides with the (internal) stability of the core. Stochastic stability also requires internal stability – \( SRC(R) \subseteq RC(R) \) – but in addition also considers the relative accessibility of recurrent classes, i.e., the number of mistakes agents need to make in order to reach a recurrent class from all other allocations. This notion of accessibility is captured by the stochastic potential. By also imposing a measure of accessibility, stochastic stability might serve as a selection device among recurrent classes and among core allocations.

Note that allocations in the same recurrent class have the same stochastic potential. To see this consider two allocations \( x \) and \( y \) in the same recurrent class. Now take any least resistance \( x \)-tree \( \tilde{T}_x \). Obviously, there exists a path from \( y \) to \( x \) solely consisting of edges with zero resistance, and such a path has to be part of \( \tilde{T}_x \) (otherwise \( \tilde{T}_x \) would not be resistance minimizing). Likewise, there also exists a path from \( x \) to \( y \) solely consisting of edges with zero resistance. But then we can construct a \( y \)-tree \( \tilde{T}_y \) with \( \sum_{[x',x''] \in \tilde{T}_y} r(x',x'') \) which implies that \( \gamma(x) \geq \gamma(y) \). Graphically one can obtain the \( y \)-tree \( \tilde{T}_y \) by first drawing the zero resistance path from \( x \) to \( y \) and then reattaching the missing vertices by using only branches of the original \( x \)-tree \( \tilde{T}_x \) — thus \( \tilde{T}_y \) is obtained from \( \tilde{T}_x \) by “tree surgery”. As the argument to obtain \( \gamma(x) \geq \gamma(y) \) is symmetric with respect to \( x \) and \( y \), it follows that \( \gamma(x) = \gamma(y) \).

This “tree-surgery argument” can also be applied to make the following point. Consider two allocations \( x \) and \( y \) that are members of two different recurrent classes and suppose for simplicity that there is a least resistance \( x \)-tree with a direct edge \([y,x] \). This tree can be transformed into a \( y \)-tree by reversing that edge. As \( x \) and \( y \) are members of two different recurrent classes it follows that \( r(x,y) > 0 \) and \( r(y,x) > 0 \). If, however, \( r(y,x) > r(x,y) \) we have just constructed a \( y \)-tree with a tree-resistance \( \sum_{[x',x''] \in \tilde{T}_y} r(x',x'') \) which implies \( \gamma(y) < \gamma(x) \) and, by Lemma 2, \( x \not\in SRC(R) \). Hence, if the resistance minimizing link (or, more generally, path) from \( y \) to \( x \) has a higher resistance than a link (or path) from \( x \) to \( y \), the recurrent class that includes \( x \) cannot be stochastically stable. Note that this observation, which is also the underlying idea behind Ellison’s (2000) radius/coraduis formalism, only provides a sufficient condition for stochastic stability. However, it illustrates that stochastic stability is driven by the trade-off between stability of a recurrent class and the respective accessibility from other recurrent classes of the process.

Finally, note that the stochastic potential is only determined by the resistance of a tree in the set of least resistance trees. Consider, for instance, \( R \in \mathcal{R} \) with \( RC(R) = \{ \{x\}, \{y\} \} \). To establish stochastic stability of \( \{x\} \) we only have to find one \( x \)-tree with tree-resistance \( \gamma(y) \). The number of least resistance trees for either \( \{x\} \) or \( \{y\} \) does not influence the result. This already suggests that the stochastic potential does not capture all aspects of accessibility and stability of allocations. We return to this observation in Subsection 5.2.

Computation  To compute \( SRC(R) \), Lemma 2 offers the following procedure.

1. Determine the set of recurrent classes \( RC(R) \).
2. Select an allocation \( x \) for every recurrent class and construct an \( x \)-tree that minimizes \( \sum_{[x',x''] \in T} r(x',x'') \).
Because of the identity of stochastic potentials within a given recurrent class, we simply refer to $\gamma(x)$ as the stochastic potential of the recurrent class that incudes $x$. The set of stochastically stable recurrent classes is the set of all allocations in the recurrent classes that have minimal stochastic potential.

To illustrate our methodology, we apply it to the examples discussed in Section 4.

**Example 1 (continued).** *Empty Core and Non-Singleton SRC(R)*
Recall that the only recurrent class is $\{x^2, x^3, x^4\}$. Hence by Lemmas 1 and 2, and the observation that allocations in the same recurrent class have the same stochastic potential, $SRC(R) = \{x^2, x^3, x^4\} = RC(R)$.

The following examples illustrate how to compute the stochastic potential.

**Example 2 (continued).** *SRC(R) = RC(R) $\supseteq$ Core(R)*
Recall that the recurrent classes are $\{x^2, x^3, x^4\}$ and $\{x^5\} = Core(R)$. The only other individually rational allocation is $x^1$. Moreover, all allocations in $\{x^2, x^3, x^4\}$ have the same stochastic potential. Hence, we can restrict ourselves, for instance, to the construction of a resistance minimizing $x^2$-tree. In any least resistance tree, allocations of the blocking cycle have to be connected with mistake-free edges. Note furthermore that $x^2$ can be reached from $x^1$ (and $x^3, x^4$) without mistake. Hence, we are left with $x^5$. Observe that every allocation in $\{x^2, x^3, x^4\}$ can be reached from $x^5$ with one mistake ($\{2, 3\}$ can recontract to obtain $x^2$ with a mistake by agent 2, $\{1, 2\}$ can recontract to obtain $x^3$ with a mistake by agent 1, and $\{1, 3\}$ can recontract to obtain $x^4$ with a mistake by agent 3) and there can be no other (direct or indirect) path through recontracting with fewer mistakes. Therefore $\overline{T}_{x,2} = \{[x^1, x^2], [x^2, x^3], [x^3, x^4], [x^4, x^2]\}$ is a resistance minimizing $x^2$-tree and $\gamma(x^2) = 1$.

As for $\gamma(x^5)$ observe that the grand coalition can directly recontract at any of the allocations in $\{x^2, x^3, x^4\}$ to $x^5$ with only one mistakes (starting from $x^2$ agent 3 makes a mistake, starting from $x^3$ agent 2 makes a mistake, and starting from $x^4$ agent 1 makes a mistake). As $\{x^2, x^3, x^4\}$ is a recurrent class, this is also the minimal edge (or path) resistance. Moreover, $x^5$ can be reached from $x^1$ without mistakes. Therefore $\overline{T}_{x,5} = \{[x^2, x^3], [x^3, x^4], [x^4, x^5], [x^1, x^5]\}$ is a resistance minimizing $x^5$-tree and $\gamma(x^5) = 1$. Thus, $\gamma(x^2) = \gamma(x^5)$ and $SRC(R) = \{x^5, x^2, x^3, x^4\} = RC(R)$.

**Example 3 (continued).** *SRC(R) = RC(R) = Core(R) and $|Core(R)| > 1$*
Recall that the recurrent classes only contain the core allocations $x^2, x^3$, and $x^4$. Note that whenever the p.d.r. process is in one of these allocations, each single agent can recontract with himself to obtain $x^1$ by making one mistake. Furthermore, any of the allocations in $\{x^2, x^3, x^4\}$ $\{1, 2, 3\}$-blocks $x^1$. Hence, recontracting between any two recurrent classes can be achieved with one mistake. Therefore, $\overline{T}_{x,2} = \{[x^1, x^2], [x^3, x^1], [x^4, x^1]\}$ is a resistance minimizing $x^2$-tree, $\overline{T}_{x,3} = \{[x^1, x^3], [x^2, x^1], [x^4, x^1]\}$ is a resistance minimizing $x^3$-tree, and $\overline{T}_{x,4} = \{[x^1, x^4], [x^2, x^1], [x^3, x^1]\}$ is a resistance minimizing $x^4$-tree, $\gamma(x^2) = \gamma(x^3) = \gamma(x^4) = 2$, and $SRC(R) = \{x^2, x^3, x^4\}$.

The last example nicely illustrates a weakness of stochastic stability as a selection device for recurrent classes. Observe that a direct transition from $x^2$ to $x^3$ needs two mistakes while all other direct transitions between core allocations only ask for one mistake. This indicates that $x^2$ is more difficult to exit for the p.d.r. process than other core allocations, and $x^3$ is more difficult to access. This suggests $x^2$ as a more accessible and more stable allocation,
or simply a better prediction for the final allocation. But due to the indirect paths from one recurrent class via \( x^1 \) to another recurrent class, all core allocations have a stochastic potential of 2 and the conjectured superiority of \( x^2 \) can not be established. We formalize this observation as our first main result.

**Theorem 2. Stochastic Stability and Recurrent Classes**

Let \((X, M(R))\) be an u.d.r. process. Then,

(i) for each \( x \in RC(R) \), \( \gamma(x) = |RC(R)| - 1 \) and

(ii) \( RC(R) = SRC(R) \).

We first prove the following auxiliary result.

**Claim 1.** For each pair \( \{x, x'\} \subseteq IR(R) \), there is a directed sequence of edges from \( x \) to \( x' \) with at most one mistake.

**Proof of Claim 1.** Observe that the claim is trivially fulfilled if \( x = x' \). Hence, assume that \( x \neq x' \). To prove the claim we have to distinguish three cases. First, suppose that \( x = e \). Then individual rationality of \( x' \) implies that \( r(x, x') = 0 \). Second suppose that \( x' = e \). As \( x \neq e \) there has to be a pair of agents \( i, j \in N \) that have to trade to achieve allocation \( x \) (i.e., there is a commodity \( l \) with \( x_j(l) = i \)). When the singleton coalition \( \{i\} \) forms, agent \( i \) can claim back his endowment of commodity \( l \) without the consent of the other agents. This requires at one mistake (by agent \( i \)). As \( x_{N \setminus \{i\}} \) is no longer feasible (recall \( x_j(l) = i \)), all agents in \( N \setminus \{i\} \) receive their endowment. Hence, we have established an edge from \( x \) to \( x' = e \) with a resistance of at most one (such that \( r(x, x') \leq 1 \)). Finally suppose that neither \( x \) nor \( x' \) equals the endowment \( e \). Then the previous findings imply that the sequence of edges \( \{ [x, e], [e, x'] \} \) constitutes a path of resistance of at most one, which concludes the proof of the claim.

**Proof of Theorem 2.** Using Claim 1, it is now easy to construct least-resistance trees.

Let \( R \in \mathcal{R} \) and denote the respective (finite) set of recurrent classes by \( \mathcal{RC}(R) = \{ \mathcal{RC}_1 \}, \ldots, \{ \mathcal{RC}_m \} \). It follows directly from the definition of a recurrent class that \( \gamma(x) \geq (m - 1) \) for each allocation \( x \in X \). Next, we prove that for all \( x \in RC(R) \), \( \gamma(x) \leq (m - 1) \) which implies \( \gamma(x) = (m - 1) \) for all \( x \in RC(R) \) (Statement (ii)) and implies – together with Lemma 2 – Statement (i).

For any allocation \( x \in RC(R) \) we construct an \( x \)-tree as follows. Consider first the recurrent class \( \mathcal{RC}_k \) with \( x \in \mathcal{RC}_k \). By the definition of a recurrent class \( r(x', x) = 0 \) for any \( x' \in \mathcal{RC}_k \), i.e., there exists a set of edges that constitute the restriction of an \( x \)-tree to \( \mathcal{RC}_k \) and has zero resistance. Suppose that all \( x' \in \mathcal{RC}_k \) are connected through that set of edges. Then pick another recurrent class \( \mathcal{RC}_l \neq \mathcal{RC}_k \) and consider an allocation \( y \in \mathcal{RC}_l \). Again, the definition of a recurrent class implies that \( r(y', y) = 0 \) for each \( y' \in \mathcal{RC}_l \) such that there exists the restriction of a \( y \)-tree to \( \mathcal{RC}_l \) which has zero resistance. Suppose that all \( y' \in \mathcal{RC}_l \) are connected through that tree. Repeat this procedure for all remaining recurrent classes. With this procedure, we have constructed graph of resistance zero that connects all allocations within a given recurrent class but does not establish edges between recurrent classes. Claim 1, however, implies that \( r(y, x) \leq 1 \) for any \( x, y \in X \). Hence, for every recurrent class \( \mathcal{RC}_m \in \mathcal{RC}(R) \) there is an allocation \( z \in \mathcal{RC}_m \) and a path from \( z \) to \( x \) which has a resistance of at most one. If we establish such a path between the respective allocations in each recurrent class and \( x \), we have constructed an \( x \)-tree with a resistance of at most
\(m - 1\) mistakes. Hence, \(\gamma(x) = m - 1\) (Statement (i)). But then Lemma 2 implies that \(SRC(R) = RC(R)\). \(\square\)

This result offers two insights. First, the requirement of stochastic stability does not work as a selection device for recurrent classes. In addition to coalitional stability, the concept of stochastic stability also incorporates a notion of coalitional accessibility of recurrent classes (i.e., it compares the number of mistakes needed to reach an allocation in a certain recurrent class from all other recurrent classes) as captured in the stochastic potential. However, as emphasized before, this notion of accessibility is rather limited as it only values the existence of least resistance trees, but not their number and overall probability (see e.g., the different stability and accessibility features of allocations \(x^2\) and \(x^3\) in Example 3). This limitation drives the lack of selective power of stochastic stability as documented in Theorem 2.

Second, the prominent role of paths via \(x^1\) in Example 3 and the proof of Theorem 2 points at a modeling problem of the type of recontracting processes we investigate. (Serrano and Volij, 2005, Footnote 6) argue that their recontracting processes are robust to alternative specifications. In particular it is robust to the requirement that any agent in \(N \setminus S\) whose bundle remain feasible keep it instead of being automatically sent back to their endowments. We see here that this robustness may not hold for the extension to multiple-type housing markets with \(\ell > 1\). Notice that when preferences are strict, Theorem 2 fails only if \(n > 3\). Thus the lack of selection in our examples remains even if we modify Assumption 3. Notice also that if \(n > 3\) but \(RC(R)\) contains no cycle, Theorem 2 remains also unaffected by a change in Assumption 3. Moreover, in order to compare our results to the “Serrano and Volij (2005) benchmark” we decided to adopt their modeling choice with respect to how allocations form. Then, Theorem 2 (ii) indicates that the selective power of stochastic stability as established by Serrano and Volij (2005) for housing markets with indifferences does not carry over to multiple-type housing markets with strict preferences – unlike one could have conjectured.

The following example (a modified version of Example 2) shows that when the assumption of strict preferences is relaxed in the multiple-type housing model, then stochastic stability may regain its discrimination power among recurrent classes.

**Example 6. Weak Preferences and SRC(R) \(\subsetneq\) RC(R)**

Reconsider Example 2 with preferences modified as follows (agent 1’s preferences are not strict anymore)

\[
\begin{align*}
(3, 1) & \ P_1 (2, 2) P_1 (1, 2) I_1 (1, 1) P_1 \ anything, \\
(2, 1) & \ P_2 (3, 3) P_2 (3, 2) P_2 (2, 2) P_2 \ anything, \\
(2, 3) & \ P_3 (1, 1) P_3 (1, 3) P_3 (3, 3) P_3 \ anything.
\end{align*}
\]

Recall that \(IR(R) = \{x^1, x^2, x^3, x^4, x^5\}\) with

\[
\begin{align*}
x^1 & = \{(1, 1), (2, 2), (3, 3)\}, \quad x^2 = \{(1, 1), (3, 2), (2, 3)\}, \\
x^3 & = \{(1, 2), (2, 1), (3, 3)\}, \quad x^4 = \{(3, 1), (2, 2), (1, 3)\}, \\
x^5 & = \{(2, 2), (3, 3), (1, 1)\}.
\end{align*}
\]

Next, we determine the set of recurrent classes. When moving from an allocation \(y\) to an allocation \(z\), if agents involved in the transition are all left indifferent, this should be counted
as mistakes since weak blocking requires that at least one agent be made strictly better off. Nonetheless, mistakes that leave agents indifferent may put less weight to as mistakes since weak blocking requires that at least one agent be made strictly better off, e.g., if the u.d.r. process is in allocation $x^3$ and the singleton coalition $\{1\}$ becomes active, the u.d.r. process still does not permit a transition to $x^1$ – even though agent 1 is indifferent. Hence, $\mathcal{RC}(R) = \{\{x^2, x^3, x^4\}, \{x^5\}\}$. However, if a lower weight is put on this mistake by agent 1 than e.g., on the mistake by the same agent to ask for his endowment when the process is at allocation $x^5$ (which leaves him with the bundle $(1, 1)$ instead of the strictly preferred bundle $(2, 2)$) we end up with a modified analysis of stochastic stability.

One of the least resistance tree for $x^2$ is $\bar{T}_{x^2} = \{[x^3, x^4], [x^4, x^2], [x^5, x^1], [x^1, x^2]\}$. Only the transition from $x^5$ to the endowment involves a mistake that makes agent 1 worse-off. On the other hand, $\bar{T}_{x^5} = \{[x^4, x^2], [x^2, x^3], [x^3, x^1], [x^1, x^5]\}$ is the unique least-resistance tree for allocation $x^5$. The only mistake is indifference-based. When going from $x^3$ to $x^1$, agent 1 claims his endowment back and the complement $N \setminus 1$ is sent back to its respective endowment. But agent 1 is indifferent between the bundles $(1, 2)$ and $(1, 1)$. In consequence $\gamma(x^5) < \gamma(x^2)$. The only stochastically stable allocation is $x^5$ because it is easier to access $x^5$ from the cycle than to access the cycle from $x^5$.

This example demonstrates how the introduction of a single indifference leads to a breakdown of Theorem 2. Hence, small manipulations of preferences lead to major changes in the set of solutions.

This section has shown that stochastic stability does not serve as a proper selection device for recurrent classes in our setting. Either, stochastic stability does not shrink the set of recurrent classes or the respective selection is highly sensitive to small manipulations of the underlying preferences.

The consequences of Theorem 2 clearly motivate the search for a more robust solution concept that can deliver sharper predictions on allocations that will be visited by the system in the long-run. We take one further step in the analysis of stability and accessibility of allocations. In the following subsection we will analyze the limit invariant distribution which captures not only the number of mistakes needed to switch between recurrent classes but also accounts for details of stability and accessibility of an allocation such as the number of coalitions that can agree or decide to improve upon it.

### 5.2 The Limit Invariant Distribution

So far we have offered a method to identify the set of stochastically stable allocations $\text{SRC}(R)$, i.e., the support of the limit invariant distribution $\mu^*(R) = \lim_{\epsilon \to 0} \mu_{\epsilon}(R)$. But the fact that an allocation $x \in X$ is in the support of the limit invariant distribution (i.e., $\mu^*(x) > 0$) does of course not indicate anything about the actual probability that the system will be in allocation $x \in X$ in the long run. This is captured by the value of $\mu^*(x)$. The next theorem presents a method to assess the limit invariant distribution that makes explicit use of the

\footnote{One way to implement this would be to follow Serrano and Volij (2005) and let $r(y, z) = |S|/|N|$ when all agents involved in $S$ in the transition from $y$ to $z$ are left indifferent. Otherwise, if at least one agent in $S$ is made worse-off in the transition from $y$ to $z$, then $r(y, z)$ is simply the number of mistakes in which agents were made worse-off. Formally, this requires an analysis of state-dependent mistakes as spelled out in Serrano and Volij (2005).}
polynomial structure of \(M^\epsilon(x', x'') = \sum_{k=0,...,n} m_k(x', x'') \epsilon^k\). For further reference we define a least resistance transition \(\tilde{m}(x', x'')\) as the component \(m_k(x', x'') > 0\) that minimizes \(k\) (i.e., for any two allocations \(x'\) and \(x''\), \(\tilde{m}(x', x'')\) depicts the respective transition probability with the smallest possible number of mistakes).

**Theorem 3.** For each p.d.r. process \((X, M^\epsilon(R))\) it holds that \(\mu^*(x) = \frac{p(x)}{\sum_{y \in X} p(y)}\) with \(p(x) = \sum_{\{T \in \mathcal{T}_x | \Pi_{[x', x''] \in T} r(x', x'') = \gamma(x)\}} \Pi_{[x', x''] \in T} \tilde{m}(x', x'')\).

**Proof.** Freidlin and Wentzell (2004) have shown that the (unique) invariant distribution of an irreducible Markov chain \((X, M)\) is given by \(\mu(x) = \frac{q(x)}{\sum_{y \in X} q(y)}\) with \(q(x) = \sum_{T \in \mathcal{T}_x} \Pi_{[x', x''] \in T} M(x', x'')\). If the Markov process is a p.d.r. process \((X, M^\epsilon(R))\) such that \(M^\epsilon(x', x'') = \sum_{k=0,...,n} m_k(x', x'') \epsilon^k\), \(q(x)\) is a polynomial of degree \(n(|X| - 1)\) in the mistake probability \(\epsilon\) (there can be only one mistake per agent per edge, and there are not more than \((|X| - 1)\) edges in a spanning tree in \(X\)), i.e., we can rewrite \(q(x) \equiv \sum_{k=0,...,n(|X| - 1)} q_k(x) \epsilon^k\).

By Theorem 2 (ii), \(\lim_{\epsilon \to 0} q(x) > 0\) if and only if \(x \in RC(R)\). Moreover, \(q_k(x) = 0\) for all \(k < \gamma(x)\) and \(q_{\gamma(x)}(x) > 0\) (by definition of the stochastic potential there are no spanning trees with less than \(\gamma(x)\) mistakes). Finally, observe that \(q_{\gamma(x)}(x) = \sum_{\{T \in \mathcal{T}_x | \Pi_{[x', x''] \in T} r(x', x'') = \gamma(x)\}} \Pi_{[x', x''] \in T} \tilde{m}(x', x'')\) (a contribution to \(q(x)\) of lowest order in \(\epsilon\) is the sum over least resistance spanning trees i.e., trees that only contain least resistance edges).

Consider a multiple-type housing market with one recurrent class. Then, for all \(x \in RC(R)\), \(\lim_{\epsilon \to 0} q(x) = q_0(x) > 0\) and \(\mu^* = \lim_{\epsilon \to 0} \mu^\epsilon(x) = \frac{q_0(x)}{\sum_{y \in RC(R)} q_0(y)}\). When \(q_0(x) = \sum_{\{T \in \mathcal{T}_x | \Pi_{[x', x''] \in T} r(x', x'') = 0\}} \Pi_{[x', x''] \in T} m_0(x', x'')\), which equals \(p(x)\) for a unique recurrent class (or \(\gamma(x) = 0\)).

Suppose from now on that \(|RC(R)| \geq 2\). Recall from Theorem 2 (i) that \(\gamma(x) = (|RC(R)| - 1)\). From \(q_j(x) > 0\) for all \(x \in RC(R)\) it follows that for all \(j = 0, \ldots, (\gamma(x) - 1)\), \(\lim_{\epsilon \to 0} \frac{\partial q_j(x)}{\partial \epsilon^j} = 0\) and \(\lim_{\epsilon \to 0} \frac{\partial \gamma(x) q_j(x)}{\partial \epsilon^j} = \frac{1}{\gamma(x)} q_j(x)\) (which is strictly positive if and only if \(x \in RC(R)\)). Then, iterated application of de l’Hospitals rule implies that

\[
\mu^*(x) = \lim_{\epsilon \to 0} \mu^\epsilon(x) = \lim_{\epsilon \to 0} \frac{q(x)}{\sum_{y \in X} q(y)} = \frac{\lim_{\epsilon \to 0} \frac{\partial \gamma(x) q(x)}{\partial \epsilon^j} q_j(x)}{\lim_{\epsilon \to 0} \sum_{y \in RC(R)} q_j(x)} = \frac{q_j(x)}{\sum_{y \in RC(R)} q_j(x)}.
\]

\(\square\)

**Computation** Theorem 3 offers the following recipe to elicit the limit invariant distribution \(\mu^*(x)\) for an allocation \(x \in RC(R)\).

1. Construct all least-resistance \(x\)-trees (i.e., determine \(\{T \in \mathcal{T}_x | \Pi_{[x', x''] \in T} r(x', x'') = \gamma(x)\}\). By Theorem 2 (i) this amounts to a construction of all trees of resistance \(|RC(R)| - 1\).

2. Compute the product of transition probabilities for all edges in a given least-resistance \(x\)-tree (i.e., \(\Pi_{[x', x''] \in T} \tilde{m}(x', x'')\)).

3. Sum over all least resistance \(x\)-trees.
We illustrate the method by revisiting some of our examples. Our findings are the second main result of the paper. We start with Example 4 to show that the limit invariant distribution serves as a selection device among core (and Walrasian) allocations.

**Example 4 (continued). Multiple Walrasian Allocations**

Recall
\[(1, 2) P_1 (3, 3) P_1 (2, 3) P_1 (1, 1) P_1 \text{ anything}, \]
\[(1, 3) P_2 (1, 2) P_2 (3, 3) P_2 (2, 2) P_2 \text{ anything}, \]
\[(2, 1) P_3 (3, 1) P_3 (3, 3) P_3 \text{ anything}, \]

\[IR(\mathcal{R}) = \{x^1, x^2, x^3, x^4\}\]

\[x^1 = \{(1, 1), (2, 2), (3, 3)\}, \quad x^2 = \{(2, 3), (1, 2), (3, 1)\}, \]
\[x^3 = \{(3, 3), (1, 2), (2, 1)\}, \quad x^4 = \{(1, 2), (3, 3), (2, 1)\}, \]

and \[RC(\mathcal{R}) = SRC(\mathcal{R}) = Core(\mathcal{R}) = W(\mathcal{R}) = \{x^3, x^4\}.\]

Theorem 2 (ii) furthermore indicates that \[\gamma(x^3) = \gamma(x^4) = 1\] (i.e., every least resistance tree includes only one mistake). But at least one mistake is needed for every edge that leads away from one of the core allocations. The only edges leading away from \(x^3\) with not more than one mistake are given by the grand coalition agreeing upon \(x^4\) (with a mistake by agent 2) or all three singleton coalitions that by mistake block via \(x^1\). In the same way, \(x^3\) can only be left towards \(x^4\) (with a blocking by the grand coalition and a mistake by agent 1) or towards \(x^1\) with mistakes by the respective singleton coalitions. The only edges with zero mistakes – that have to be exploited to form a spanning tree in the space of allocations – are from \(x^1\) to any other individually rational allocation and from \(x^2\) to \(x^3\) (all by the grand coalition). Figure 1 depicts the three different types of \(x^3\)-least resistance trees. Two more trees can be constructed by attaching \(x^1\) to \(x^2\) and \(x^4\), respectively, instead of attaching it to \(x^3\) in the left tree. We compare this with the set of \(x^4\)-least resistance trees. As depicted in Figure 2, the tree on the left again symbolizes three different trees through the possible ways to attach \(x^1\). Because \(x^3\) can \(\{1, 2, 3\}\)-block \(x^2\) with no mistake, but \(x^4\) cannot, there is no \(x^4\)-analogue to the middle tree in Figure 1.

![Figure 1: Least Resistance \(x^3\)-trees in Example 4](image)

---

\(^{17}\)For each of the example, we recapitulate the complete set of preferences and of allocations to help the reader follow the reasoning construction of the limit invariant distribution.
According to Theorem 3, 
\[
\frac{\mu^*(x^3)}{\bar{m}(x^2, x^3)} = \frac{\bar{m}'(x^4, x^3)(\bar{m}(x^1, x^2) + \bar{m}(x^1, x^3) + \bar{m}(x^1, x^4) + \bar{m}(x^4, x^1)(\bar{m}(x^1, x^3) + \bar{m}(x^1, x^2)))}{\bar{m}(x^2, x^3)},
\]
and 
\[
\frac{\mu^*(x^4)}{\bar{m}(x^2, x^3)} = \bar{m}(x^1, x^2) + \bar{m}(x^1, x^3) + \bar{m}(x^1, x^4) + \bar{m}(x^3, x^1)\bar{m}(x^1, x^4).
\]

Now recall that the probability to make a mistake does not depend on the identity of the agent (see Assumption 4) and suppose that the probability for a certain coalition to form does not depend on the current allocation. Then, \(\bar{m}(x^3, x^4) = \bar{m}(x^4, x^3)\) and \(\bar{m}(x^3, x^1) = \bar{m}(x^4, x^1)\). Finally, suppose that \(\bar{m}(x^1, x^3) = \bar{m}(x^1, x^4)\) (i.e., the grand coalition agrees upon \(x^3\) and \(x^4\) if the p.d.r. process is in \(x^1\) with the same probability). Then, \(\mu^*(x^3) > \mu^*(x^4)\) such that the system is more likely to be found in allocation \(x^3\) then in \(x^4\). Observe that this result is driven by the fact that \(x^3\) can be agreed upon by the grand coalition if the system is in allocation \(x^2\) while this is not feasible for allocation \(x^4\). This difference in mistake-free edges – or accessibility – leads to a larger set of least resistance \(x^3\)-trees and thereby enhances the respective component in the limit invariant distribution.

\[\Box\]

This example shows that the various core (Walrasian) allocations of a multiple-type housing market do not necessarily have the same probabilities in the long-run. Accordingly, one may wonder whether core (Walrasian) allocations are at least more likely final allocations than non-core (non-Walrasian) allocations. To elaborate on this point we reconsider Example 3.

**Example 3 (continued).** \(\text{SRC}(R) = \text{RC}(R) = \text{Core}(R) \supseteq \text{W}(R)\)

Recall 
\[
(1, 2) P_1 (3, 3) P_1 (2, 3) P_1 (1, 1) P_1 \text{ anything},
\]
\[
(1, 3) P_2 (2, 1) P_2 (3, 3) P_2 (2, 2) P_2 \text{ anything},
\]
\[
(3, 1) P_3 (2, 1) P_3 (3, 3) P_3 \text{ anything},
\]
\[
IR(R) = \{x^1, x^2, x^3, x^4\} \text{ with}
\]
\[
x^1 = \{(1, 1), (2, 2), (3, 3)\}, \quad x^2 = \{(2, 3), (1, 2), (3, 1)\},
\]
\[
x^3 = \{(1, 2), (3, 3), (2, 1)\}, \quad x^4 = \{(3, 3), (1, 2), (2, 1)\},
\]
the core consists of the allocations \(\{x^2, x^3, x^4\}\), and the unique Walrasian allocation is \(\{x^3\}\) such that \(\text{RC}(R) = \text{SRC}(R) = \text{Core}(R) \supseteq \text{W}(R)\).
Observe that a direct edge from $x^2$ to $x^3$ takes two mistakes while all other transitions from one core allocation to another need only one mistake. This suggests that $x^2$ is less likely to be left than $x^3$ and $x^4$ (i.e., is more stable) while $x^3$ is less likely to be reached by the p.d.r. process than $x^2$ and $x^4$ (i.e., is less accessible). In Appendix A we prove that this intuition is true and indeed $\mu^*(x^2) > \mu^*(x^4) > \mu^*(x^3)$ if the probability that a certain coalition forms does not depend on the allocation and a given coalition agrees upon each improvement with the same probability. Hence, $x^2$ is the most accessible and stable allocation while the $x^3$, the (unique) Walrasian allocation, is the worst prediction for the long-run behavior of the u.d.r. process.

The next example shows that the limit invariant distribution does not only allow for a distinction between different singleton recurrent classes but does also allow for an analysis of the probability that a recontracting process ends up in a cycle of individually rational allocations.

**Example 2 (continued).** $SRC(R) = RC(R) \supseteq Core(R)$ Recall

\[(3, 1) P_1 (2, 2) P_1 (1, 2) P_1 (1, 1) P_1 \text{ anything,} \]
\[(2, 1) P_2 (3, 3) P_2 (3, 2) P_2 (2, 2) P_2 \text{ anything,} \]
\[(2, 3) P_3 (1, 1) P_3 (1, 3) P_3 (3, 3) P_3 \text{ anything,} \]

$IR(R) = \{x^1, x^2, x^3, x^4, x^5\}$ with,
\[x^1 = \{(1, 1), (2, 2), (3, 3)\}, \quad x^2 = \{(1, 1), (3, 2), (2, 3)\}, \]
\[x^3 = \{(1, 2), (2, 1), (3, 3)\}, \quad x^4 = \{(3, 1), (2, 2), (1, 3)\}, \]
\[x^5 = \{(2, 2), (3, 3), (1, 1)\}, \]

and $Core(R) = W(R) = \{x^5\}$ and $RC(R) = SRC(R) = \{x^2, x^3, x^4, x^5\}$.

Theorem 2 (ii) indicates that $\gamma(x^2) = \gamma(x^3) = \gamma(x^4) = \gamma(x^5) = 1$. This implies that every least resistance tree can only contain one mistake, which is also necessary to connect –directly or indirectly– the cycle and the core-allocation. As a consequence, $x^2$, $x^3$, and $x^4$ have to be connected with the mistake-free and unique recontracting-cycle in any least resistance tree. Any $x^5$-tree needs a sequence of edges from an element of the cycle to $x^5$ with one mistake. Figure 3 depicts examples of the two distinct types of least resistance $x^5$-trees. Either, an element of a cycle is directly connected with $x^5$ (i.e., the grand coalition forms and one agent makes a mistake – see Tree (a)) or the cycle-element is connected with $x^1$ (a singleton coalition forms and the respective agents makes one mistake) and the grand coalition agrees upon $x^5$ afterwards (see Tree (b)). The different permutations of cycle elements and the feasible connections between $x^1$ and the rest of the tree lead to 15 trees in total (12 of type (a) and 3 of type (b)) that are listed in Appendix A.

Likewise any least resistance tree of a cycle-element (for instance $x^2$) has to include a (sequence of) edge(s) from $x^5$ to an element of the cycle. Figure 4 depicts examples of the two distinct types of least resistance $x^2$-trees. Either, $x^5$ is directly connected with an element of the cycle (i.e., a coalition of two agents forms and one of these agents makes a mistake – see Tree (a)) or $x^5$ is connected with $x^1$ (a singleton coalition forms, the respective agent makes one mistake, and the grand coalition or $\{2, 3\}$ agree upon $x^2$ afterwards) which is depicted in
Figure 3: Least Resistance $x^5$-trees

Figure 4: Least Resistance $x^2$-trees

Tree (b). The different permutations of cycle elements and the feasible connections between $x^1$ and the rest of the tree lead to 15 trees in total (12 of type (a) and 3 of type (b)) that are listed in the Appendix.

In contrast to Example 4 (see above) the number of least resistance trees for $x^2$ and $x^5$ is identical. However, different edges turn out to be bottle-necks. In the following, we present an informal discussion of the relevant effects - a complete treatment is relegated to Appendix A. To start out, observe that $x^2$ and $x^5$-trees differ in three aspects. First, any least resistance $x^5$-tree of type (a) in Figure 3 contains a blocking of a cycle allocation by the grand coalition to reach $x^5$, while $x^2$-trees of type (a) in Figure 4 need a blocking of $x^5$ by a particular coalition of two agents to reach a cycle-allocation. If the probability that the grand coalition is chosen equals the probability that any coalition with two agents is chosen, this difference does not matter. Second, any least resistance $x^5$-tree of type (b) in Figure 3 contains a blocking of a cycle allocation by a singleton coalition to reach $x^1$ (by mistake), while $x^2$-trees of type (b) in Figure 4 need a blocking of $x^5$ by a singleton coalition to reach an element of the cycle. Now recall that in every cycle allocation one of the agents is already
at his endowment, hence if he alone is given the opportunity to recontract the system can not switch to \( x^1 \). Hence, \( \tilde{m}(x^5, x^1) > \tilde{m}(x^2, x^1) \) if the probability that a singleton coalition forms does not depend on the allocation. This leads to a larger contribution from this type of trees to \( \mu^*(x^2) \) than to \( \mu^*(x^5) \). Third, \( x^5 \)-trees of type (b) in Figure 3 also contain a blocking of \( x^1 \) by \( x^5 \), while \( x^2 \)-trees of type (b) in Figure 4 need a blocking of \( x^1 \) by \( x^2 \). Note that for the former allocation to be agreed upon the grand coalition has to form while the latter allocation can also be achieved by coalitions of two agents. Hence, \( \tilde{m}(x^1, x^2) > \tilde{m}(x^1, x^5) \). In Appendix A we prove that this intuition holds and \( \mu^*(x^2) > \mu^*(x^5) \) if \( \tilde{m}(x, x^1) < \tilde{m}(x^5, x^1) \) for every \( x \in \{x^2, x^3, x^4\} \), \( \tilde{m}(x^1, x^2) > \tilde{m}(x^1, x^5) \), and coalitions of size three form with the same probability as coalitions of size two. Hence, the probability that the system is found in one cycle allocation is higher than the respective probability to find it in the core.\(^{18}\) Note, however, that this result is not driven by the lack of mistake-free edges that lead to the core allocation \( x^5 \) (as in the previous examples), but by details of the process that determine the opportunities to reach and exit the endowment.

The above-mentioned examples highlight the following features of the limit invariant distribution. First, the probability that a dynamic recontracting process is in a certain recurrent class does not depend on its topology (singleton or cycle). Neither is it necessarily enhanced by the fact that a certain allocation is in the core or Walrasian. The limit invariant distribution rather depends on two issues: (i) The number and relative transition probabilities of resistance minimizing paths from one recurrent class to another. Consider as an example the importance of the edges to and from the endowment in Example 2. (ii) The accessibility of a recurrent class from individually rational allocations that are not member of a recurrent class. An example is allocation \( x^2 \) in Example 4.

6 Concluding Remarks

6.1 Solution Concepts for Dynamic Recontracting Processes

Recurrent Classes We first model economic interaction in multiple-type housing markets by an unperturbed dynamic recontracting process. Then, our first non-empty solution concepts for any given multiple-type housing market \( R \) is its set of recurrent classes \( RC(R) \) and the set of associated allocations \( RC(R) \). The set of recurrent classes of the dynamic recontracting process is non-empty and contains all sets of allocations that cannot be blocked by any allocation outside the respective recurrent class. Every element of the core forms a singleton recurrent class. Moreover, non-singleton recurrent classes exist (in settings with an empty core as well as vis-a-vis a non-empty core).

Stochastically Stable Allocations Next, we allow agents to make mistakes and model economic interaction in multiple-type housing markets by a perturbed dynamic recontracting process. The second non-empty solution concepts for any given multiple-type housing market \( R \) then is its set of stochastically stable allocations.

Agents make mistakes only with a small probability. If the perturbation of the dynamic recontracting process through mistakes becomes sufficiently small (by letting mistake probabilities converge to zero), the process will – in the long-run – converge to a subset of the recurrent classes which determine the set of stochastically stable allocations.

\(^{18}\)In Appendix A we show that \( \mu^*(x^2) = \mu^*(x^3) = \mu^*(x^4) \) under these conditions due to symmetry.
Consider, for instance, a market \( R \) with two recurrent classes (as in Example 6). One recurrent class can be left (towards the other recurrent class) only if at least two agents make a mistake, while the other class can be left if just one agent makes a mistake. Then, the process will converge to the first recurrent class with probability one – this recurrent class is the only stochastically stable recurrent class of \( R \). Hence, while recurrent classes are characterized by the (core-like) feature of internal stability (no allocation in a recurrent class can be blocked by recontracting among the members of a coalition upon an allocation outside the recurrent class), stochastically stable allocations are in general also characterized by their accessibility from other recurrent classes. We formalize the interplay of stability and accessibility through the stochastic potential that indicates the minimal number of mistakes needed to reach a recurrent class from all other recurrent classes.

For our specification of the perturbed dynamic recontracting process we show that every recurrent class is stochastically stable as long as preferences are strict (and the dynamic recontracting process is modeled with weak blocking).

**Limit Invariant Distributions** Finally, to better understand which allocations are likely to be final allocations of our dynamic process of trade, we introduced a method to directly access the limit invariant distribution. By construction, the limit invariant distribution \( \mu^*(R) \) of a dynamic recontracting process is the unique probability distribution over allocations that the dynamic recontracting process will converge to in the long-run. Compared to the set of (stochastically stable) recurrent classes, we now obtain valuable information, namely a probability distribution, that might help to discriminate between allocations. Moreover, by considering the complete set of least resistance trees in the computation of the limit invariant distribution the interplay between stability and accessibility becomes clear (as illustrated in several examples). However, the extra information obtained clearly comes at the cost of having to compute all least resistance trees (compared to the construction of one tree for a stochastic stability analysis). Since any least resistance tree only consists of least resistance edges, this essentially boils down to a combinatorial exercise (as also demonstrated in several examples).

Apart from our analysis of dynamic recontracting processes for multiple-type housing markets, our paper also aims at an illustration of the method to compute the limit invariant distribution. The particular features of multiple-type housing markets proved useful to demonstrate its applicability and intuitive appeal. However, it should be obvious that results similar to Theorem 3 are also feasible and desirable for other Markov processes, for instance in evolutionary (asymmetric) non-cooperative games or models of social and economic network formation.

### 6.2 Relation to Other Solution Concepts

**Dynamic Recontracting and the Core** The concepts proposed in this paper distinguish themselves from the core in two respects. First, both \( RC(R) \) and \( SRC(R) \) are non-empty due to the finiteness of the Markov process. In particular, cycles as in Example 1 emerge naturally as recurrent classes (and stochastically stable allocations). Moreover, core allocations are the only singleton recurrent classes of the process. It therefore seems natural to invoke the requirement of stochastic stability to select among recurrent classes in the very same way as the literature on evolutionary equilibrium selection in non-cooperative game theory used...
stochastic stability to select among Nash equilibria. However, this kind of selection is only applicable if indifference in preferences are possible or if strong blocking is used. Nonetheless, its major weakness is the sole consideration of the stochastic potential (i.e., the smallest number of mistakes needed to reach an allocation from all other allocations) while not taking into account the respective number of paths that lead to an allocation with the smallest number of mistakes. A direct analysis of the limit invariant distribution as conducted in Section 5.2 proceeds one step further as it requires internal stability and accessibility through all least resistance trees. As the components of the limit invariant distribution depict the probability that the system converges to a certain allocation, it thereby ultimately solves the problem of multi-valuedness of the set of recurrent classes (and the core). Our examples illustrate the intuitive appeal of such a core-selection procedure that trades-off stability and accessibility and thereby enriches the static concept of the core with the dynamic aspects of recontracting processes.

Dynamic Recontracting and Walrasian Allocations In contrast to the set of (stochastically stable) recurrent classes, the set of Walrasian allocations can be empty, even in the presence of a non-empty core. By Theorem 2, Walrasian allocations are always stochastically stable if preferences are strict. An analysis of the limit invariant distributions, however, suggests that (i) Walrasian allocations may be bad predictors for the final allocation of a dynamic recontracting process and (ii) different Walrasian allocations of the same market do not share the same properties in terms of coalitional accessibility: some are easier to reach and exit than others and thereby have different weights in the long-run probability distribution. This underlines that there may be frictions in the market and the trading process that are indeed not captured by the Walrasian equilibrium concept.

6.3 Weak and Strong Blocking
To keep the exposition as simple as possible we focus on weak blocking. Strong blocking has been investigated for one-type housing markets by Serrano and Volij (2005). From a modeling perspective, the key difference turns out to be the special state dependent probability of mistakes which is necessary for strong blocking but is not needed in our setting. In particular, Serrano and Volij (2005) assume that contracts which leave an agent indifferent are signed with probability $\epsilon$ while contracts which make an agent (strictly) worse off are signed with probability $\epsilon^\lambda$ with $\lambda > 1$. In Example 6 we demonstrated that the introduction of weak preferences or strong blocking reestablishes the selective power of stochastic stability that is absent otherwise.

For expositional ease in the presentation of the computational method to elicit the limit invariant distribution we have chosen to model dynamic recontracting with strict preferences and weak blocking. In contrast to the concept of stochastic stability that crucially relies on these model specification, it should be obvious from Section 5.2 that our method to compute the limit invariant distribution offers enough flexibility to be applicable regardless of the strictness of preferences and the blocking notion.

6.4 Is this the end of the story?
Our analysis has shown that neither the core nor Walrasian allocations might be good predictors for dynamic recontracting processes. We have seen some instances where a cycle of
allocations may be the best predictor for the final allocation of our dynamic trading process. Clearly, this conclusion is in part driven by the myopia of agents. Recall that at each period agents agree to trade if it is beneficial to do so. In accepting such trades, agents do not envision the possible blockings along the trading path. Nevertheless, it is clear from our analysis that there are aspects of stability and in particular of accessibility of allocations that are not captured by the core. Some core allocations turn out to be harder to reach than others while non-core allocations may emerge naturally through a sequence of trades. This indicates that modeling the dynamics of recontracting only implicitly by a cooperative solution concept as the core will not work (even though we have modeled our dynamic recontracting process with a core bias by never transferring property rights and using weak blocking).

Finally, several possible extensions of this work can be considered. First, myopic agents could be substituted by farsighted agents. We conjecture that such a replacement would eliminate cycles of allocations. Second, an analysis of dynamic processes for general NTU games could be a fruitful path of research.

Appendix

A Examples

Example 1: An Empty Core Recall

\[ (3, 1) P_1 (1, 2) P_1 (1, 1) P_1 \text{ anything}, \]
\[ (2, 1) P_2 (3, 2) P_2 (2, 2) P_2 \text{ anything}, \]
\[ (2, 3) P_3 (1, 3) P_3 (3, 3) P_3 \text{ anything}, \]

and \( IR(R) = \{x^1, x^2, x^3, x^4\} \) with

\[ x^1 = \{(1, 1), (2, 2), (3, 3)\}, \quad x^2 = \{(1, 1), (3, 2), (2, 3)\}, \]
\[ x^3 = \{(1, 2), (2, 1), (3, 3)\}, \quad x^4 = \{(3, 1), (2, 2), (1, 3)\}. \]

Hence, \( Core(R) = W(R) = \emptyset \). Furthermore, \( RC(R) = SRC(R) = \{x^2, x^3, x^4\} \).  

Example 2: The Unique Walrasian Allocation Equals the Core Allocation Recall

\[ (3, 1) P_1 (2, 2) P_1 (1, 2) P_1 (1, 1) P_1 \text{ anything}, \]
\[ (2, 1) P_2 (3, 3) P_2 (3, 2) P_2 (2, 2) P_2 \text{ anything}, \]
\[ (2, 3) P_3 (1, 1) P_3 (1, 3) P_3 (3, 3) P_3 \text{ anything}, \]

and \( IR(R) = \{x^1, x^2, x^3, x^4, x^5\} \) with

\[ x^1 = \{(1, 1), (2, 2), (3, 3)\}, \quad x^2 = \{(1, 1), (3, 2), (2, 3)\}, \]
\[ x^3 = \{(1, 2), (2, 1), (3, 3)\}, \quad x^4 = \{(3, 1), (2, 2), (1, 3)\}, \]
\[ x^5 = \{(2, 2), (3, 3), (1, 1)\}. \]
Clearly, \(x^3\{2,3\}\)-blocks \(x^1\), \(x^3\{1,2\}\)-blocks \(x^2\), \(x^4\{1,3\}\)-blocks \(x^3\), and \(x^2\{2,3\}\)-blocks \(x^4\). Since \(x^5\) cannot be blocked by any coalition, \(Core(R) = \{x^5\}\). Next, we prove that allocation \(x^5\) is Walrasian. It is easy to check that the following (in)equalities hold, e.g., for price system \(p \equiv (p_1, p_2)\) such that \(p_1 = (0, \frac{1}{2})\) and \(p_2 = (1, 0, \frac{1}{2})\).

(1) \(p_1(1) + p_2(1) = p_1(2) + p_2(2),\)
(2) \(p_1(2) + p_2(2) = p_1(3) + p_2(3),\)
(3) \(p_1(3) + p_2(3) = p_1(1) + p_2(1),\)
(4) \(p_1(3) + p_2(1) > p_1(1) + p_2(1),\)
(5) \(p_1(2) + p_2(1) > p_1(2) + p_2(2),\)
(6) \(p_1(2) + p_2(3) > p_1(3) + p_2(3).\)

Hence, \(W(R) = \{x^5\}\). Furthermore, \(RC(R) = SRC(R) = \{x^2, x^3, x^4, x^5\}\).

In the following we indicate the set of least resistance trees for each of the recurrent classes. As it follows from Theorem 2(i) that \(\gamma(x^5) = \gamma(x^k) = 1\) for \(k = 2, 3, 4\) it suffices to consider those transitions that need at most one mistake. In the following table we indicate the respective (set of) coalition(s) that facilitates a transition from an initial state (row) to a final state (column) with zero mistakes (we omit the diagonal elements as they are irrelevant for cycle-free graphs). For expositional ease we introduce the following abbreviations.

\[
N \equiv \{1, 2, 3\}
\]
\[
C_1 = \{\{1\}, \{2\}, \{3\}\}
\]
\[
C_2 = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}
\]

<table>
<thead>
<tr>
<th>(x^1)</th>
<th>(x^2)</th>
<th>(x^3)</th>
<th>(x^4)</th>
<th>(x^5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x^1)</td>
<td>({N, {2,3}})</td>
<td>({N, {1,2}})</td>
<td>({N, {1,3}})</td>
<td>(N)</td>
</tr>
<tr>
<td>(x^2)</td>
<td>(\emptyset)</td>
<td>(x)</td>
<td>({1,2})</td>
<td>(\emptyset)</td>
</tr>
<tr>
<td>(x^3)</td>
<td>(\emptyset)</td>
<td>(\emptyset)</td>
<td>(x)</td>
<td>({1,3})</td>
</tr>
<tr>
<td>(x^4)</td>
<td>({2,3})</td>
<td>(\emptyset)</td>
<td>(\emptyset)</td>
<td>(x)</td>
</tr>
</tbody>
</table>

The next table lists the respective coalitions that facilitate a transition if a certain member makes a mistake.\(^{19}\)

<table>
<thead>
<tr>
<th>(x^1)</th>
<th>(x^2)</th>
<th>(x^3)</th>
<th>(x^4)</th>
<th>(x^5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x^1)</td>
<td>(\emptyset)</td>
<td>(\emptyset)</td>
<td>(\emptyset)</td>
<td>(\emptyset)</td>
</tr>
<tr>
<td>(x^2)</td>
<td>({C_1, {1,2}, {1,3}})</td>
<td>(x)</td>
<td>(\emptyset)</td>
<td>(\emptyset)</td>
</tr>
<tr>
<td>(x^3)</td>
<td>({C_1, {1,3}, {2,3}})</td>
<td>({2,3})</td>
<td>(x)</td>
<td>(N)</td>
</tr>
<tr>
<td>(x^4)</td>
<td>({C_1, {1,2}, {2,3}})</td>
<td>(\emptyset)</td>
<td>(\emptyset)</td>
<td>(x)</td>
</tr>
<tr>
<td>(x^5)</td>
<td>(C_1)</td>
<td>({2,3})</td>
<td>({1,2})</td>
<td>({1,3})</td>
</tr>
</tbody>
</table>

With the transition opportunities as depicted in the previous tables, we can now list all least resistance trees for the (unique) core allocation \(x^5\) and a cycle allocation \(x^2\), say. We start with the set of \(x^5\)-trees. If every least resistance tree includes exactly one mistake (i.e., \(\gamma(x^5) = 1\)), this mistake has to be made on a (cycle-free) sequence of edges from an element of the cycle to \(x^5\). From any allocation in the cycle there are exactly two such paths. Either, the grand coalition forms and the agent who is worse off at allocation \(x^5\) (note that this is

\(^{19}\)In this example this agent is uniquely determined.
exactly one agent for every allocation in the cycle) agrees upon \(x^5\) by mistake, or a singleton coalition forms and the respective agent asks for his endowment even though he is then worse off (note that in every cycle allocation there are two such singleton allocations, e.g., at \(x^2\) only agent 2 and 3 can recontract on their endowment and thereby induce a transition to \(x^1\)). Agent 1, in contrast, is already at his endowment). These two paths induce the two types of least resistance trees depicted in Figure 3. Next, observe that the allocations of the cycle have to be connected via the respective (unique) mistake-free edges. As for type (a) note that there are three different allocations from which the cycle can be left to \(x^5\). Moreover, there can be a (mistake-free) edge from \(x^1\) to any other of the four allocations. This leaves as with the following 12 trees of type (a).

\[
\{[x^3, x^4], [x^4, x^2], [x^2, x^5], [x^1, x^2]\};\{[x^3, x^4], [x^4, x^2], [x^2, x^5], [x^1, x^3]\};
\{[x^3, x^4], [x^4, x^2], [x^2, x^5], [x^1, x^4]\};\{[x^3, x^4], [x^4, x^2], [x^2, x^5], [x^1, x^5]\};
\{[x^4, x^2], [x^2, x^3], [x^3, x^5], [x^1, x^2]\};\{[x^4, x^2], [x^2, x^3], [x^3, x^5], [x^1, x^3]\};
\{[x^4, x^2], [x^2, x^3], [x^3, x^5], [x^1, x^4]\};\{[x^4, x^2], [x^2, x^3], [x^3, x^5], [x^1, x^5]\};
\{[x^2, x^3], [x^3, x^4], [x^4, x^5], [x^1, x^2]\};\{[x^2, x^3], [x^3, x^4], [x^4, x^5], [x^1, x^3]\};
\{[x^2, x^3], [x^3, x^4], [x^4, x^5], [x^1, x^4]\};\{[x^2, x^3], [x^3, x^4], [x^4, x^5], [x^1, x^5]\}.
\]

Likewise there are three different allocations in the cycle from which \(x^1\) can be reached (and subsequently left towards \(x^5\)). Hence, we have to add the following three trees.

\[
\{[x^3, x^4], [x^4, x^2], [x^2, x^1], [x^1, x^5]\};
\{[x^2, x^3], [x^3, x^4], [x^4, x^1], [x^1, x^5]\};
\{[x^4, x^2], [x^2, x^3], [x^3, x^1], [x^1, x^5]\}.
\]

In a similar way, we can construct the set of \(x^2\)-trees. If every least resistance tree includes exactly one mistake (i.e., \(\gamma(x^2) = 1\)), this mistake has to be made on a (cycle-free) sequence of edges from \(x^5\) to an element of the cycle. To any allocation in the cycle there are two such paths. Either, a particular coalition of size two forms and the agent who is worse off at the cycle-allocation agrees upon the cycle allocation by mistake. Note that there is exactly one coalition of size two that can actually contract upon each cycle allocation (the agent who is left at his endowment can not be a member of such a coalition). Or one of the singleton coalition forms and the respective agent asks for his endowment even though he is then worse off, afterwards the grand coalition or \(\{2, 3\}\) agree upon \(x^2\). These two types of paths induce the two types of least resistance trees depicted in Figure 4 (again the allocations of the cycle have to be connected via the respective (unique) mistake-free edges). As for type (a) note that there are three different allocations from which the cycle can be accessed from \(x^5\). Moreover, there can be a (mistake-free) edge from \(x^1\) to any other of the four allocation. This leaves as with the following 12 \(x^2\)-trees of type (a).

\[
\{[x^3, x^4], [x^4, x^2], [x^2, x^5], [x^1, x^2]\};\{[x^3, x^4], [x^4, x^2], [x^2, x^5], [x^1, x^3]\};
\{[x^3, x^4], [x^4, x^2], [x^2, x^5], [x^1, x^4]\};\{[x^3, x^4], [x^4, x^2], [x^2, x^5], [x^1, x^5]\};
\]
\{[x^4,x^2],[x^2,x^3],[x^5,x^3],[x^1,x^2]\};\{[x^4,x^2],[x^2,x^3],[x^5,x^3],[x^1,x^3]\};
\{[x^4,x^2],[x^2,x^3],[x^5,x^3],[x^1,x^4]\};\{[x^4,x^2],[x^2,x^3],[x^5,x^3],[x^1,x^5]\};
\{[x^2,x^3],[x^5,x^4],[x^1,x^2]\};\{[x^2,x^3],[x^3,x^4],[x^5,x^4],[x^1,x^3]\};
\{[x^2,x^3],[x^3,x^4],[x^5,x^4],[x^1,x^4]\};\{[x^2,x^3],[x^3,x^4],[x^5,x^4],[x^1,x^5]\}.

Likewise there are three different allocations in the cycle that can be reached from \(x^1\) (after \(x^5\) has been left towards \(x^1\)). Hence, we have to add the following three trees.

\{[x^1,x^2],[x^1,x^2],[x^1,x^5],[x^1,x^1]\};
\{[x^2,x^3],[x^2,x^4],[x^1,x^4],[x^5,x^1]\};
\{[x^4,x^2],[x^2,x^3],[x^1,x^3],[x^5,x^1]\}.

This is sufficient information to apply the formula for \(\mu^*\) in Theorem 3. For expositional ease let us make the following assumptions. First, recall from Assumption 4 that the probability for an agent to commit a mistake does not depend on his identity, the coalition, or the allocation. Moreover, suppose the probability that a certain coalition forms does not depend on the identity of the agents and that a coalition chooses each improving allocation with the same probability. Then,

\[
\tilde{m}(x^2,x^3) = \tilde{m}(x^3,x^4) = \tilde{m}(x^4,x^2),
\tilde{m}(x^1,x^2) = \tilde{m}(x^1,x^3) = \tilde{m}(x^1,x^4),
\tilde{m}(x^2,x^1) = \tilde{m}(x^3,x^1) = \tilde{m}(x^4,x^1),
\tilde{m}(x^2,x^5) = \tilde{m}(x^3,x^5) = \tilde{m}(x^4,x^5),
\tilde{m}(x^5,x^2) = \tilde{m}(x^5,x^3) = \tilde{m}(x^5,x^4).
\]

With this we get

\[
\frac{\mu^*(x^5)}{\mu^*(x^2)} = \frac{9\tilde{m}(x^2,x^5)\tilde{m}(x^1,x^2) + 3\tilde{m}(x^1,x^5)\tilde{m}(x^2,x^5) + 3\tilde{m}(x^1,x^5)\tilde{m}(x^2,x^1)}{9\tilde{m}(x^5,x^2)\tilde{m}(x^1,x^2) + 3\tilde{m}(x^1,x^5)\tilde{m}(x^5,x^2) + 3\tilde{m}(x^5,x^1)\tilde{m}(x^1,x^2)}.
\]

Hence, the ratio of \(\mu^*(x^5)\) and \(\mu^*(x^2)\) is determined by the relation \(\tilde{m}(x^2,x^5)/\tilde{m}(x^5,x^2)\), \(\tilde{m}(x^1,x^5)/\tilde{m}(x^1,x^2)\), and \(\tilde{m}(x^2,x^1)/\tilde{m}(x^5,x^1)\). As for \(\tilde{m}(x^2,x^5)/\tilde{m}(x^5,x^2)\) recall from the second table (see above) that a least resistance edge from \(x^2\) to \(x^5\) needs the grand coalition to form and agent 3 making a mistake, while the least resistance edge from \(x^5\) to \(x^2\) needs the coalition \(\{2,3\}\) to form and agent 2 making a mistake. Suppose coalitions of size two and size three form with the same probability, than \(\tilde{m}(x^2,x^5) = \tilde{m}(x^5,x^2)\). As for \(\tilde{m}(x^1,x^5)/\tilde{m}(x^1,x^2)\) recall from the first table (see above) that \(x^2\) can be recontracted upon at \(x^1\) by the grand coalition \textit{and} \(\{2,3\}\), while the grand coalition is needed to agree upon \(x^5\). If all improving allocations are chosen from a coalition with the same probability, we therefore get \(\tilde{m}(x^1,x^2) > \tilde{m}(x^1,x^5)\). But these conditions also ensure that \(\tilde{m}(x^2,x^1) < \tilde{m}(x^5,x^1)\) which implies that \(\mu^*(x^2) > \mu^*(x^5)\). Finally the symmetry of the preferences with respect to a permutation of allocations \(x^2, x^3\), and \(x^4\) implies that \(\mu^*(x^2) = \mu^*(x^3) = \mu^*(x^4)\) whenever coalition formation does not depend on allocations and the identity of the agents.

\(\diamondsuit\)
Example 3: Multiple Core Allocations and Unique Walrasian Allocation Recall

\[ (1, 2) P_1 (3, 3) P_1 (2, 3) P_1 (1, 1) P_1 \text{ anything}, \]
\[ (1, 3) P_2 (1, 2) P_2 (3, 3) P_2 (2, 2) P_2 \text{ anything}, \]
\[ (3, 1) P_3 (2, 1) P_3 (3, 3) P_3 \text{ anything}, \]

and \( IR(R) = \{x^1, x^2, x^3, x^4\} \) with

\[ x^1 = \{(1, 1), (2, 2), (3, 3)\}, \quad x^2 = \{(2, 3), (1, 2), (3, 1)\}, \]
\[ x^3 = \{(1, 2), (3, 3), (2, 1)\}, \quad x^4 = \{(3, 3), (1, 2), (2, 1)\}. \]

Clearly, \( x^2 \{1, 2, 3\}\)-blocks \( x^1 \). Since \( x^2, x^3 \), and \( x^4 \) cannot be blocked by any coalition, \( Core(R) = \{x^2, x^3, x^4\} \). Next, we check if any of the core allocations is Walrasian.

Allocation \( x^2 \) is not Walrasian. It if was, then the following (in)equalities would hold:

\[ (1) \quad p_1(1) + p_2(1) = p_1(2) + p_2(3), \quad (2) \quad p_1(2) + p_2(2) = p_1(1) + p_2(2), \]
\[ (3) \quad p_1(3) + p_2(3) = p_1(3) + p_2(1), \quad (4) \quad p_1(3) + p_2(3) > p_1(1) + p_2(1), \]
\[ (5) \quad p_1(1) + p_2(2) > p_1(1) + p_2(1), \quad (6) \quad p_1(1) + p_2(3) > p_1(1) + p_2(2). \]

By (2) and (6), we obtain \( p_1(1) + p_2(3) > p_1(1) + p_2(2) \). By (5), \( p_2(2) > p_2(1) \) and by (3), \( p_2(1) = p_2(3) \). Hence, \( p_2(3) > p_2(2) > p_2(1) = p_2(3) \); a contradiction.

Allocation \( x^3 \) is Walrasian. It is easy to check that the following (in)equalities hold, e.g., for price system \( p \equiv (p_1, p_2) \) such that \( p_1 = (2, \frac{1}{2}, 1) \) and \( p_2 = (1, 1, \frac{1}{2}) \):

\[ (1) \quad p_1(1) + p_2(1) = p_1(1) + p_2(2), \quad (2) \quad p_1(2) + p_2(2) = p_1(3) + p_2(3), \]
\[ (3) \quad p_1(3) + p_2(3) = p_1(2) + p_2(1), \quad (4) \quad p_1(1) + p_2(2) > p_1(2) + p_2(2), \]
\[ (5) \quad p_1(1) + p_2(3) > p_1(2) + p_2(2), \quad (6) \quad p_1(3) + p_2(1) > p_1(3) + p_2(3). \]

Allocation \( x^4 \) is not Walrasian. If it was, then the following (in)equalities would hold:

\[ (1) \quad p_1(1) + p_2(1) = p_1(3) + p_2(3), \quad (2) \quad p_1(2) + p_2(2) = p_1(1) + p_2(2), \]
\[ (3) \quad p_1(3) + p_2(3) = p_1(2) + p_2(1), \quad (4) \quad p_1(1) + p_2(2) > p_1(1) + p_2(1), \]
\[ (5) \quad p_1(1) + p_2(3) > p_1(2) + p_2(2), \quad (6) \quad p_1(3) + p_2(1) > p_1(3) + p_2(3). \]

By (2) and (5), we obtain \( p_1(1) + p_2(3) > p_1(1) + p_2(2) \). Thus, \( p_2(3) > p_2(2) \). By (4), \( p_2(2) > p_2(1) \) and by (6), \( p_2(1) > p_2(3) \). Hence, \( p_2(3) > p_2(2) > p_2(1) > p_2(3) \); a contradiction.

Hence, \( W(R) = \{x^3\} \). Furthermore, \( RC(R) = SRC(R) = Core(R) = \{x^2, x^3, x^4\} \).

To compute the limit invariant distribution, we now indicate the set of least resistance trees for each of the recurrent classes. As it follows from Theorem 2(i) that \( \gamma(x^k) = 2 \) for \( k = 2, 3, 4 \) it suffices to consider those edges that need at most one mistake (otherwise no spanning tree between three recurrent classes can be formed). In in the following table we indicate the respective (set of) coalition(s) that facilitates a transition from an initial state (row) to a final state (column) with zero mistakes (we omit the diagonal elements as they are irrelevant for cycle-free graphs).
The next table lists the respective coalitions that facilitate a transition if exactly one member makes a mistake.\(^{20}\)

<table>
<thead>
<tr>
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<th>(x^1)</th>
<th>(x^2)</th>
<th>(x^3)</th>
<th>(x^4)</th>
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<tr>
<td>(x^4)</td>
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</table>

The crucial symmetry breaking in the transition matrix is due to the fact that it takes two mistakes for a direct edge from \(x^2\) to \(x^3\) while all other direct edges between core allocations need only one mistake (and \(x^1\) can be accessed from any core allocation by one mistake of the respective singleton coalition). Hence, the dynamic recontracting process is less likely to exit \(x^2\) than to exit \(x^3\) or \(x^4\) and also less likely to enter \(x^3\) than to enter \(x^2\) or \(x^4\). Hence, one might suspect that \(\mu^*(x^2) > \mu^*(x^4) > \mu^*(x^3)\).

To prove this claim, we list the complete set of least resistance trees for each core-allocation. There are the following 16 \(x^2\)-trees.\(^{21}\)

\[
\begin{align*}
\{[x^3, x^2], [x^4, x^3], [x^1, x^2]\} & \{[x^3, x^2], [x^4, x^3], [x^1, x^3]\} & \{[x^3, x^2], [x^4, x^3], [x^1, x^4]\} \\
\{[x^4, x^2], [x^3, x^4], [x^1, x^2]\} & \{[x^4, x^2], [x^3, x^4], [x^1, x^3]\} & \{[x^4, x^2], [x^3, x^4], [x^1, x^4]\} \\
\{[x^3, x^2], [x^4, x^2], [x^1, x^2]\} & \{[x^3, x^2], [x^4, x^2], [x^1, x^3]\} & \{[x^3, x^2], [x^4, x^2], [x^1, x^4]\} \\
\{[x^1, x^2], [x^4, x^3], [x^3, x^1]\} & \{[x^1, x^3], [x^3, x^2], [x^4, x^1]\} \\
\{[x^1, x^2], [x^3, x^4], [x^4, x^1]\} & \{[x^1, x^2], [x^3, x^4], [x^4, x^1]\} \\
\{[x^1, x^4], [x^3, x^4], [x^4, x^1]\} & \{[x^3, x^2], [x^1, x^2], [x^4, x^1]\} \\
\{[x^4, x^2], [x^1, x^2], [x^3, x^1]\} & \{[x^1, x^2], [x^4, x^1], [x^3, x^1]\} \\
\{[x^1, x^2], [x^4, x^1], [x^3, x^1]\} \\
\end{align*}
\]

We continue with the following 8 \(x^3\)-trees.

\[
\begin{align*}
\{[x^4, x^3], [x^2, x^4], [x^1, x^2]\} & \{[x^4, x^3], [x^2, x^4], [x^1, x^3]\} & \{[x^4, x^3], [x^2, x^4], [x^1, x^4]\} \\
\{[x^4, x^3], [x^2, x^1], [x^1, x^3]\} \\
\end{align*}
\]

\(^{20}\)The agent who makes a mistake is uniquely determined in this example.

\(^{21}\)Trees listed in one row only differ through a different edge from the endowment.
\{[x^2, x^4], [x^4, x^1], [x^1, x^3]\};
\{[x^2, x^1], [x^1, x^4], [x^4, x^3]\};
\{[x^4, x^3], [x^2, x^1], [x^1, x^3]\};
\{[x^2, x^1], [x^4, x^1], [x^1, x^3]\};

Finally, there are 12 \(x^4\)-trees.
\{[x^3, x^2], [x^2, x^4], [x^1, x^4]\}; \{[x^3, x^2], [x^2, x^4], [x^1, x^3]\};
\{[x^2, x^4], [x^3, x^4], [x^1, x^3]\}; \{[x^2, x^4], [x^3, x^4], [x^1, x^4]\};
\{[x^2, x^1], [x^1, x^3], [x^3, x^4]\};
\{[x^1, x^4], [x^2, x^4], [x^2, x^3]\};
\{[x^1, x^2], [x^2, x^4], [x^3, x^4]\};
\{[x^2, x^4], [x^1, x^4], [x^2, x^3]\};
\{[x^1, x^4], [x^2, x^4], [x^3, x^4]\};
\{[x^1, x^4], [x^2, x^4], [x^3, x^4]\};

To simplify the computation of \(\mu^*(x)\) recall that the probability of a mistake does not depend on the allocation or the identity of the agent. Suppose further that the probability for a certain coalition to become active does not depend on the allocation. Then, \(\widetilde{m}(x^2, x^1) = \widetilde{m}(x^3, x^1) = \widetilde{m}(x^4, x^1)\). Moreover, assume that the grand coalition agrees upon recontracting on each core-allocation with the same probability if the p.d.r. process is at \(x^1\) (i.e., \(\widetilde{m}(x^1, x^2) = \widetilde{m}(x^1, x^3) = \widetilde{m}(x^1, x^4)\)). Finally, suppose that all transitions between core allocations that only need one mistake have the same probability (i.e., \(\widetilde{m}(x^2, x^4) = \widetilde{m}(x^3, x^2) = \widetilde{m}(x^3, x^4) = \widetilde{m}(x^4, x^2) = \widetilde{m}(x^4, x^3)\)). This is, for instance, the case if the grand coalition agrees upon each improvement with equal probability. The formula in Theorem 3 then leads to the following expressions for the respective components of the limit invariant distribution.

\[
\mu^*(x^2) = 9\widetilde{m}(x^1, x^2)(\widetilde{m}(x^2, x^4))^2 + 6\widetilde{m}(x^2, x^1)\widetilde{m}(x^1, x^2)\widetilde{m}(x^2, x^4) + \widetilde{m}(x^1, x^2)(\widetilde{m}(x^2, x^4))^2
\]

\[
\mu^*(x^3) = 3\widetilde{m}(x^1, x^2)(\widetilde{m}(x^2, x^4))^2 + 4\widetilde{m}(x^2, x^1)\widetilde{m}(x^1, x^2)\widetilde{m}(x^2, x^4) + \widetilde{m}(x^1, x^2)(\widetilde{m}(x^2, x^4))^2
\]

\[
\mu^*(x^4) = 6\widetilde{m}(x^1, x^2)(\widetilde{m}(x^2, x^4))^2 + 5\widetilde{m}(x^2, x^1)\widetilde{m}(x^1, x^2)\widetilde{m}(x^2, x^4) + \widetilde{m}(x^1, x^2)(M(x^2, x^4))^2
\]

which indicates that \(\mu^*(x^2) > \mu^*(x^4) > \mu^*(x^3)\) confirming our initial intuition motivated by the different accessibility and stability of the various core allocations. Note in particular that the unique Walrasian allocation \(x^3\) is the worst predictor for the long-run behavior of the process.

\textbf{Example 4: Multiple Walrasian Allocations} Recall

\((1, 2) P_1 (3, 3) P_1 (2, 3) P_1 (1, 1) P_1 \text{ anything},\)

\((1, 3) P_2 (1, 2) P_2 (3, 3) P_2 (2, 2) P_2 \text{ anything}\)
(2, 1) \ P_3 \ (3, 1) \ P_3 \ (3, 3) \ P_3 \ anything,

and \ IR(R) = \{x^1, x^2, x^3, x^4\} with,

\[ x^1 = \{(1, 1), (2, 2), (3, 3)\}, \quad x^2 = \{(2, 3), (1, 2), (3, 1)\}, \]
\[ x^3 = \{(3, 3), (1, 2), (2, 1)\}, \quad x^4 = \{(1, 2), (3, 3), (2, 1)\}. \]

Clearly, \( x^2 \{1, 2, 3\}\)-blocks \( x^1 \) and \( x^3 \{1, 2, 3\}\)-blocks \( x^2 \). Since \( x^3 \) and \( x^4 \) cannot be blocked by any coalition, \( Core(R) = \{x^3, x^4\} \). Next, we check if any of the core allocations is Walrasian.

Allocation \( x^3 \) is Walrasian. It is easy to check that the following (in)equalities hold, e.g., for price system \( p \equiv (p_1, p_2) \) such that \( p_1 = (2, 2, 0) \) and \( p_2 = (1, 2, 3) \):

\( p_1(1) + p_2(1) = p_1(3) + p_2(3), \quad p_1(2) + p_2(2) = p_1(1) + p_2(2), \)
\( p_1(3) + p_2(3) = p_1(2) + p_2(1), \quad p_1(1) + p_2(2) > p_1(1) + p_2(1), \)
\( p_1(1) + p_2(3) > p_1(2) + p_2(2). \)

Allocation \( x^4 \) is Walrasian. It is easy to check that the following (in)equalities hold, e.g., for price system \( p \equiv (p_1, p_2) \) such that \( p_1 = (2, 1, 2) \) and \( p_2 = (1, 1, 1) \):

\( p_1(1) + p_2(1) = p_1(3) + p_2(3), \quad p_1(2) + p_2(2) = p_1(1) + p_2(2), \)
\( p_1(3) + p_2(3) = p_1(2) + p_2(1), \quad p_1(1) + p_2(2) > p_1(2) + p_2(2), \)
\( p_1(1) + p_2(3) > p_1(2) + p_2(2). \)

Hence, \( W(R) = \{x^3, x^4\} \).

By Theorem 1 (iii) only individually rational allocations are candidates to be elements of a recurrent class. By Theorem 1 (ii), \( \{x^3\} \) and \( \{x^4\} \) are recurrent classes. Since \( x^3 \{1, 2, 3\}\)-blocks \( x^2 \) and \( x^4 \{1, 2, 3\}\)-blocks \( x^1 \), \( \{x^3\} \) and \( \{x^4\} \) are the only recurrent classes. Therefore, \( \text{RC}(R) = \text{SRC}(R) = Core(R) = \{x^3, x^4\} \).

**Example 5: No Walrasian Allocations and a Multi-Valued Core** Recall

\( (1, 2) \ P_1 \ (3, 3) \ P_1 \ (2, 3) \ P_1 \ (1, 1) \ P_1 \ anything, \)
\( (3, 2) \ P_2 \ (1, 2) \ P_2 \ (3, 3) \ P_2 \ (2, 2) \ P_2 \ anything, \)
\( (1, 3) \ P_3 \ (3, 1) \ P_3 \ (2, 3) \ P_3 \ (2, 1) \ P_3 \ (3, 3) \ P_3 \ anything, \)

and \( IR(R) = \{x^1, x^2, x^3, x^4, x^5\} \) with,

\[ x^1 = \{(1, 1), (2, 2), (3, 3)\}, \quad x^2 = \{(1, 1), (3, 2), (2, 3)\}, \]
\[ x^3 = \{(2, 3), (1, 2), (3, 1)\}, \quad x^4 = \{(3, 3), (1, 2), (2, 1)\}, \]
\[ x^5 = \{(1, 2), (3, 3), (2, 1)\}. \]

Clearly, \( x^2 \{2, 3\}\)-blocks \( x^1 \), \( x^2 \{2, 3\}\)-blocks \( x^4 \), and \( x^5 \{2, 3\}\)-blocks \( x^5 \). Since \( x^2 \) and \( x^3 \) cannot be blocked by any coalition, \( Core(R) = \{x^2, x^3\} \). Next, we check if any of the core allocations is Walrasian.

Allocation \( x^2 \) is not Walrasian. If it was, then the following (in)equalities would hold:
Hence, $W \{classes. Since $RC$ A Markov Process Dictionary

By (5), $p_1(1) + p_2(3) > p_1(1) + p_2(1)$, and therefore, $p_1(1) + p_2(3) = p_1(2) + p_2(3)$.

By (7), $p_1(1) + p_2(3) > p_1(3) + p_2(3)$.

Next, by (1), $p_1(2) = p_1(3)$ and therefore, $p_1(3) + p_2(3) = p_1(2) + p_2$. By (7), $p_1(1) > p_1(3)$ and therefore, $p_1(1) + p_2(1) > p_1(3) + p_2(1)$. Using the previous (in)equalities, (3) implies $p_1(3) + p_2(3) > p_1(3) + p_2(1)$. Hence, $p_2(3) > p_2(1)$; a contradiction to (8).

Allocation $x^3$ is not Walrasian. If it was, then the following (in)equalities would hold:

By (7), $p_1(1) > p_1(3)$ and by (6), $p_1(3) > p_1(2)$. By (2), $p_1(2) = p_1(1)$. Hence, $p_1(1) > p_1(3) + p_1(2) = p_1(1)$; a contradiction.

By Theorem 1 (iii) only individually rational allocations are candidates to be elements of a recurrent class. According to Theorem 1(ii) $\{x^2\}$ and $\{x^3\}$ are (singleton) recurrent classes. Since $x^2 \{1, 2, 3\}$-block $x^1$ and $x^2 \{2, 3\}$-blocks $x^4$ and $x^5$, $\{x^2\}$ and $\{x^3\}$ are the only recurrent classes. Therefore, $RC(R) = SRC(R) = Core(R) = \{x^2, x^3\}$.

\[\Phi\]

B A Markov Process Dictionary

- A Markov process (or Markov chain) is a collection $(X, M)$ of a discrete state space $X$ and a mapping $M : X \times X \rightarrow [0, 1]$ where $M(x, x')$ describes the probability that the state equals allocation $x' \in X$ in period $t + 1$ whenever it was in $x \in X$ in period $t$. Clearly, $\sum_{x' \in X} M(x, x') = 1$. In this contribution we restrict ourselves to finite, homogeneous Markov processes where $X$ is a finite set (the set of allocations) and transition probabilities induced (and captured in $M$) do not depend on time. A path of such a Markov process is a mapping from a countable time set $T$ (usually $\mathbb{Z}^+$) to the state space that depicts the evolution of this dynamic process $x(t) : T \rightarrow X$.

- A recurrent class (or absorbing set or limit set) $A \subseteq X$ is a minimal set of allocations that once entered throughout the dynamic process is never abandoned, i.e., for all $x \in A$ and $x' \notin A$, $M(x, x') = 0$. We denote the set of recurrent classes by $RC$.

- A singleton recurrent class or an absorbing state is an allocation $a \in X$ with $M(a, a) = 1$ (and as a consequence, for all $x \in X \setminus \{a\}$, $M(a, x) = 0$). Allocations that do not belong to any recurrent class are called transient.
• A recurrent class is \textit{aperiodic} whenever it does not contain any deterministic and non-trivial cycle, \textit{i.e.}, there is no sequence of at least two allocations \( \{x_1, x_2, \ldots, x_n\} \) such that for all \( i = 1, \ldots, n-1 \), \( M(x_i, x_{i+1}) = M(x_n, x_1) = 1 \). Note that a sufficient condition for the aperiodicity of a recurrent class \( A \) is that there is an \( x \in A \) such that \( M(x, x) > 0 \), \textit{i.e.}, that the Markov process exhibits sufficient inertia. Any singleton recurrent class is obviously aperiodic. An u.d.r. process is aperiodic.

• Every Markov process induces (a set of) \textit{invariant distributions} \( \mu : X \to [0,1] \) with \( \sum_{x \in X} \mu(x) = 1 \) and \( \mu \cdot M = \mu \). Every recurrent class \( A \subseteq X \) corresponds to exactly one invariant distribution with support \( A \), \textit{i.e.}, \( \sum_{x \in A} \mu(x) = 1 \). The set of all invariant distributions of a Markov process is the convex hull of the invariant distributions of all its recurrent classes. The support of an invariant distribution \( \mu \) is therefore a (non-empty) set of recurrent classes.

• By the \textit{ergodic theorem} an invariant distribution which is induced by a recurrent class \( A \subseteq X \) describes the time-average behavior of the system once it reached \( A \). That is, \( \mu(x) \) depicts the fraction of time that the system spends in allocation \( x \in A \) if the initial probability distribution over allocations has support \( A \).

• A Markov process is \textit{ergodic} if it has a unique recurrent class. Note that the invariant distribution of an ergodic Markov process is unique and thus depicts the time-average behavior independent of the initial probability distribution over allocations.

• A Markov process is called \textit{irreducible} if it is ergodic and the unique recurrent class coincides with the state space \( X \).

• By the \textit{fundamental theorem of Markov processes} an invariant distribution which is induced by an aperiodic recurrent class \( A \subseteq X \) describes the probability that the system will be in allocation \( x \) if it reached an allocation in \( A \) and propagated forever, \textit{i.e.}, for all \( x \in A \) and all probability distributions over allocations \( \nu : X \to [0,1] \) whose support is contained in \( A \), \( \mu(x) = \lim_{T \to \infty} \nu \cdot P^T \).

• A \textit{perturbed Markov process} \((X, M')\) is a Markov process such that all transition probabilities \( M'(x,x') \) are continuous in \( \epsilon \), and for all \( x, x' \in X \), \( \lim_{\epsilon \to 0} M'(x,x') = M(x,x') \). More specifically, \( M'(x,x') > 0 \) for \( \epsilon > 0 \) implies that there is an \( r \geq 0 \) such that \( \infty > \lim_{\epsilon \to 0} \epsilon^{-r} M'(x,x') \). Hence, we restrict ourselves to regular perturbations and \( M' \) is polynomial in \( \epsilon \) (\textit{i.e.}, we can write \( M'(x,x') = \sum_{i=0,\ldots,\hat{i}} m_i(x,x')\epsilon^i \) where \( \hat{i} \) is a finite number and \( m(x,x') > 0 \) for at least some \( i \)).

• The \textit{limit invariant distribution} \( \mu^* \) of a Markov process \((X, M)\) is the invariant distribution \( \mu^* \) of a perturbed process \((X, M')\) in the limit of \( \epsilon \to 0 \). Note that any perturbed Markov process is irreducible, hence its invariant distribution is unique. Moreover, \( \lim_{\epsilon \to 0} \mu^* \equiv \mu^* \) exists and is an invariant distribution of \((X, M)\) (\textit{e.g.}, Young, 1993, Theorem 4 (i)). Hence, the support of the limit invariant distribution (denoted by \( SRC \)) is a set of recurrent classes of the unperturbed process \( SRC \equiv \{A \in RC \mid \text{for all } x \in A, \mu^*(x) > 0\} \subseteq RC \).

• An allocation in the support of \( \mu^* \) is called a \textit{stochastically stable allocation}. 

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References


