The Quantity and Quality of Teachers: A Dynamic Trade-off

Gregory Gilpin\textsuperscript{a} and Michael Kaganovich\textsuperscript{b,c}

Abstract
We study the dynamics of the quantity and quality of teachers in the framework of dynamic general equilibrium OLG model. The quantity and quality are jointly set by a government agency wishing to maximize the quality of basic education per student while bound by teachers’ collective bargaining agreement which equalizes teacher pay. Our model features two stages of education: basic and advanced (college), the latter being required of teachers. The cost of hiring teachers is influenced by the outside opportunities that college graduates have in the production sector. We show that this factor strengthens in the process of endogenous growth and moreover that it pushes the optimal trade-off between quantity and quality of teachers in the direction of the former. Namely, the number of teachers hired will grow over time while their relative (but not the absolute) human capital attainment will fall. This evolution of human capital accumulation is accompanied by increasing inequality, within the group of college educated workers in particular. Further, we consider the comparative dynamics effect of exogenous skill biased technological change represented by a positive shock to productivity of the skilled workers, hence to the college premium. We show that this will exacerbate the negative trends in the quality of basic education in relation to GDP growth. Countering this trend would therefore require an increase in the share of GDP spent on basic education, assuming that the institutional setup of the school system remains unchanged.

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1. Introduction

A reduction of class sizes in K-12 schooling has been one of the main education policy priorities in the United States over the last several decades as a means to raising education quality. This is evidenced by the dynamics of student-teacher ratio which fell from 25.8 in 1960 to 15.7 in 2005. Remarkably, this decline has persisted through the contemporaneous ups and downs of the enrollment dynamics (Digest of Education Statistics, 2007, table 61, reproduced in Table 1 below). Research, however, has shown that students' test scores have not risen despite increased individualized instruction. This has compelled policy makers and researchers to question the role of the quantity of teachers vs. their quality as factors in student performance (see Hanushek et al. (2005)), particularly in light of a possible trade-off between the two.

Indeed the changes in education statistics between 1955 and 2005 displayed in our Table 1 suggest the possibility of a quantity-quality trade-off in the supply of teachers over this period: a remarkable growth in the total number of teachers consistently outpacing the growth of student enrollments was accompanied by declining relative teacher salaries, despite essentially steady overall K-12 public education expenditures as a share of GDP since 1970. Another significant trend observed over about the same period is the decline in the aptitude of teachers relative to

<table>
<thead>
<tr>
<th>Year</th>
<th>Enrollment</th>
<th>Teachers</th>
<th>Pupil / teacher ratio</th>
<th>Real Expenditures per Pupil</th>
<th>Expenditures per Pupil</th>
<th>Expenditures to GDP</th>
<th>Relative Teacher Salary</th>
</tr>
</thead>
<tbody>
<tr>
<td>1955</td>
<td>30,680</td>
<td>1,141</td>
<td>26.9</td>
<td>234.2</td>
<td>294.2</td>
<td>3.3%</td>
<td>43.0</td>
</tr>
<tr>
<td>1960</td>
<td>36,281</td>
<td>1,408</td>
<td>25.8</td>
<td>2606.3</td>
<td>375.1</td>
<td>3.6%</td>
<td>44.0</td>
</tr>
<tr>
<td>1965</td>
<td>42,173</td>
<td>1,710</td>
<td>22.3</td>
<td>3440.0</td>
<td>537.7</td>
<td>3.9%</td>
<td>44.0</td>
</tr>
<tr>
<td>1970</td>
<td>45,894</td>
<td>2,059</td>
<td>20.4</td>
<td>6534.6</td>
<td>1503.6</td>
<td>4.6%</td>
<td>41.0</td>
</tr>
<tr>
<td>1975</td>
<td>48,819</td>
<td>2,198</td>
<td>18.7</td>
<td>5985.3</td>
<td>2501.6</td>
<td>4.0%</td>
<td>41.0</td>
</tr>
<tr>
<td>1980</td>
<td>40,877</td>
<td>2,184</td>
<td>17.9</td>
<td>7045.4</td>
<td>3755.6</td>
<td>3.8%</td>
<td>35.0</td>
</tr>
<tr>
<td>1985</td>
<td>39,422</td>
<td>2,206</td>
<td>17.2</td>
<td>8014.7</td>
<td>5257.8</td>
<td>4.3%</td>
<td>35.0</td>
</tr>
<tr>
<td>1990</td>
<td>41,217</td>
<td>2,398</td>
<td>17.3</td>
<td>8121.8</td>
<td>6146.9</td>
<td>4.3%</td>
<td>36.5</td>
</tr>
<tr>
<td>1995</td>
<td>44,840</td>
<td>2,598</td>
<td>16.7</td>
<td>9214.8</td>
<td>7930.7</td>
<td>4.5%</td>
<td>36.5</td>
</tr>
<tr>
<td>2000</td>
<td>47,204</td>
<td>2,941</td>
<td>15.7</td>
<td>10041.5</td>
<td>9788.4</td>
<td>4.6%</td>
<td>36.5</td>
</tr>
<tr>
<td>2005</td>
<td>49,113</td>
<td>3,137</td>
<td>15.7</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Source:
1 Digest of Education Statistics 2007, Table 61.
2 Digest of Education Statistics 2007, Table 25.
3 Hanushek & Rivkin (2003).
4 Digest of Education Statistics 2007, Table 32.

Notes:
* In thousands
b In Percent
c 1959 data.
d College educated females, age 20-29, earning less than average female teacher, age 20-29.
e Deflated by CPI and in 2006 dollars.
other educated workers. Hoxby and Leigh (2004) estimate that in 1963, 41% of all teachers were of the “middle” aptitude relative to their educated peers, with 17% above and 42% below the average; by comparison, in 2000, 28% of all teachers were of the “middle” aptitude with 5% above and 67% below average. Corcoran et al. (2004) provide similar results.

Much of the empirical literature analyzing the factors of student performance focuses on estimating the returns to either quality or quantity of teachers while attempting to control the other variable. For example, Aaronson et al. (2007), Clotfelter et al. (2007), Rivkin et al. (2005), Goldhaber and Anthony (2007)) estimate the effect of teacher characteristics on student achievement while partially controlling the class size based on geographic and time variation of class size mandates. The second empirical strategy aims to estimate how class size affects student achievement while attempting to control for teacher quality. Angrist and Lavy (1999), Krueger and Whitmore (2001), Krueger (1999), Jepsen and Rivkin (2002) work with data from policy experiments producing random assignment of students to smaller and larger classes to obtain unbiased estimates of the effects of class size on student achievement controlling for teacher quality. Using data from North Carolina, Clotfelter et al. (2007) conclude that teacher experience, test scores and regular licensure all have greater positive effects on student achievement, whether compared to the effects of changes in class size or to the socioeconomic characteristics of students. However, Goldhaber and Anthony (2007) using the same North Carolina data find that the evidence is mixed on whether improved observable teacher credentials have positive impact on student achievement. These results are similar to Rivkin et al. (2005) who use the UTD Texas Schools Project. On the other hand, Angrist and Lavy (1999) use Israel’s mandated class size limits estimate class size effects on student achievement. They find that reducing class size causes significant and substantial increase in test scores for fourth and fifth graders, although not for third graders. Krueger (1999) analyzes data from Tennessee Project STAR and finds that students’ scores on standardized tests increase by four percentage points in the first year they attend smaller classes while in subsequent years the test scores grow by about one percentage point per year. These findings are, however, challenged by Hanushek (1999) who suggests that the project design has resulted in the upward bias in the data in favor of the class size effect.

Thus the results of the empirical literature do not seem to provide a conclusive justification for the policy focus on reducing teacher-pupil ratio, on the grounds of benefits to
student performance. This appears to be due in part to the fact that the pursuit of this policy has been accompanied by a decline of teacher quality, at least in relation to their college-educated peers. In fact, the presence of this quantity-quality trade-off in the dynamics of the two variables evokes the challenge of endogeneity in the estimation exercises. Indeed, in response to Card and Krueger (1992) finding, based on longitudinal data, of positive effect of decreasing class size on student performance, Jepsen and Rivkin (2002) argue that using mandated class size reduction programs as natural experiments for estimating the class size effect is problematic when these changes are accompanied by a trade-off between the quantity and quality of teachers, since it is likely to be endogenous. They point out that California’s class size reduction program came at the expense of hiring lower quality teachers to staff additional classrooms, which offset the benefits of smaller classes. Similarly, Hoxby (1996) finds that in unionized public school higher measured inputs may produce no gains to student achievement: the unions engage in rent-seeking which leads to lower productivity, via teacher quality or effort, enough to offset any gains from the additional resources, smaller student-teacher ratios in particular.

Thus, despite a significant attention in the literature, the understanding of the striking structural change in the allocation of educational resources over the recent decades remains incomplete. This underscores the need for a broader theoretical framework, which would capture the dynamic interaction between inputs in education as it is influenced by the labor market in the production economy. We note in this regard a branch of recent literature which has studied how outside job market opportunities have affected the quality of teachers. Flyer and Rosen (1997) report that the three-fold increase in direct costs of education per student is attributable to the growing market opportunities for women. Hanushek and Rivkin (1997) document the decline in the earnings of women teachers relative to women in other occupations and suggest that the expansion of alternative opportunities reduced teacher quality. Hanushek and Rivkin (2003) estimate that in 1955, 50% of all educated male workers earned less than male teachers, compared to 36% in 2000. Likewise, in 1955, 48% of all educated female workers earned less than female teachers compared to 29% in 2000. Similar analyses concerning the effect of the outside work opportunities on teacher quality are offered by Goldhaber and Liu (2003), Stoddard (2003), and Bacolod (2006). Lakdawalla (2006) demonstrates that a rising skill premium of educated workers due to faster technological change coupled with low productivity growth of
skilled teachers, has lead to the substitution of unskilled teachers for increasingly expensive skilled teachers.

Our paper develops a theoretical framework for analyzing the teacher quantity-quality trade-off and offers an explanation to the observed trend biased in favor of quantity. We present a model which incorporates the factors of education quality discussed above, in a dynamic general equilibrium framework where government education policy decisions affect and are affected by individual education and employment decisions, whereas the dynamics of human capital accumulation and labor productivity has a feedback effect on both. In our model, the government agency wishes to maximize the quality of basic education per student and, due to a budget constraint, faces a trade-off between the quality and quantity of teachers to be hired. Furthermore, we assume that the agency is bound by teachers’ collective bargaining agreement which equalizes teacher pay. It is, indeed, well documented that teachers’ unions significantly contribute to the phenomenon of compression of seniority adjusted wages: unions provide tenure to teachers and tie their salary primarily to experience rather than performance; administrators wishing to hire higher quality teachers are forced by the unions to then provide matching raises to teachers across the board.\footnote{It should be noted that unionization is not the sole factor responsible for the compression of teacher salaries. It is also due in part to the difficulty of measuring teacher productivity, especially in terms of educational value added given unobservable student characteristics. But even if such characteristics were observable, there still exists the challenge of determining criteria for performance based pay for teachers. For example, low ability students exhibit relatively low average gains in learning throughout the year, therefore an approach based on marginal improvement of students’ performance would not fairly compensate teachers for working with lower ability children.}

The wage compression in public schools imposes similar wage rigidity on the private school teacher market (Lakdawalla, 2006).

In our model, a government education agency has two policy variables: teacher salary, which is uniform according to the collective bargaining regime, and the number of teachers to be hired. The model features two stages of education: basic and advanced (college), the latter being required of teachers. College graduates can also take jobs in the skilled labor force of the production sector and get paid a competitive wage according to their human capital attainment. This opportunity cost implies that the level of teacher salary set by the government agency will determine the top quality (human capital level) of a teacher who can be hired at this salary. All college graduates whose human capital is below this level will be motivated to take a teaching
job at this same salary. Therefore, given the top quality cut-off determined by the government-set teacher salary, the number of teachers the government decides to hire will determine the lowest human capital cut-off among teachers. Thus the total cost of hiring teachers is affected in our model by the outside opportunities available to skilled individuals in the production sector. We show, moreover, that in the process of endogenous growth this effect strengthens and that it pushes the optimal trade-off between quantity and quality of teachers in the direction of the former. Namely, in the face of rising over time cost of highly able skilled workers, the government agency will find it optimal to opt for increasing the number of teachers hired while reducing the overall relative quality of the pool of teachers. (The absolute human capital attainment of teachers, however, will rise along with the overall human capital accumulation, while sliding toward the lower tail of the distribution of college educated population.)

Furthermore, we show that this human capital dynamics is characterized by increasing inequality within the group of college educated workers as well as between it and the unskilled.

Thus this paper offers a theory explaining the trend in education policy in favor of lower student-teacher ratios (i.e., higher quantity of teachers) combined arguably with deteriorating teacher quality, despite growing per student schooling expenditures.

We then build on these results to further analyze the impact of exogenous technological change biased toward skill, i.e. augmenting productivity of skilled workers and thereby the college premium. We show that such technological change will exacerbate the negative trends in the quality of basic education in relation to GDP growth. Specifically, the comparative dynamics effect (relative to the benchmark trajectory) will be detrimental to the aggregate quality of teachers as well as to the quality of basic education, due to the upward shocks to the cost of skilled labor. This is consistent with the aforementioned observation by Lakdawalla (2006) as well as the discussion by Rangazas (2002) of Baumol’s Disease in education sector as a result of technological change driving up the sector’s real labor costs, which leads him to conclude that school expenditures must be deflated by the rate technological change in addition to the rate of inflation. Our model adds a dimension to this understanding: we show that in the process of human capital driven economic growth the rise in premium for high ability will outpace that for the average even besides the effect of technological change, hence an additional downward pressure on the “real” quality of education inputs.

The paper is organized as follows. Section 2 develops a dynamic general equilibrium
model with unionized public schools. Section 3 defines a competitive equilibrium, Section 4 derives main results, and Section 5 concludes. Appendix 1 contains most of the proofs. Appendix 2 presents a glossary of notation.

2. The Model

We develop a general equilibrium growth model of an economy populated by overlapping generations of individuals whose life consists of three periods: childhood, young adulthood, and old age. We identify a generation with the period when its members are young adults, thus the individuals born in period $t-1$ form a generation $G_t$. We assume that population size is constant in each generation $G_t$ and that it forms a continuum on the interval $[0,1]$. Let $\mu(.)$ be the induced Lebesgue measure on this set, so that $\mu([0,1]) = 1$ for all $t$.

Children make no decisions of their own and receive basic (or first stage) education which is provided publicly. Young adults are endowed with a unit of time and face an option of devoting a fixed fraction $n$ of it to acquiring advanced education (which we will also refer to as college or second stage education); the balance of time not spent on education is inelastically devoted to work. Specifically, the individuals without college education will work for the full unit of time in the “unskilled” production workforce. Those with college education either work for the remaining fraction of time $1-n$ in the “skilled” production workforce or, if qualified by the government, can work as public school teachers. Individuals derive income from work. They spend part of it on consumption when young and invest the rest to use the returns to finance their consumption in retirement, the last period of life.

2.1. Production

The production sector of the economy consists of private perfectly competitive firms producing a homogeneous capital/consumption good by means of a constant returns technology which uses three factors of production - physical capital as well as unskilled and skilled human capital. The aggregate production function is given by

$$Y_t = D K_t^\alpha \left[ H_t^\rho + \theta_t H_t^\psi \right]^{1-\alpha},$$

(1)

See Appendix 2 for the glossary of notation.
where $\alpha \in [0,1]$, $D > 0$, while $K_t$, $H_t^u$, $H_t^s$ stand, respectively, for aggregate supply of physical capital, unskilled human capital, and skilled human capital employed in the production sector in period $t$. The coefficient $\theta_t$ characterizes the net productivity augmentation of skilled human capital (adjusted for the shorter employment duration due to the time spent in college) which is imbedded in technology. The sequence of $\{\theta_t\}_{t=0}^\infty$ characterizing the evolution (or stationarity, as a special case) of the skill premium in the process of technological change is assumed to be exogenously given.

2.2. Households

All individuals $\omega$ of generation $G_t$, $t = 0, 1, 2, \ldots$ have identical intertemporal preferences over consumption as young adults and retirees given by

$$\ln c_{t,t} (\omega) + \beta \ln c_{t,t+1} (\omega)$$

subject to the life-time budget constraint

$$c_{t,t} (\omega) + (1 + r_{t+1})^{-1} c_{t,t+1} (\omega) \leq (1 - \tau_t) I_t (\omega)$$

where $r_{t+1}$ is the market interest rate, $I_t (\omega)$ is the individual’s wage income derived from human capital, while $\tau_t$ is the uniform rate of labor income tax collected by the government. We leave the process by which the tax rates are determined outside the scope of this study and assume that $\tau_t$ are exogenous and may either be constant or vary over time (see a discussion of a rationale for the latter possibility in the concluding Section 5 of the paper).

According to production function (1) individuals working in the production economy receive the wage at competitive rates $w_t$ and $\theta_t w_t$, respectively, per unit of their unskilled or skilled human capital, whichever applies. Thus the income of individual $\omega$ who receives only basic education and attains the level of unskilled human capital $h_t^u (\omega)$ will be

$$I_t^u (\omega) = w_t h_t^u (\omega)$$

The individual $\omega$ who obtains college education, attains the level of skilled human capital $h_t^s (\omega)$ and is employed in the production sector, will receive income

$$I_t^s (\omega) = \theta_t w_t h_t^s (\omega)$$

College educated individuals who become teachers will receive income $I_t^h$ to be specified later.
2.3. Human Capital Formation

The human capital received by each child \( \omega \) of generation \( G_t \) at the first (basic) stage of his education is produced in period \( t-1 \) by combining children's random innate ability with public education according to

\[
\eta^*(\omega) = C a(\omega) E_{t-1}
\]

where \( C \) is a positive constant, \( E_{t-1} \) is a uniform quality of public schooling received by each child in period \( t-1 \) while \( a(\omega) \) is the child’s innate ability.

**Assumption 1.** Innate ability \( a(\omega) \) is distributed independently and identically in each generation (the time indexation is thus omitted); the distribution is uniform in the interval \([a, A]\).

The uniformity assumption, which we make following the example of Galor and Moav (2000), is aimed at obtaining similarly tractable analytical results but does not appear to be essential for their overall intuition. To further simplify the exposition (but at no cost to the essence of the matter) in deriving our main results in Section 4 we will let \( a = 0 \).

We will now introduce the human capital production function for the advanced (college) stage of education. Consistent with Ben-Porath (1967) and Rosen (1976) we assume that the gains from college education depend on one’s prior preparation, which in turn depends on innate ability. Moreover, we assume that college education has a pre-requisite human capital threshold \( h^* \). Rather than an *ad hoc* admission requirement (we assume that all individuals are free to choose to go to college but base this decision purely on income considerations) we view this threshold as a set of benchmark skills, such as adequate language and mathematical proficiency whose deficit would preclude any benefit from learning at the advanced stage. Specifically, we postulate that if an individual \( \omega \) of generation \( G_t \) chooses to go to college, he will become a

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3 See Su (2004) for a similar approach to college eligibility. One can envision that this knowledge threshold may evolve over time with changes in learning technology. For example, it now tends to incorporate computer literacy; while applicants are not tested on it for admission decisions, their progress in many college specialties will critically depend on it. For the purposes of our analysis \( h^* \) is assumed fixed. Note the important distinction between this knowledge threshold, which determines a student’s true performance, and the concept of an *ad hoc* admission threshold addressed in the “educational standards” literature, such as Betts (1998), an education policy variable that serves as a sorting device and employability signal.
“skilled” agent with the level of human capital given by

\[ h_i^*(\omega) = bh_i^u(\omega) + B[h_i^*(\omega) - h^*] \]  (7)

where \( b \in (0,1) \) and \( B > 0 \) are given constants. Thus according to the expression (7) the gains from college education depend on the extent to which the individual’s pre-college human capital attainment \( h_i^u(\omega) \) exceeds the threshold \( h^* \).

The college education production function (7) also reflects a partial loss of pre-college human capital, according to the coefficient \( b \), for the purposes of skilled human capital. While this loss is counteracted by the net productivity augmentation \( \theta_i \) of skilled human capital according to the economy’s production function (1), we impose a condition

\[ b\theta_i < 1 \]  (8)

which indeed implies that individuals whose pre-college human capital \( h_i^u(\omega) \) is below, at, or even slightly above the threshold \( h^* \), will not gain from attending college and therefore will not choose to do so. It is likewise logical to assume that highly able individuals, particularly those with the highest ability level \( A \), will benefit from attending college. According to equations (6) and (7) this will be true in generation \( G \) iff the following inequality holds:

\[ (b + B)CAE_{i-1} \geq Bh^* \]

In Section 4, and in more detail in Appendix 1, we will state parametric assumptions which in particular ensure that the above inequality does hold at all times.

According to the expressions (6) and (7) human capital attainment, and therefore the corresponding income level, is an increasing function of the innate ability. Therefore if a certain individual decides to attend college then all agents with higher ability will also do so. Thus in each period \( t \) there is an ability cut-off level \( a_i^* \) such that an individual \( \omega \) in generation \( t \) will choose to attend college if and only if his ability \( a(\omega) \) exceeds \( a_i^* \). Without the loss of generality we’ll make a convention that individuals with ability level \( a_i^* \) do choose to go to college. We will later show that this college attendance ability cut-off level is given by the formula

\[ a_i^* = \frac{1}{CE_{t-1}} \frac{\theta_i Bh^*}{\theta_i (b + B) - 1} \]  (9)

which has a straightforward meaning: an individual will choose to attend college if and only if
his wage income derived from skilled human capital given by formula (7) and adjusted for the net productivity augmentation $\theta$, will exceed his wage based on the unskilled human capital obtained at the first stage of education according to its production function (6).

The kinked form of the college human capital production function (7), combined with pre-college preparation given by (6), implies that individual’s advanced human capital attainment exhibits increasing returns to ability, for which the quality of basic publicly provided education is a complementary input. This allows us to capture an important and arguably realistic property of the “ability premium” of college education: the skill upgrade that it provides to a highly able student is disproportionately larger than the one gained by a less able peer. Furthermore, while higher quality public basic education “lifts all boats”, more able students will derive disproportionately greater benefits from it. This “ability premium” argument is used in some of the recent literature to explain the evidence of increasing dispersion of earnings. For example, Huggett et al. (2006) use life-cycle framework with a multi-stage Ben-Porath type model of human capital accumulation, which exhibits increasing returns to ability at higher stages of education, to explain the evolution of wage dispersion in the U.S. Another strand of models represented by Galor and Zeira (1993) is able to explain intergenerational persistence of inequality by the presence of credit constraints. The underlying mechanism, however, is fundamentally similar to ours: the consequence of the borrowing constraints is that investment in education exhibits increasing returns to agents’ endowments (within a certain range). Restuccia and Urrutia (2004) use a calibrated model which includes explicit early and college stages of education to apportion the factors, including individual ability and borrowing constraints, responsible for the intergenerational persistence of income inequality. By contrast, in most models of public education, such as by Glomm and Ravikumar (1992), human capital accumulation exhibits decreasing returns to private inputs, which leads to vanishing relative variation of income.

2.4. Quality of Basic Education

We shall now introduce the per student basic education quality $E_t$, i.e. the public input in the

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4 Cunha and Heckman (2007) and related recent work appear to provide a unified framework for these approaches.
basic education production function (6) provided in period $t$, as a function of the quality and quantity of teachers chosen by a government agency. Recall that only college educated individuals are eligible to be employed as teachers. Let $\Sigma_t$ be the set of individuals $\omega$ in generation $t$ employed as teachers. Let $z_t$ be the total number of teachers. Since population size was normalized to 1 in all generations, $z_t$ is also the fraction of teachers in the overall population in generation $t$, as well as the teacher-student ratio for generation $t+1$ students.

We define the aggregate teacher quality as the aggregate human capital of teachers

$$q_t = \int_{\omega \in \Sigma_t} h^t(\omega) d \mu_t(\omega)$$

Likewise, the average teacher quality is given by $z_t^{-1} \int_{\omega \in \Sigma_t} h^t(\omega) d \mu_t(\omega) = z_t^{-1} q_t$. The explicit account for the heterogeneity of teachers’ human capital attainment reflected in this formula is obviously an essential element of our model. Eckstein and Zilcha (1994), to our knowledge the first to explicitly incorporate teacher human capital as an input in (compulsory) schooling, assumed that it equals to the average human capital of their generation.\(^5\)

We now define the per student quality of basic education as a Cobb-Douglas function of the quantity and aggregate quality of teachers:

$$E_t = z_t^{-1} q_t^\gamma$$

An education production function similar to (10) was used by Tamura (2001) to estimate the factors of observed conditional convergence of incomes across the US states. He, however, included parental human capital along with that of teachers’ among the factors in the production of children’s human capital, such that it exhibited constant returns to the unbounded factors, enabling the model to feature sustained endogenous growth of incomes. Much of the literature incorporating education technology in models of economic growth, following Glomm and Ravikumar (1992), includes education expenditure as a factor separate from the effect of human capital of parents or teachers, such that assuming constant returns to the two ensures sustained endogenous growth of incomes.

\(^5\) Hatsor (2008) contrasts such regime, which implies that teachers are selected at random from the population, with the one where the quality of teachers is an optimal policy decision traded off against their quantity. She focuses on comparing the implications of these regimes for growth and welfare in the framework of strategic interaction between the education and budgeting authorities of the government.
growth. Since the human capital of teachers is the only potentially unbounded educational input in our simplified model, which is in this respect similar to human capital production in Lucas (1988), we must likewise require non-decreasing returns to this factor alone, i.e. \( \nu \geq 1 \), in order to also ensure that the model can exhibit aggregate growth of human capital across generations.

The special case of (10) with \( \gamma = \nu = 1 \), i.e.

\[
E_t = z_t \int_{\omega \in \Omega_t} h_t^p(\omega)d\mu_t(\omega)
\]

has a particularly straightforward interpretation. Assume that all teachers are perfectly sorted across classes, each class of equal size \( z_t^{-1} \), such that each student is exposed through his classes to a cross-section of teachers which perfectly represents their distribution of quality. Then the expression (11) which is equivalent to

\[
E_t = \int_{\omega \in \Omega_t} \frac{h_t^p(\omega)}{z_t}d\mu_t(\omega)
\]

can be interpreted as per student average teacher quality.

2.5. Government

The government funds and administers public education at the basic level with the goal of maximizing its quality \( E_t \), as defined above, subject to the budget constraint given by the revenue from a uniform labor income tax at a flat rate \( \tau_t \). To this end in each period \( t \), the government must set teacher salary \( I_t^h \) and the number of teachers to be hired \( z_t \). As discussed in the Introduction, we postulate that all teachers in generation \( G_t \) receive equal salary, according to a collective bargaining agreement. Since college educated individuals have an option to work in the production sector for a competitive wage as defined by the expression (5), the government’s choice of teacher salary \( I_t^h \) will uniquely determine the highest level of human capital attainment \( \bar{h}_t \) among individuals who will choose to become teachers. Indeed it should satisfy the equation\(^6\)

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\(^6\) Since one’s work career is summarily represented in our model by one time period, we do not model the wage dynamics over the course of a worker’s or teacher’s career as he accumulates seniority and experience. The appropriate understanding of the income variables in this framework is that they represent aggregates over the entire career, such as respective present values at the career’s outset. While teachers’ union collective bargaining agreements stipulate wage differentials based on seniority, equation (12) should be understood as the comparison of respective aggregates over the course of the alternative careers in question.
\[ \theta^*, \bar{h}_i = I_i^h \] (12)

Thus all college graduates with human capital level \( h_i^*(\omega) \) at or below \( \bar{h}_i \) will be obviously motivated to accept employment as a teacher rather than work in the production sector. However, the government’s goal to maximize the overall education quality for a set number of teachers \( z_i \) implies that the set \( \Sigma_i \) of teachers the government will hire consists of all individuals whose level of human capital \( h_i^*(\omega) \) attained in college falls into the interval \([h_i, \bar{h}_i]\) where the minimum teacher qualification threshold \( h_i \) is determined by the intended number of teachers, i.e. the measure

\[ z_i = \mu(\omega \mid h_i \leq h_i^*(\omega) \leq \bar{h}_i) \] (13)

where the top cut-off \( \bar{h}_i \) is determined, according to (12), by the teacher salary \( I_i^h \) set by the government.

Recalling the production functions of basic and advanced education given, respectively, by the expressions (6) and (7), we define the cut-off innate ability levels \( \bar{a}_i \) and \( \bar{a}_i \) which characterize the teachers who possess, respectively, the cut-off levels of human capital \( h_i \) and \( \bar{h}_i \) induced by the government policy choice. In other words (we again refer the reader to the glossary of notation available in Appendix 2),

\[ a_i = \frac{h_i + Bh_i^*}{(b + B)CE_{t-1}} \quad \text{and} \quad \bar{a}_i = \frac{\bar{h}_i + Bh_i^*}{(b + B)CE_{t-1}} \] (14)

Accordingly, the number of teachers to be hired, as defined by the formula (13), can be expressed as

\[ z_i = \frac{\bar{a}_i - a_i}{A - a} \] (15)

For the government policy choice of \( I_i^h, \ z_i, \) to be feasible, the minimum teacher qualification threshold \( h_i \) defined by (13) must belong to the range of human capital levels attained by college graduates. In other words, the corresponding ability level \( a_i \) must exceed the college attendance cut-off level \( a_i^* \).

Thus according to (10) the government’s basic education quality optimization problem can be stated as

14
\[
\max_{z_t, \tilde{h}_t} E_t
\]

subject to (13)

\[z_t \theta_t w_t \tilde{h}_t = T_t \quad \text{and} \quad a_t \geq a_t^*\]

where \(T_t\) is the tax revenue collected by the government in period \(t\).

Using the expressions (6), (7), (14), (15) this problem can be restated in terms of decision variables \(z_t\) and \(a_t\), which will be helpful for providing a closed form solution to the problem. We place this technical analysis in Appendix 1.

Figure 1 below offers an illustration for the government's education quality optimization problem in period \(t\). The horizontal and vertical axes represent, respectively, the ability of current workers and their wage income. The wage income of unskilled workers is given by \(w_t h_t^n\), while for the skilled production sector workers it is \(w_t h_t^s\). The sloped line represents the distribution of incomes of production sector workers in generation \(t\) as a function of individual ability according to the formulas (6) and (7), where the kink corresponds to the ability cut-off for college attendance \(a_t^*\).

![Figure 1: Basic Education Optimization Problem](image-url)
The vertical bars in the figure illustrate several feasible government education policy combinations of the number of teachers to be hired and teacher salaries satisfying the government's budget constraint: \( (z_t^1, I_t^h), (z_t^2, I_t^{h2}), (z_t^3, I_t^{h3}) \), from left to right. The width of the bar corresponds to the number of teachers \( z_t \) while its height stands for teacher salary \( I_t^h \), so the bar’s area is the total expenditure on basic education. Policy \( (z_t^1, I_t^h) \) represented by the left-most bar in the figure is characterized by the lowest student-teacher ratio (the largest quantity of teachers) but also the lowest teacher quality. This policy option is highly inefficient in terms of education quality derived from the given budget revenue, as shown by the large deadweight loss represented by the triangular portion of the bar located above the sloped line: this is the excess salary paid to teachers above their opportunity cost value. The deadweight loss is the highest here because this policy involves hiring an inefficiently large number of low quality teachers who nevertheless have to be paid at the same rate as their best peer. The policy \( (z_t^3, I_t^{h3}) \) (the right-most bar in the figure) delivers the highest teacher quality, relative to the other two policy combinations. This education policy has the lowest deadweight loss associated with the collective bargaining regime but it still has low per pupil education quality due to small number of teachers, i.e. high student-teacher ratio. The interior policy option \( (z_t^2, I_t^{h2}) \) is where the per student education quality is the highest among the three options depicted, providing the balance in terms of student-teacher ratio and the deadweight loss associated with the span of teacher quality.

3. General Equilibrium and Optimal Policy

We can now summarize the fundamental elements of the model and their relationships in a general equilibrium framework. We will first define the dynamic general equilibrium for given government education policy parameters and then incorporate the government’s optimal policy into the recursive dynamic general equilibrium framework.

Given the sequence of tax rates \( \{\tau_t\}_{t=0}^\infty \) and the sequence of government education policy parameters \( \{I_t^h, z_t\}_{t=0}^\infty \), i.e. teacher salaries and the numbers of teachers hired in each period, respectively, as well as the initial period \( t = 0 \) aggregate supply of capital \( K_0 \), the distributions
of the retirees’ consumption levels \( c_{-1,0}(\omega) \), and per student basic education quality \( E_{-1} \) provided to generation \( G_0 \) individuals as children, we define the dynamic general equilibrium as a collection of sequences of

(a) factor prices \( \{ (1 + r_{t+1}), w_t, \theta_t w_t \}_{t=0}^{\infty} \) respectively of physical, unskilled and skilled human capital as inputs in production in period \( t \);

(b) aggregate variables \( \{ Y_t, K_t, H^u_t, H^s_t, T_t, E_t, \alpha_t \}_{t=0}^{\infty} \), i.e., respectively, aggregate output, inputs of physical, unskilled and skilled human capital in production, government’s tax revenue, the quality of basic education provided to each student in period \( t \), as well as the endogenous innate ability cut-off for college attendance;

(c) distributions of individual consumption and education decisions

\( \{ c_{t,t}(\omega), c_{t+1,t}(\omega), h^u_t(\omega), h^s_t(\omega) \}_{t=0}^{\infty} \) as well as employment decisions by college graduates such that

(i) the factor prices are determined competitively, i.e. are set equal to the marginal products of respective inputs:

\[
1 + r_{t+1} = \alpha DK_\alpha \left[ H^u_t + \theta_t H^s_t \right]^{-\alpha}, \quad w_t = (1 - \alpha) DK_\alpha \left[ H^u_t + \theta_t H^s_t \right]^{-\alpha}
\]

(ii) each individual \( \omega \in [0,1] \) in generation \( G_t \) makes a decision whether to go to college and if so whether to be employed as a teacher or in the production sector so as to maximize his income while taking as given basic education quality \( E_{t-1} \), production sector wage rates \( w_t \) and \( \theta_t w_t \) (for unskilled and skilled labor, respectively), teacher salary \( I^b_t \) and the number of teachers \( z_t \) to be hired, whereas his human capital level \( h^u_t(\omega) \) or \( h^s_t(\omega) \) (depending on his college attendance decision) is determined according to the education production functions (6) and (7); (note that according to equation (12) and the collective bargaining agreement, teacher salary will exceed production sector wage for all but the top quality teacher, so the government teacher employment limit \( z_t \) will bind;)

(iii) based on his income \( I_t(\omega) \) determined according to (ii), each individual \( \omega \in [0,1] \) in generation \( G_t \) makes his young- and old-age consumption decisions \( c_{t,t}(\omega) \),
by solving the optimization problem (2)- (3) while taking the rates of interest $1 + r_{t+1}$ and tax $\tau_t$ as given;

(iv) the quality of basic education $E_t$ provided to generation $G_{t+1}$ individuals (as children) is determined by the expression (10) while the set of teachers $\Sigma_t$ is defined by individual employment decisions according to (ii) and the number of teachers hired $z_t$ is as given by the government’s policy;

(v) the markets for goods, physical capital, and skilled and unskilled labor clear in each period:

\[
Y_t = \int_{\omega \in [0,1]} c_{t,t}(\omega) d\mu_t(\omega) + \int_{\omega \in [0,1]} c_{t,t-1}(\omega) d\mu_{t-1}(\omega),
\]

\[
K_t = (1 + r_{t+1})^{-1} \int_{\omega \in [0,1]} c_{t,t-1}(\omega) d\mu_t(\omega),
\]

\[
H_t^u = \int_{a \leq a(\omega) \leq a^*_t} h_t^u(\omega) d\mu_t(\omega),
\]

\[
H_t^w = \int_{a^*_t \leq a(\omega) \leq A} h_t^w(\omega) d\mu_t(\omega) - \int_{\omega \in \Sigma_t} h_t^w(\omega) d\mu_t(\omega),
\]

where the ability cut-off for college attendance $a_t^*$ is determined by individual college attendance decisions as defined in (ii);

(vi) the aggregate tax revenue is composed of labor income taxes collected from all categories of employees, i.e.

\[
T_t = \tau_t \left( w_t H_t^u + \theta_t w_t H_t^w + z_t I_t^h \right)
\]

We can now define the government’s optimal education policy in period $t$ recursively, based on the above general equilibrium construct. Namely, the government chooses teacher salaries $I_t^h$ and the number of teachers $z_t$ for period $t$ by solving the optimization problem (16) where the top teacher quality $\bar{h}_t$ is determined by equation (12), while taking as given the economy’s general equilibrium values of production sector wage rate $w_t$, aggregate tax revenue $T_t$ and the distribution of skilled human capital attainment $h_t^s(\omega)$ by generation $G_t$ individuals.
Noting the mutual dependence of the general equilibrium variables in period $t$ and the government’s optimal education policy we define the Education-Economy recursive dynamic equilibrium (RDE for brevity) as a fixed point of this relationship, recursively determined for each period $t$.

4. Main Results

To simplify the exposition we will assume henceforth without any substantive loss of generality that parameter $a = 0$, thus innate ability in each generation is distributed uniformly on the interval $[0, A]$. Furthermore, we impose some conditions on parameters of both basic and college education systems, as well as on the level of education tax rates, skill premium in the production sector, as well as on per student basic education quality $E_{-1}$ provided to students in the initial generation $G_0$.

**Assumption 2.** Education funding is sufficiently high and both basic and college education technologies are sufficiently productive to make sustained growth of human capital possible. Specifically, $\nu \geq 1$ (returns to the human capital of teachers are non-decreasing)\(^7\) and the productivity coefficients $C$ and $(b + B)$ are sufficiently large, while education tax rates $^8 \tau$, are not too low while not exceeding $1 - \frac{\nu}{2\nu}$ at all times.

The gist of all of the parameter restrictions imposed by Assumption 2 is ensuring that the

\(^7\) The specific technical role of the non-decreasing returns requirement is to ensure growth of basic education quality with a sustained growth factor $g > 1$ stated in Lemma 1 below. In case of $\nu < 1$, Lemma 1 and ensuing main results of the paper will remain qualitatively correct in transition to a steady state if the initial stocks of human capital in the economy are low.

\(^8\) Variability of education tax rates over time is not essential for our results. We explicitly allow for this possibility so as to (a) reflect the rise in per student public education funding in relation to GDP growth apparent in Table 1 and (b) make the results applicable to a further discussion in the paper of government policies which may indeed involve such rise in education funding. The explicit formulation of Assumption 2 in Appendix 1 shows that it imposes a uniform lower as well as upper bound on $\tau$. 
funding (determined by the tax rates) and productivity of both basic and college level education systems (determined by respective parameters C and \((b+B)\) of the education production functions (6) and (7) under the condition of non-decreasing returns to teacher quality \(\nu \geq 1\) discussed in Section 2.4) are sufficient to make sustained growth of human capital attainments across generations possible. The fact that such sustained growth of education quality, particularly at the basic level, actually takes place under these conditions is established in Lemma 1 below.

The additional condition \(\tau_t \leq 1 - \frac{\gamma}{2\nu}\) in Assumption 2 will be certainly met if \(\nu \geq \gamma\), i.e. if the relative importance of teacher quality in schooling effectiveness is not lower than that of the teacher-student ratio, which will be clearly the case here under the natural requirement that \(\gamma \leq 1\), i.e. that the there are non-increasing returns to the number of teachers.

More specific technical restrictions on the parameters involved in Assumption 2 are spelled out in Appendix 1 where we then prove the existence and uniqueness of the recursive dynamic equilibrium (RDE) in the model. We do so by deriving the explicit solution of the optimal education policy problem (16) in terms of decision variables \(z_i\) and \(\bar{a}_i\), as well as the optimal lower ability cut-off among those hired as teachers \(\bar{a}_i\) and the resulting recursive dynamic equilibrium sequence of these variables. Specifically, we prove the following

**Proposition.**

(i) In each period \(t = 0,1,\ldots\) the basic quality optimization problem (16), where the government’s education budget \(T_t\) is defined by the general equilibrium relationship (21) while the prior period’s education quality \(E_{t-1}\) and individual educational and employment decisions are given, has a unique solution characterized by the following expressions, respectively, for the optimal quantity of teachers and the upper and lower ability cut-offs for teachers to be hired:

\[
z_t = \left(\frac{\gamma}{(2\nu - \gamma)} \left(\frac{\tau_t}{1 - \tau_t}\right)\right)^{1/2} \left(1 - \frac{2B^*h^*}{(b + B)ACE_{t-1}} + \frac{\theta_i (B^*h^*)^2}{(\theta_i (b + B) - 1) (b + B)(ACE_{t-1})^2}\right)^{1/2}
\]

\[
\bar{a}_t = \frac{B^*h^*}{(b + B)CE_{t-1}} + A_z \left(\frac{\nu(1 - \tau_t)}{\gamma} + \frac{1}{2}\right)
\]
This optimal solution for period $t$ education policy uniquely determines the general equilibrium college attendance and employment decisions by generation $t$ individuals, and as a consequence, period $t$ basic education quality $E_t$. This recursion defines the unique recursive dynamic equilibrium in the model for all $t = 0, 1, \ldots$ given the initial generation’s basic education quality $E_{t-1}$.

The following important Lemmas are proven in Appendix 1:

**Lemma 1 (Growth of Basic Education Quality).** The recursive equilibrium dynamics exhibits sustained growth of the quality of per student basic education. Specifically, there is a factor $g > 1$ such that $E_t > gE_{t-1}$ is true for all $t = 0, 1, \ldots$

**Lemma 2 (The Interiority Property).** In the recursive dynamic equilibrium, the ability of the least qualified teacher exceeds the college attendance cut-off ability in all time periods, i.e. $a_t > a_t^*$ is true for $t = 0, 1, \ldots$. Thereby the human capital of the least qualified teacher will not be the lowest among his contemporary college graduates.

**Lemma 3.** The ability cut-off for college attendance $a_t^*$ satisfies equality (9), i.e.

$$a_t^* = \frac{1}{CE_{t-1}} \left( \frac{\theta_i B h^*}{\theta_i (b + B)} - 1 \right)$$

which means that an individual will choose to attend college if and only if his resulting skilled human capital given by formula (7) adjusted for the net productivity augmentation $\theta_i$ will exceed his unskilled human capital derived from the basic stage of education according to its production function (6).

**Remark.** Since we assumed that individuals make a decision whether to attend college solely on the basis of income motive, it is clear that the ability cut-off for college attendance $a_t^*$ should satisfy inequality...
\[
\alpha_t^* \leq \frac{1}{CE_{t-1}} \frac{\theta_t Bh^*}{\theta_t (b + B) - 1}
\]  
(22)

Indeed, according to (6), (7) and (4), (5), an individual with ability exceeding the right hand side of (22) will certainly increase his income by going to college. Lemma 3 demonstrates that inequality (22) is in fact satisfied as equality in equilibrium. The reason this fact is less than obvious is that as a result of collective bargaining based pay structure for teachers, some are paid above their production sector opportunity cost, hence a potential distortion in the cost-benefit analysis of college attendance.

The result of Lemma 1 provides for an underlying endogenous growth feature of our model where the rise of human capital attainments at the basic level is the “engine of growth”. Our main focus is on the dynamics of teacher quantity and quality which accompanies and is entailed in this process of growth. The following Lemma proven in Appendix 1 addresses a step in this recursion: it characterizes the effect of rising basic education quality \( E_{t-1} \) in period \( t-1 \) on education decision variables in period \( t \).

**Lemma 4.** The effects of increased quality of basic education \( E_{t-1} \) in period \( t-1 \) are expanded college attendance (lower college attendance ability cut-off) in the next generation, larger quantity of teachers, and their lower quality given by their lower ability cut-off values. Specifically, the following is true for all \( t = 0,1, \ldots \) in the recursive dynamic equilibrium:

\[
\frac{\partial \alpha_t^*}{\partial E_{t-1}} < 0, \quad \frac{\partial z_t}{\partial E_{t-1}} > 0, \quad \frac{\partial \alpha_t}{\partial E_{t-1}} < 0, \quad \frac{\partial a_t}{\partial E_{t-1}} < 0
\]

Combining the last two facts, \( \frac{\partial \alpha_t}{\partial E_{t-1}} < 0, \frac{\partial a_t}{\partial E_{t-1}} < 0 \) for \( t = 0,1, \ldots \), with Lemma 1, which shows that education quality \( E_{t-1} \) does in fact grow over time, yields our central result.

**Theorem 1 (Dynamics of the Quantity and Quality of Teachers).** The recursive dynamic equilibrium (RDE) exhibits the following evolution of education policy variables:

- the quantity of teachers \( z_t \) grows over time;
- the relative quality of teachers characterized by the range of their innate abilities falls: both the upper and the lower cut-offs $\bar{a}, \underline{a}$ decrease over time;

- the college attendance ability cut-off $a^*$ also drops over time (thus the college-bound population expands); $a^*$ remains consistently below the lower ability cut-off for teachers $\underline{a}$ (according to Lemma 2).

The decline of the relative quality of teachers measured against the contemporaneous college educated cohort does not imply the fall in their absolute levels of human capital. As shown by the following fact proven in Appendix 1 the absolute human capital levels of teachers will, in fact, grow. This is due to the overall growth in education established in Lemma 1, which permits relatively less-able teachers in the new generations to attain higher human capital level than did relatively more able teachers of previous generations.

**Corollary.** While the relative quality of teachers falls over time in the RDE (according to the Theorem), the absolute quality of teachers characterized by their human capital attainment grows: both the human capital of the top teacher and the least qualified one, $\bar{h}_i$, $h_i$, rise over time.

**Discussion.** The intuition for the above results derives from the mechanics of economic growth in our model. Rising per student quality of basic education (Lemma 1) opens up the opportunity to pursue higher education for an expanding group of students. Namely, college attendance becomes worthwhile for an ever broader population, adding on students with relatively low ability. At the same time, the human capital attainment of highly able students increases disproportionately relative to their less able peers due to increasing returns to ability exhibited by the college education production function (7). Thus economic growth drives the rise of income inequality within the group of college graduates (we will formally prove this fact in Theorem 2 below). As a result, the opportunity cost of highly able college graduates measured by their increasing relative earnings in the production sector rises over time. Therefore hiring high ability individuals as teachers is becoming a relatively more expensive option (one can characterize this
as an *effect of rising talent premium*), which pushes the quality-quantity trade-off for the education policy-makers in favor of the latter. In other words, hiring a larger number of teachers of relatively lower quality is becoming an increasingly more cost-effective policy.

Figure 2 below illustrates the evolution of the optimal education policy choices along with the evolution of income distribution in successive time periods, $t$ and $t+1$. The part of the graph pertaining to period $t$ corresponds to the illustration given in Figure 1 of Section 3: the dashed sloped line represents the income distribution of production sector workers in period $t$, while the light shaded bar depicts the optimal education policy (the bar’s width $z_t$ is the number of teachers while its height is teacher salary in period $t$).

The solid sloped line represents wage income distribution in period $t+1$. It exhibits a kink further to the left, i.e. a lower college attendance ability cut-off than in period $t$, consistent with Theorem 1. Furthermore, the comparison of the solid and dashed lines confirms the fact discussed above that the benefit derived by skilled individuals from the quality of basic education grows disproportionately with their ability. This implies the rising relative cost of highly able teachers and results in the shift of the quantity-quality trade-off depicted by the
position of the dark shaded bar, the government optimal education policy choice in period $t + 1$, to the left of the period $t^*$’s light shaded bar, as stated in Theorem 1.

The argument in the above discussion concerning the growing relative market cost of high ability individuals is made explicit by the following result which characterizes the evolution of income inequality in our model.

**Theorem 2 (The Evolution of Income Inequality).** The recursive equilibrium dynamics exhibits growing inequality within the group of skilled individuals, as well as the rise in inequality between this group and the unskilled.

**Proof.** Based on the income formulas (4)-(5), the human capital accumulation formulas (6)-(7) and using the uniform distribution of abilities as well as the formula (9) for the threshold ability between the groups, we can obtain the mean income of unskilled individuals:

$$T_u^t = \frac{I_u^t(a^*_t)}{2} = \frac{w_t \theta_t B h^*_t}{2 (\theta_t (b + B) - 1)} = \frac{w_t C E_{t-1} a^*_t}{2}$$

and the mean income of the skilled (ignoring the distortion due to collective bargaining in the education sector):

$$T_s^t = \frac{I_s^t(a^*_t) + I_s^t(A)}{2} = \frac{w_t \theta_t \left( (b + B) \theta_t B h^*_t \right) + A (b + B) C E_{t-1} - 2 B h^*_t}{2 (\theta_t (b + B) - 1)}$$

Thus the inequality between the groups can be characterized by

$$\sigma_{i_{un}}^t = \frac{T_s^t}{T_u^t} = \frac{A (b + B) C E_{t-1} (\theta_t (b + B) - 1) + (2 - \theta_t (b + B))}{B h^*_t}$$

This expression obviously increases in basic education quality, which according to Lemma 1 rises over time.

The inequality within the skilled group (ignoring the aforementioned distortion) is characterized by

$$\sigma_s^t = \frac{I_s^t(A)}{I_s^t(a^*_t)} = \frac{(b + B) A C E_{t-1} - B h^*_t}{(b + B) a^*_t C E_{t-1} - B h^*_t} = \left( \theta_t (b + B) - 1 \right) \frac{(b + B) A C E_{t-1} - B h^*_t}{B h^*_t}$$

which also grows with the rise of basic education quality. ■

Using formula (9) we can rewrite expression (23) as
This demonstrates that the upward trend of the absolute disparity between the highest and lowest incomes of skilled workers characterized by Theorem 2 can be attributed to two factors:

(i) the rise of basic education quality $E_{t-1}$, which increases the human capital and thereby the incomes of all workers, but disproportionately more so at the high end of the ability distribution;

(ii) falling, according to Lemma 2, college attendance ability cut-off $a^*_i$, which brings lower ability workers into the fold of the skilled hence increasing the intra-group inequality.

As we discussed above and in Section 2, the results of Theorem 2 are due to the absolute growth of the quality of per student publicly provided basic education, which has unequal impact on individuals across the distribution of abilities because of the complementary relationship between individual ability and quality of education,\(^9\) hence the “rising talent premium” effect.

Much of the recent growth literature which presents the evidence of increasing dispersion of incomes of skilled workers over the last four decades of the XX century attributes this to the skill biased nature of technological change (see e.g., Acemoglu (1998), (2000), Galor and Moav (2000)). A competing view, supported, e.g., by the findings of Eckstein and Nagypal (2004), gives more credence to the evidence of increasing dispersion in educational attainments. Note that our Theorem 2 results on income inequality are based on the “rising talent premium” effect: they were derived exclusively from the growing provision of education, i.e., they remain valid even when skill premium coefficients $\theta_i$ are held constant over time. Of course one should expect that a rise in skill premium $\theta_i$ will further exacerbate the expansion in income inequality.

We will next verify this conjecture, that there is an additional “SBTC effect” on the cost of talent and thereby on the optimal quality and quantity of teachers, distinct from the effect of the rising talent premium.

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\(^9\) A somewhat similar argument for the magnifying effect that greater public education funding may have on income inequality is advanced by Glomm and Kaganovich (2003) in the presence of complementarity between public and parental private inputs, imperfect altruism, and borrowing constraints. The fact of such complementary relationship and its implications for inequality was documented for the case of Britain by LeGrand (1982).
We will introduce the exogenous skill biased technological change into the model given by positive shocks to the skill premium coefficients $\theta$, and will explore its effects on the education policy variables and the quality of education. Specifically, we consider the recursive dynamic equilibrium (RDE) corresponding to the original exogenously given sequence $\{\theta_t\}_{t=0}^\infty$ (the benchmark sequence, which may, in particular, be stationary) and assume that the productivity augmentation of skilled labor receives a positive shock from time $t_0$ on, i.e. that for $t = t_0, t_1, \ldots$ the values $\theta_t$ are replaced with some $\theta_t' > \theta_t$. We will characterize the effect of this exogenous change on the RDE, particularly on the education policy variables. We obtain the following comparative dynamics result (see Appendix 1 for the proof).

**Theorem 3 (The Effect of Skill Biased Technological Change).** Consider the comparative dynamics experiment described above where skill premium coefficients $\theta_t$ receive a positive shock from period $t_0$ on. The corresponding recursive dynamic equilibrium, relative to the benchmark RDE, will be characterized, for $t \geq t_0$, by

- lower quantity of teachers $z_t$;
- lower aggregate quality of teachers $q_t$ and therefore
- lower quality of basic education $E_t$.

Note the negative effect on both the number and aggregate quality of teachers which is due to an upward shock to the cost of skilled labor. The Theorem thus shows that the technological change biased toward skilled labor will have a detrimental (SBTC) effect on the absolute quality of basic education, exacerbating the negative rising talent premium effect of a secular downward trend in the relative quality of teachers stated in Theorem 1. These results will apply, in particular, when education tax rate $\tau$ stays constant, which means that education budget grows at the rate of GDP growth.

This leads to an important implication of our analysis: given the negative impact of rising skill premium on the quality of education, a policy aimed at neutralizing this effect would require an increase in funding of education (assuming no change in the institutional setup of the school system and teachers’ labor market) at a rate faster than GDP growth, i.e. raising the fraction of
GDP devoted to education. Note that Rangazas (2002), in his accounting of sustained growth of the U.S. economy since 1870, while attributing its biggest share to human capital accumulation, distinguishes between an unsustainable (“transitional”) component due to the rise of education expenditures as a share of GDP, and the sustainable one caused by their balanced growth proportionately to that of GDP, as well as the other two “sustainable” factors: intergenerational human capital externality and a secular trend of technological change. Our results, however, suggest that these factors are intertwined. Indeed, the rising “talent premium” and technological progress lead to increasing relative cost of maintaining a given teacher quality standard, thereby requiring educational expenditures to rise faster than GDP if a steady growth of human capital is to be sustained. While this may call into question the separation between the “transitional” and “sustainable” factors in human capital accumulation, the distinction is limited to the current education technology, which is itself plausibly undergoing transition to a new paradigm.

5. Conclusions

Over the last forty years, education policy in the U.S. has changed significantly, focusing in particular on lowering the student-teacher ratio. We have developed a model which offers an insight into this evolution by relating it to the changes in the US economy characterized by rising skill premium and overall income inequality. We argue that teacher wage compression due in large part to collective bargaining agreements has a significant effect on decisions concerning quantity-quality trade-offs in hiring teachers. Our model predicts that as incomes of college educated individuals rise and become more dispersed, education policy-makers are forced to adjust relative teacher salaries and thereby quality standards. Education quality is optimized by lowering relative quality of teachers while increasing their numbers. This causes the higher ability college educated people to choose private sector employment which offers higher reward to skilled workers.

We argue moreover that a rise in skill premium caused, in particular, by skill biased technological change will exacerbate the negative trends in the relative quality of education. Indeed, the labor of college graduates will further appreciate relative to the average wage and hence relative to the tax revenue. Countering this trend would therefore require an increase in the
share of GDP spent on basic education, assuming that the institutional setup of the school system remains unchanged.

Our finding that skill biased technological change can have a negative effect on the quality of education is an interesting case of negative feedback, since SBTC literature points to the rise in the supply of skill due to growing availability of education as its underlying cause. Furthermore, this leads to an issue which appears important for future research on the aggregate long term effects of SBTC: as the technical change brings about productivity gains, one needs to factor in its effects on the cost and quality of education and the corresponding policy responses in order to assess the full long-term impact.
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Appendix 1

The Government’s Basic Education Quality Optimization Problem

Thanks to Assumption 1 of the uniform distribution of innate ability on the interval \([a, A]\) and according to the basic and advanced education production functions (6) and (7) we can simplify expressions (10) and (13), respectively, as

\[
E_t = z_t^q q_t^e = \frac{z_t^q [\bar{h}_m^2 - h^2]^{\nu}}{[2(A-a)(b+B)CE_{t-1}]^{\nu}} \tag{A1}
\]

\[
z_t = \frac{\bar{h}_m - h_t}{(A-a)(b+B)CE_{t-1}} \tag{A2}
\]

and therefore problem (16) to maximize the quality of basic education \(E_t\) subject to the government budget constraint can be restated as

\[
\max_{z_t, \bar{h}_m} \frac{z_t^q [\bar{h}_m^2 - h^2]^{\nu}}{[2(A-a)(b+B)CE_{t-1}]^{\nu}} \tag{A3}
\]

subject to

(A2)

\[
z_t \theta_i w_i \bar{h}_m = T_i
\]

\[
a^*_t \geq a_t^e
\]

or equivalently, according to (A2), as

\[
\max_{z_t, \bar{h}_m} 2^{-\nu} \left( \bar{h}_m + h_t \right)^{\nu} z_t^{\nu+\nu} \tag{A3}
\]

subject to

\[
z_t \theta_i w_i \bar{h}_m = T_i
\]

\[
a^*_t \geq a_t^e
\]

Note that the optimal lower and upper cut-off levels of teachers’ human capital \(\bar{h}_m, h_t\) are related through the optimal choice of their number \(z_t\) according to equation (A2). The optimization in problem (A3) thus expresses the trade-off between the quantity and quality of
teachers to be hired. The quality of the top teacher \( \bar{h}_t \) will not only determine his salary \( I^h_t = \theta_t w_t \bar{h}_t \) due to his outside option as a skilled worker, but will set the identical salary for all other teachers according to the equal pay based collective bargaining agreement. Conversely, teacher salary \( I^h_t \) set by the government will uniquely determine the top teacher quality \( \bar{h}_t \). Therefore the total teachers’ wage bill in the government budget constraint is given by \( z_t \theta_t w_t \bar{h}_t \).

Therefore, using relationships (14) to express \( \bar{h}_t \) and \( \bar{h}_t \) and then eliminate \( a_t \) according to formula (15), we obtain

\[
q_t = \frac{\bar{h}_t^2 - \bar{h}_t^2}{2(A - a_t)(b + B)CE_t-1} = \left[ (b + B)CE_{t-1} \bar{a}_t - Bh^* - \frac{1}{2} z_t(A - a_t)(b + B)CE_{t-1} \right] z_t
\]

so we can restate the government’s education quality optimization problem (A3) as

\[
\max_{z_t, \bar{a}_t} E_t = 2^{-v} \left[ 2(b + B)CE_{t-1} \bar{a}_t - 2Bh^* - z_t(A - a_t)(b + B)CE_{t-1} \right] z_t^{\gamma + v}
\]

subject to \( z_t \theta_t w_t \left[ (b + B)CE_{t-1} \bar{a}_t - Bh^* \right] = T_t \) and

\[
\bar{a}_t - z_t(A - a_t) \geq a_t^*
\]

Assumptions of the Model

We will now spell out specific conditions on the model’s parameters behind the Assumption 2 outlined in Section 4 of the paper. We impose the following restrictions on the economy’s parameters, where \( E_{t-1} \) is an exogenously given per student basic education quality provided to generation \( G_0 \) individuals.

Assumption 2. The returns to the human capital of teachers are non-decreasing: \( \nu \geq 1 \). Furthermore, the following inequalities are true for \( t = 0, 1, \ldots \):

(i) \( \left( \frac{\nu}{\gamma} (b + B) AC(1 - \tau_t) \right)^{1 + \gamma / \nu} \left( \left( \frac{\gamma}{2\nu + \gamma} \right) \left( \frac{\tau_t}{1 - \tau_t} \right) \right)^{1 + \gamma / 2\nu} \left( 1 - \frac{Bh^*}{(b + B)ACE_{t-1}} \right) > 1 \)

(ii) \( \left( \frac{\nu(1 - \tau_t)}{\gamma} - \frac{1}{2} \right) \left( \frac{\gamma}{2\nu + \gamma} \right) \left( \frac{\tau_t}{1 - \tau_t} \right)^{1/2} \left( 1 - \frac{Bh^*}{(b + B)ACE_{t-1}} \right) > \frac{1}{\theta_t(b + B)} \)
The main thrust of the above conditions concerns the parameters which characterize educational gains. Inequality (i) is satisfied, if parameter $C$ characterizing the human capital gains in basic education according to (6) is sufficiently large. Inequality (ii) will hold if $(b + B)$, a productivity characteristic of the college education production function (7), is large enough.

Assumption 2 also requires that education taxes $\tau_i$ were not too small (the above inequalities imply a uniform lower bound for $\tau_i$) while not exceeding $1 - \frac{\gamma}{2\nu}$. Indeed, $\tau_i \leq 1 - \frac{\gamma}{2\nu}$ must be true in order for the term $\left(\frac{\nu(1 - \tau_i) - 1}{\gamma} - \frac{1}{2}\right)$ in the above condition (ii) to be positive. As discussed in Section 4, the inequality $\tau_i \leq 1 - \frac{\gamma}{2\nu}$ imposes a requirement that $\gamma$, the relative importance of the teacher-student ratio for schooling effectiveness, should not be substantially greater than $\nu$, the relative importance of the teacher quality.

We will now proceed to prove the Proposition, the Lemmas, Corollaries, and Theorems. We will first prove the Proposition and Lemmas 1 and 2 under the hypothesis that Lemma 3 is correct, i.e. that the cut-off ability $a_i^*$ of college attendees satisfies formula (9). We will then prove that Lemma 3 is indeed correct in the recursive dynamic equilibrium, and thereby the imposition of the hypothesis does not diminish the generality of (or create circularity problems with) the argument.

**Proof of the Proposition**

According to the teacher salary equation (12) and the tax revenue formula (21), the government budget constraint in the above formulations of the education quality optimization problem can be stated as

\[(1 - \tau_i)z_i\bar{\theta}_i H_i = \tau_i \left(\nu H_i^u + \theta V_i^\nu\right)\]  \hspace{1cm} (A5)

Using the education production functions (6) and (7) and the assumption that innate ability is uniformly distributed on $[0, A]$ we can rewrite the general equilibrium relationships (19), (20) as

\[H_i^u = CE_{t-1} \int_0^A a da = \frac{(a_i^*)^2}{2A} CE_{t-1}\]  \hspace{1cm} (A6)
Therefore expressing \( \bar{t}_i \) through \( \bar{a}_i \) according to the relationship in (14) we can rewrite the budget constraint (A5) as

\[
(1-\tau_i)\theta_i z_i \left((b+B)CE_{t-1}\bar{a}_i - Bh^*\right) = \\
\frac{\tau_i}{2A} \left[(a_i^*)^2 + \theta_i (b+B) \left(A^2 - (a_i^*)^2 - (\bar{a}_i)^2 + (a_i)^2\right)\right] - \frac{\tau_i B h^*}{A} \left[A - a_i^* - \bar{a}_i + a_i\right] 
\]

We now eliminate variables \( a_i^* \) and \( a_i \) from (A8) by substituting the value of \( a_i^* \) given by (9) according to Lemma 3, and using the expression \( a_i = \bar{a}_i - Az_i \) which follows from the relationship since we set \( a = 0 \). This immediately turns expression (A8) into a linear equation in terms of variable \( \bar{a}_i \) which yields

\[
\bar{a}_i = \frac{z_i \tau_i A}{2} + \frac{Bh^*}{(b+B)CE_{t-1}} + \frac{\tau_i}{2z_i} \left[1 - \frac{2Bh^*}{A(b+B)CE_{t-1}} + \frac{\theta_i BH^*}{\left(\theta_i (b+B) - 1\right)(b+B)ACE_{t-1}} \right] 
\]

Recall that this expression incorporates the government budget constraint of the optimization problem (A4). That problem’s objective function, upon substituting the expression (A5) for \( a_i \), becomes a function of a single variable \( z_i \). We will first solve for its unconstrained maximization and then discuss the verification that its solution satisfies the only remaining constraint \( \bar{a}_i - Az_i \geq a_i^* \) in the optimization problem (A4).

Thus we are looking at the unconstrained maximization of the following function:

\[
F(z_i) = q_i z_i^\gamma = \left(\frac{\tau_i (b+B)ACE_{t-1}}{2} - \tau_i Bh^* + \frac{\tau_i B^2 h^*}{2\left(\theta_i (b+B) - 1\right)ACE_{t-1}} - \frac{(1-\tau_i)(b+B)ACE_{t-1}z_i^2}{2}\right)z_i^\gamma 
\]

Its first order necessary condition is given by the equation

\[
\gamma z_i^\gamma q_i^{\gamma-1} - \nu (1-\tau_i)(b+B)ACE_{t-1}z_i^{\gamma+1} q_i^{-1} = 0 
\]

yielding unique non-negative solution:
\[ z_t = \left( \frac{\gamma}{2\nu + \gamma} \left( \frac{\tau_t}{1 - \tau_t} \right) \right)^{1/2} \left( 1 - \frac{2Bh^*}{(b+B)ACE_{t-1}} + \frac{\theta_t (Bh^*)^2}{(\theta_t (b+B) - 1)(b+B)(ACE_{t-1})^2} \right)^{1/2} \]  

(A11)

It is straightforward to verify that this solution also satisfies the second order sufficient condition of the maximization problem. Substituting expression (A11) back into formula (A9) we obtain

\[
\overline{a}_t = \frac{\tau A z_t}{2} + \frac{Bh^*}{(b+B)CE_{t-1}} + \frac{\tau A}{2z_t} \left( 1 - \frac{2Bh^*}{(b+B)ACE_{t-1}} + \frac{\theta_t (Bh^*)^2}{(\theta_t (b+B) - 1)(b+B)(ACE_{t-1})^2} \right)
\]

which simplifies, by using equation (A11) again, into

\[
\overline{a}_t = \frac{Bh^*}{(b+B)CE_{t-1}} + Az_t \left( \frac{\nu(1 - \tau_t)}{\gamma} + \frac{1}{2} \right)
\]

(A12)

Recall that \( a_t = \overline{a}_t - Az_t \) according to (15) since we set \( a = 0 \). Applying this to (A12) we obtain

\[
a_t = \frac{Bh^*}{(b+B)CE_{t-1}} + Az_t \left( \frac{\nu(1 - \tau_t)}{\gamma} - \frac{1}{2} \right)
\]

(A13)

As discussed earlier, in order to ascertain that the expressions (A11), (A12), (A13) represent the solution of the constrained optimization problem (A4), it remains to verify that the constraint \( a_t > a_t^* \) does hold for \( a_t \) given by (A13). This will be indeed demonstrated in the proof of Lemma 2 below.

Observe that the education policy optimization as well as the individuals’ and the production sector’s general equilibrium reactions are determined recursively. Indeed, according to expressions (A11)–(A13), education quality \( E_{t-1} \) uniquely determines optimal education policy in period \( t \), i.e. the number of teachers, as well as the range of their innate abilities and thereby, due to (14), the range of their human capital attainment. This in turn will uniquely determine college attendance and employment decisions by generation \( t \) individuals, hence their incomes and their allocations. Government’s education policy will also determine the current period’s basic education quality \( E_t \), so the recursion continues.

Proof of Lemma 1

Recall that according to (16) \( E_t = \left( z_t (b+B)CE_{t-1} \overline{a}_t - z_t Bh^* - \frac{1}{2} (b+B) ACE_{t-1} z_t^2 \right)^\nu z_t^\gamma \).
Substituting the expression for \( t \) given in (A12), we obtain

\[
E_t = \left( A(b + B)CE_{t-1} \left( \frac{v(1 - \tau_t)}{\gamma} \right) \right)^\nu z_t^{2\nu + \gamma}, \text{ or according to (A11)}
\]

\[
E_t = \left( \frac{A(b + B)CE_{t-1}v(1 - \tau_t)}{\gamma} \right)^\nu \left( \frac{\gamma \tau_t}{2\nu + \gamma - 1 - \tau_t} \left( 1 - \frac{2Bh^*}{(b + B)ACE_{t-1}} + \frac{\theta_t(Bh^*)^2}{(\theta_t(b + B) - 1)(b + B)(ACE_{t-1})} \right) \right)^\nu z_t^{2\nu + \gamma/2}
\]

Note that since \( \frac{\theta_t(b + B)}{\theta_t(b + B) - 1} > 1 \), the following inequality is true

\[
1 - \frac{2Bh^*}{(b + B)ACE_{t-1}} + \frac{\theta_t(Bh^*)^2}{(b + B)(\theta_t(b + B) - 1)(ACE_{t-1})^2} > \left[ 1 - \frac{Bh^*}{(b + B)ACE_{t-1}} \right]^2 \tag{A14}
\]

Therefore we can write

\[
E_t > \left( \frac{v}{\gamma} (b + B) ACE_{t-1} (1 - \tau_t) \right)^\nu \left( \left( \frac{\gamma}{2\nu + \gamma} \right) \left( \frac{\tau_t}{1 - \tau_t} \right) \right)^{\nu + \gamma/2} \left( 1 - \frac{Bh^*}{(b + B)ACE_{t-1}} \right)^{2\nu + \gamma/2}
\]

Thus, in order to prove the Lemma it is sufficient to show that for all \( t = 0, 1, ... \)

\[
\left( \frac{v}{\gamma} (b + B) AC (1 - \tau_t) \right)^{1 - \nu/2} \left( \left( \frac{\gamma}{2\nu + \gamma} \right) \left( \frac{\tau_t}{1 - \tau_t} \right) \right)^{1 - \gamma/2\nu} \left( 1 - \frac{Bh^*}{(b + B)ACE_{t-1}} \right) > 1
\]

which is indeed true according to Assumption 2(i) and by the induction argument. ■

Proof of Lemma 2

Based on Lemma 3 we use expression (9) for \( a_t^* \). Then according to (A13) our task of proving the inequality \( a_t > a_t^* \) is equivalent to verifying the inequality

\[
\frac{Bh^*}{(b + B)CE_{t-1}} + Az_t \left( \frac{v(1 - \tau_t)}{\gamma} - \frac{1}{2} \right) > \frac{1}{CE_{t-1}} \frac{\theta_t Bh^*}{\theta_t(b + B) - 1} \text{ or } Az_t \left( \frac{v(1 - \tau_t)}{\gamma} - \frac{1}{2} \right) > \frac{1}{(b + B)CE_{t-1}} \frac{Bh^*}{\theta_t(b + B) - 1}
\]

Upon substituting the expression (A11) for \( z_t \), the last inequality becomes

\[
A \left( \frac{v(1 - \tau_t)}{\gamma} - \frac{1}{2} \left( \left( \frac{\gamma}{2\nu + \gamma} \right) \left( \frac{\tau_t}{1 - \tau_t} \right) \right)^{1/2} \left( 1 - \frac{2Bh^*}{(b + B)ACE_{t-1}} + \frac{\theta_t(Bh^*)^2}{(\theta_t(b + B) - 1)(b + B)(ACE_{t-1})^2} \right) \right)^{1/2} \frac{Bh^*}{(b + B)CE_{t-1}} \theta_t(b + B) - 1
\]

\[
> \frac{1}{(b + B)CE_{t-1}} \theta_t(b + B) - 1
\]

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Under Lemma 3 the right hand side in (A15) is smaller than $\frac{A}{\theta_i(b+B)}$ since $a_i^* < A$. Therefore according to (A14) in order to prove inequality (A15) it is by far sufficient to establish

$$\left(\frac{\nu(1-\tau_i)}{\gamma} - \frac{1}{2}\right)^{1/2} \left(1 - \frac{Bh^*}{(b+B)ACE_t}\right) > \frac{1}{\theta_i(b+B)}$$

(A16)

which is indeed true for all $t = 0, 1, \ldots$ according to Assumption 2(ii) combined with Lemma 1.

Proof of Lemma 3

The above proofs were based on the hypothesis that Lemma 3 is correct, i.e. that the ability cut-off for college attendance $a_i^*$ satisfies equality (9), i.e. we proved that if college attendance cut-off ability is $a_i^* = \frac{1}{CE_{t-1}} \frac{\theta_i Bh^*}{\theta_i(b+B) - 1}$ then the optimal education policy requires that all teachers’ ability strictly exceed this threshold. This in turn means that the marginal college graduate will be employed in the production sector. As we explained after stating equality (9), if an individual with ability below $a_i^*$ attended college his skilled human capital adjusted for the net productivity augmentation $\theta_i$ will be inferior to his unskilled human capital derived from the first stage of education, therefore a job in production sector’s skilled labor force would not compel such individual to attend college. Thus the only way the violation of Lemma 3 could occur is if such individual had an opportunity to be hired as a teacher. Compare, however, optimization problem (16) where $a_i^* < \frac{1}{CE_{t-1}} \frac{\theta_i Bh^*}{\theta_i(b+B) - 1}$ to the one with $a_i^* = \frac{1}{CE_{t-1}} \frac{\theta_i Bh^*}{\theta_i(b+B) - 1}$. One can easily see that the only difference would be lower tax revenue $T_i$ in the former case. Therefore such government policy would be inferior to the one where $a_i^* = \frac{1}{CE_{t-1}} \frac{\theta_i Bh^*}{\theta_i(b+B) - 1}$. Thus the latter indeed characterizes the recursive dynamic equilibrium optimum, i.e. Lemma 3 is correct.

Proof of Lemma 4

By differentiating expressions (9) and (A11) we immediately obtain:
\[
\frac{\tilde{c}a_{i}^*}{\tilde{c}E_{i-1}} = \frac{-\tilde{\theta}Bh^*}{(\tilde{\theta}_i(b+B)-1)CE_{i-1}^2} < 0
\]  
(A17)

\[
\frac{\tilde{c}\gamma_{i}}{\tilde{c}E_{i-1}} = \frac{1}{z_i} \left( \gamma \left( \frac{\tau_i}{1-\tau_i} \right) + \frac{Bh^*}{(b+B)ACE_{i-1}^2} \left( 1 - \frac{a_{i}^*}{A} \right) \right) > 0
\]  
(A18)

According to (A12) and (A13), respectively, we can write

\[
\frac{\tilde{c}a}{\tilde{c}E_{i-1}} = -\frac{Bh^*}{(b+B)CE_{i-1}^2} + \frac{A}{z_i} \left( \frac{2v(1-\tau_i) + \gamma}{2\gamma} \left( \frac{\gamma}{2\nu + \gamma} \right) \left( \frac{\tau_i}{1-\tau_i} \right) \frac{Bh^*}{(b+B)ACE_{i-1}^2} \left( 1 - \frac{a_{i}^*}{A} \right) \right)
\]  
(A19)

\[
\frac{\tilde{c}a_{i}}{\tilde{c}E_{i-1}} = -\frac{Bh^*}{(b+B)CE_{i-1}^2} + A \left( \frac{2v(1-\tau_i) - \gamma}{2\gamma} \left( \frac{\gamma}{2\nu + \gamma} \right) \left( \frac{\tau_i}{1-\tau_i} \right) \frac{Bh^*}{(b+B)ACE_{i-1}^2} \left( 1 - \frac{a_{i}^*}{A} \right) \right)
\]  
(A20)

Note that since \( \frac{\tilde{\theta}_i(b+B)}{\tilde{\theta}_i(b+B)-1} > 1 \), the following inequality is true

\[
1 - \frac{2Bh^*}{(b+B)ACE_{i-1}} + \frac{\tilde{\theta}_i(Bh^*)^2}{(b+B)(\tilde{\theta}_i(b+B)-1)(ACE_{i-1})^2} > \left[ 1 - \frac{Bh^*}{(b+B)ACE_{i-1}} \right]^2
\]  
(A21)

Therefore according to (A11)

\[
\begin{align*}
\gamma & > \left( \frac{\gamma}{2\nu + \gamma} \left( \frac{\tau_i}{1-\tau_i} \right) \right)^{1/2} \left( 1 - \frac{Bh^*}{(b+B)ACE_{i-1}} \right) > \left( \frac{\gamma}{2\nu + \gamma} \left( \frac{\tau_i}{1-\tau_i} \right) \right)^{1/2} \left( 1 - \frac{\tilde{\theta}_iBh^*}{(\tilde{\theta}_i(b+B)-1)(ACE_{i-1})} \right) \\
& = \left( \frac{\gamma}{2\nu + \gamma} \left( \frac{\tau_i}{1-\tau_i} \right) \right)^{1/2} \left( 1 - \frac{a_{i}^*}{A} \right)
\end{align*}
\]

Thus expression (A19) will be negative as long as the inequality

\[
\begin{align*}
\left( \frac{\gamma}{2\nu + \gamma} \left( \frac{\tau_i}{1-\tau_i} \right) \right)^{1/2} > \left( \frac{\gamma}{2\nu + \gamma} \left( \frac{\tau_i}{1-\tau_i} \right) \right)^{1/2} \left( \frac{2v(1-\tau_i) + \gamma}{2\gamma} \right)
\end{align*}
\]  
(A22)
is true, or equivalently

\[
\frac{\gamma}{2\nu + \gamma} \tau_i > \left( \frac{\gamma}{2\nu + \gamma} \right)^2 \left( \frac{\tau_i}{1 - \tau_i} \right)^2 \left( \frac{2\nu(1 - \tau_i) + \gamma}{2\gamma} \right)^2
\]

which can be simplified as: \((1 - \tau_i)^2 \left( \frac{2\nu + 1}{\gamma} \right) > \tau_i \left( \frac{\nu}{\gamma} (1 - \tau_i) + \frac{1}{2} \right)^2\). This last inequality is certainly true if \((1 - \tau_i)^2 \left( \frac{2\nu + 1}{\gamma} \right) > \tau_i \left( \frac{\nu}{\gamma} (1 - \tau_i) + \frac{1}{2} \right)^2\) which reduces to \(\tau_i < \frac{4}{5 + 2\nu / \gamma}\). The latter is guaranteed by the condition \(\tau_i < 1 - \frac{\gamma}{2\nu}\), which is imposed by Assumption 2 (ii). Thus we have proven the negativity of the expression (A19).

Comparing expressions (A19) and (A20) one can see that negativity of (A19) implies the same for (A20). Therefore we can conclude that

\[
\frac{\partial a_i}{\partial E_{t-1}} < 0, \quad \frac{\partial a_i}{\partial E_{t-1}} < 0
\]

which completes the Lemma’s proof. ■

Proof of Corollary to Theorem 1

Recall that according to relationships (14)

\[
h^*_i = a_i \left( b + B \right) CE_{t-1} - Bh^* \quad \text{and} \quad \bar{h}_i = \bar{a}_i \left( b + B \right) CE_{t-1} - Bh^*
\]

Therefore due to (A12) and (A13), respectively, as well as to (A11) we can write

\[
\bar{h}_i = (b + B) ACE_{t-1} \left( \frac{\nu(1 - \tau_i)}{\gamma} - \frac{1}{2} \right) z_i
\]

\[
= (b + B) CA \left( \frac{\nu(1 - \tau_i)}{\gamma} - \frac{1}{2} \right) \left( \frac{\gamma}{2\nu + \gamma} \left( \frac{\tau_i}{1 - \tau_i} \right) \right)^{1/2} \left( E_{t-1}^2 - \frac{2Bh^*E_{t-1}}{(b + B) AC} + \frac{\theta_i \left( Bh^* \right)^2}{(\theta_i(b + B) - 1)(b + B)(AC)^2} \right)^{1/2}
\]

\[
h_i = (b + B) ACE_{t-1} \left( \frac{\nu(1 - \tau_i)}{\gamma} + \frac{1}{2} \right) z_i
\]

\[
= (b + B) AC \left( \frac{\nu(1 - \tau_i)}{\gamma} + \frac{1}{2} \right) \left( \frac{\gamma}{2\nu + \gamma} \left( \frac{\tau_i}{1 - \tau_i} \right) \right)^{1/2} \left( E_{t-1}^2 - \frac{2Bh^*E_{t-1}}{(b + B) AC} + \frac{\theta_i \left( Bh^* \right)^2}{(\theta_i(b + B) - 1)(b + B)(AC)^2} \right)^{1/2}
\]

both increasing functions of \(E_{t-1}\). Indeed, the derivative of each is proportionate to the
expression \( 1 - \frac{Bh^*}{(b + B) ACE_{t-1}} \), which is positive as shown by (A16). ■

Proof of Theorem 3

The proof will proceed by the induction argument.

Consider firstly the effect of a positive shock to coefficient \( \theta_t \) in period \( t = t_0 \) on education policy variables in this same period. According to (A11) direct derivative \( \frac{\partial}{\partial \theta_t} \) has the same sign as

\[
\frac{\partial}{\partial \theta_t} \left[ \frac{\theta_t}{\theta_t (b + B) - 1} \right],
\]

i.e. negative, so we can write

\[
\frac{\partial z_t}{\partial \theta_t} < 0
\]  

(A23)

Therefore, according to (A12) and (A13), respectively, we can write

\[
\frac{\partial q_t}{\partial \theta_t} = \frac{\partial}{\partial \theta_t} \left[ \frac{\partial z_t}{\partial \theta_t} \right] = A \left( \frac{1}{2} - \tau_t \right) \frac{\partial z_t}{\partial \theta_t} < 0
\]  

(A24)

\[
\frac{\partial q_t}{\partial \theta_t} = \frac{\partial}{\partial \theta_t} \left[ \frac{\partial z_t}{\partial \theta_t} \right] = A \left( \frac{1}{2} - \tau_t \right) \frac{\partial z_t}{\partial \theta_t} < 0
\]  

(A25)

Recall that according to the derivation of (16)

\[
q_t = \left[ (b + B) CE_{t-1} \bar{a}_t - Bh^* - \frac{1}{2} z_t A(b + B) CE_{t-1} \right] z_t
\]

since we have assumed \( a = 0 \). Using formula (A12) we can rewrite the above as

\[
q_t = A (b + B) CE_{t-1} \frac{\nu(1 - \tau_t)}{\gamma} z_t^2
\]  

(A26)

which implies, according to (A23), that

\[
\frac{\partial q_t}{\partial \theta_t} < 0
\]  

(A27)

for \( t = t_0 \). Combining (A23) with (A26) and referring to (10) we can conclude that for \( t = t_0 \)
We can now proceed to the next step of the induction and evaluate the effect born by the education policy variables in period \( t = t_0 + 1 \), keeping in mind two sources of this effect: the direct effect of higher value of \( \theta_{t+1} \) and the indirect one caused by lower education quality in the previous period \( E_{t_0} \) as established in (A28). The results in (A23), (A27) and (A28) show that the direct effects on the variables \( z_{t+1}, q_{t+1}, E_{t+1} \) in any period \( t+1 \) of a contemporaneous rise in \( \theta_{t+1} \) are negative. For the purposes of completing the induction argument it will therefore be sufficient to prove that a decline in \( E_t \) will have a negative effect on \( z_{t+1}, q_{t+1}, E_{t+1} \), in other words that the derivatives of these variables with respect to \( E_t \) are all positive.

According to (A11) the derivative \( \frac{\partial (z_{t+1}^2)}{\partial E_t} \) has the same sign as the expression

\[
1 - \frac{\theta_{t+1} Bh^*}{(\theta_{t+1} (b + B) - 1) ACE_{t-1}} \quad \text{which according to Lemma 3 is equal to} \quad 1 - A^{-1} a^*_t \quad \text{and therefore positive. Thus we can conclude that}
\]

\[
\frac{\partial z_{t+1}}{\partial E_t} > 0 \quad \text{(A29)}
\]

Rewriting expression (A6) for period \( t+1: \ q_{t+1} = A(b + B)CE_t \frac{\nu(1 - \tau_{t+1})}{\gamma} z_{t+1}^2 \), we note that a rise in \( E_t \) affects \( q_{t+1} \) directly (obviously positively) as well as indirectly through \( z_{t+1} \), also positively according to (A29). We can therefore conclude that \( \frac{\partial q_{t+1}}{\partial E_t} > 0 \). This combined with (A29) implies due to (10) that \( \frac{\partial E_{t+1}}{\partial E_t} > 0 \). Thus according to the above discussion the Theorem’s proof is complete. □
Appendix 2

Glossary of Notation

\( \beta \) Discount factor in individual intertemporal preferences

\( \tau_t \) Labor income tax rate in period \( t \)

\( T_t \) Total government revenue in period \( t \)

\( a \) Lower bound on innate ability, from Section 4 on \( a=0 \) is assumed

\( A \) Upper bound on innate ability

\( a(\omega) \) The innate ability of individual \( \omega \)

\( a^*_t \) Ability cut-off level for attending college in period \( t \)

\( h^u_t(\omega) \) Unskilled individual’s level of human capital in period \( t \)

\( H^u_t \) Aggregate unskilled human capital in goods production in period \( t \)

\( h^s_t(\omega) \) Skilled individual’s level of human capital in period \( t \)

\( H^s_t \) Aggregate skilled human capital in period \( t \)

\( H^{sy}_t \) Aggregate skilled human capital in goods production in period \( t \)

\( C \) Productivity coefficient of compulsory basic education

\( E_t \) Public education quality in period \( t \)

\( b \) Coefficient of in-college depreciation of pre-college human capital.

\( B \) Productivity coefficient of higher education

\( h^* \) Human capital threshold for admission to college

\( D \) TFP coefficient in the goods production sector

\( \alpha \) Physical capital income share in the goods production sector

\( \theta_t \) Productivity augmentation of skilled human capital (skill premium) in the production sector in period \( t \)

\( \gamma \) Returns to quantity of teachers

\( \nu \) Returns to quality of teachers

\( \Sigma_t \) Set of individuals employed as teachers in period \( t \)
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z_t$</td>
<td>Number of teachers (the share of teachers in the working population) in period t</td>
</tr>
<tr>
<td>$I_t$</td>
<td>Teacher’s salary in period t</td>
</tr>
<tr>
<td>$a_t$</td>
<td>The lowest ability level among teachers in period t</td>
</tr>
<tr>
<td>$h_t$</td>
<td>The lowest human capital level among teachers in period t</td>
</tr>
<tr>
<td>$\bar{a}_t$</td>
<td>The highest ability level among teachers in period t</td>
</tr>
<tr>
<td>$\bar{h}_t$</td>
<td>The highest human capital level among teachers in period t</td>
</tr>
<tr>
<td>$q_t$</td>
<td>Aggregate teacher quality in period t</td>
</tr>
</tbody>
</table>