Large Deviations Estimation of the Windfall and Shortfall Probabilities for Optimal Diversified Portfolios*

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Abstract

Many investors believe that they can effectively reduce risk by, among other ways, holding large combinations of investment assets. The purpose of this paper is to develop asymptotic approximations of the windfall and shortfall probabilities for an optimal portfolio of risky assets as the number of the assets becomes sufficiently large. We start by proving general large deviations theorems, then present specific results with an application to the multivariate normal case. Our theoretical results justify the diversification tenet of the allocation strategies that many hedge funds and pension funds tend to adopt nowadays.

Key words: Diversification; Large deviations; Shortfall probabilities; Windfall probabilities

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1 Introduction

The average investor will do best by diversifying among all classes of stocks. Trying to catch styles as they move in and out of favor is not only difficult, but also quite risky [Siegel (1998, p. 104)].

..., the golden number for American xenophobes – is at least fifty equal-sized and well-diversified U.S. stocks ... With such a portfolio, the total risk is reduced by over 60 percent. [Malkiel (2007, p. 190)].

The above quotations clearly imply that diversification is an important investment vehicle. Indeed, this was the primary motivation for creation of the very first hedge fund with the aim of overperforming (or beating) a given benchmark return (usually proxied by the change in a stock market price index or a bond yield), especially in the periods when markets are volatile. Hence, maximization of the windfall probability (i.e., the probability that the hedge fund return will overperform a given benchmark return) or minimization of the shortfall probability (i.e., the probability that the hedge fund return will underperform this benchmark return) has, certainly, become a ultimate objective of many investment strategies, including hedge funds (see, e.g., Fishburn (1984), Stutzer (2003) or Basak et al. (2006), and many others). It is worth mentioning at this point that the shortfall probability is a special case of the lower-partial moments proposed by Lee and Rao (1988). And the shortfall probability stimulates diversification because it is a subadditive function (i.e., $P(X + Y \leq r) \leq P(X \leq r) + P(Y \leq r)$), notwithstanding a caveat that it does not take into account the severity of an incurred damage event as required by coherent risk measures, which were proposed by Artzner et al. (1999). Nonetheless, the practical usefulness of coherent risk measures have been discussed a great deal in the risk management literature (see, e.g., Heyde et al. (2007)). Hence, in the present paper we shall not be concerned with the coherence issue, but focus on the main theme – the risk of optimal diversified portfolios.

Furthermore, one can not determine the optimal weights of assets held in a fund without the knowledge of the distribution of the returns for all possible portfolios. This issue becomes more
complicated as the number of individual assets increases; and in most cases, the closed form of this return distribution does not exist, thus one needs to rely on a variety of asymptotic approximations. A line of work has focused on using the inverse Fourier transform and integral approximation methods. For instance, Glasserman et al. (2002) propose an importance sampling algorithm to approximate an inverse Fourier integral used for computing the portfolio loss distribution when the underlying assets have a heavy-tailed distribution. However, this approach may be neither valid nor tractable when the size of portfolio is large or the heavy-tailedness assumption is relaxed. This present paper pursues a different line by developing general large deviations approximations for the optimal windfall and shortfall probabilities.

The large deviations principle has been a modus operandi for approximating the tail probabilities of an average sum of random variables. Traditionally, the theory of large deviations is formulated in terms of the asymptotic behavior of normalized logarithms of probabilities of certain tail (or extreme) events. For a detailed presentation of the large deviations theory and a complete discussion on its applications in various fields, the reader may consult den Hollander (2000), Varadhan (2008), or Touchette (2008), amongst many others. See also references therein for further information. In statistics, Bahadur (1960, 1967) defined the asymptotic efficiency of an estimator as the limit of the normalized logarithms of probability of the difference between the estimator and the true parameter exceeding a certain performance benchmark – which is essentially the large deviations probability; Jurečková (1981) employed this concept of asymptotic efficiency to develop a novel concept used to study tail behavior of location estimators.

In discrete-time finance, Stutzer (2003) adopts the large deviations principle to construct a criterion of long-term outperformance for a portfolio invested in two assets – one risky asset and 1 riskless asset. It has been shown that, as the investment horizon increases to infinity, this criterion is equivalent to the probability that the average logarithmic cumulative returns outperforms a riskfree interest rate. More interestingly, Stutzer (2003) interprets the objective minimization of the shortfall probability as the subjective maximization of an expected endogenous CRRA utility function.
In continuous-time finance, we could mention a few, certainly nonrepresentative, contributions such as Browne (1999), Browne (2000), Fleming and Sheu (2000), or Pham (2003). For instance, Pham (2003) develops a novel approach, based on a nonstandard large deviations principle, for long-term investment by proving the asymptotic duality between the probability that the terminal wealth of a portfolio invested in two assets exceeds that of an investment in a stochastic benchmark and a risk-sensitive control problem on an expected profit function.

The goal of this paper is different from the above-mentioned papers to such an extent that our focus is on one-period optimal investment in diversified portfolios. This is a practically interesting problem for two reasons: First, investors can, most of the time, reduce idiosyncratic risks by holding diversified portfolios. Second, diversification is more relevant (and indeed less costly) for short-term investments than long-term ones. Apparently, diversification is not necessary in the long term for the reason that risky assets appear safer over a longer time frame as idiosyncratic risks decline over time. And diminishing idiosyncratic risks are due to the fact that more information about firms become available at investors’ disposal – that is, the longer one times the horizon the more predictable returns become. Thus, reminiscent of the results of Barberis (2000), in a portfolio containing cash and a stock, the allocation to the stock increases with the horizon.

Therefore, the contribution of the present paper is twofold. First, we derive large deviations approximations of the windfall and shortfall probabilities for an optimal large portfolio based on dual problems on the profit and loss functions. The intuition is that there may exist a unbounded set of optimal portfolio choices that yields a maximum windfall probability (or a minimum shortfall probability) for the portfolio return, as the number of assets in these portfolios increases to infinity\(^1\), thus every investor can always do best by maximizing the expectation of a particular endogenous profit function so as to achieve the same maximum windfall probability (or minimizing the expectation of a particular endogenous loss function so as to achieve the same minimum windfall probability). In fact, we show that an investor who has an exponential or power profit function can

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\(^1\)For example, suppose that \(X \sim N(0, 1)\); \(Y \sim N(0, 1)\); and \(X\) and \(Y\) are independent. We can immediately see that \(0.2X + 0.8Y \sim N(0, 0.2^2 + 0.8^2)\) and \(0.8X + 0.2Y \sim N(0, 0.8^2 + 0.2^2)\). This implies that two portfolios, \((0.2, 0.8)\) and \((0.8, 0.2)\), yield the same windfall and shortfall probabilities.
attain the maximum windfall probability; and an investor who has an exponential or power loss function can hit the minimum shortfall probability. In this sense, our results vindicate the basic premise of optimal investment that *diversification* in a conventional portfolio optimization problem (i.e., maximizing/minimizing the profit/loss of a final wealth) helps to reduce the shortfall risk. Second, we apply the proposed approximations to derive the minimum shortfall probability in the well-known case whereby the joint distribution of risky assets’ returns is multivariate normal.

The plan of this paper is as follows. Section 2 presents general results with discussions. Section 3 presents some specific applications. Section 4 concludes the paper. Notation is tabulated as follows:

- $X_i$: the return (or the profit) of asset $i$
- $r$: a chosen benchmark return
- $\alpha = \{\alpha_1, \ldots, \alpha_i, \ldots, \alpha_n\}'$: a portfolio of $n$ risky assets
- $1(x \in A)$: an indicator function, “equal to 1 if $x \in A$, and 0 otherwise”
- $1$: a column vector of ones
- $S_n^{(\alpha)} = \sum_{i=1}^{n} \alpha_i X_i$: the return of a portfolio of $n$ risky assets
- $\tilde{\alpha}(\theta)$ (abbrev. $\tilde{\alpha}$): a $n$-dimensional optimal [portfolio] control vector [if it exists] of the supremum $\sup_{\alpha \in A} \log E[\ell(S_n^{(\alpha)}, n\theta)]$ if $\theta > 0$, and of the infimum $\inf_{\alpha \in A} \log E[\ell(S_n^{(\alpha)}, n\theta)]$ if $\theta < 0$
- $\tilde{\alpha}(\hat{\theta})$ (abbrev. $\tilde{\alpha}$): a $n$-dimensional optimal [portfolio] control vector evaluated at an exposing point, $\hat{\theta}_r$
- $\equiv$: “asymptotically equal”
- $B_\epsilon(r)$: a closed ball with the center $r$ and the diameter $\epsilon$
- $(a, b)^+$: the maximum number out of $\{a, b\}$
- $(a, b)^-$: the minimum number out of $\{a, b\}$
- $\mathbb{P}$: the joint distribution of $\{X_1, \ldots, X_n\}$
- $\mathbb{P}^*$: the conjugate distribution of $\mathbb{P}$
- $\mathbb{R}$: the real line
- $\mathbb{R}^+$: the positive real line

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the negative real line
arbitrarily small generic constants.

Note that, if $S_n^{(\alpha)}$ is the gross return, then $S_n^{(\alpha)}$ is always nonnegative for all $\alpha \in [0, 1]^n$.

2 General Results

Let $\ell(x, \theta)$ denote a nonnegative proper lower semicontinuous function of $\theta$ for every $x$ in a given marginal support of $\ell(x, \bullet)$, say $\mathcal{X}$. Supposing that the function $\ell(x, \theta)$ has the following monotonicity property:

\[
\{ \theta \in \mathbb{R} : \theta > 0 \} = \{ \theta \in \mathbb{R} : \frac{\partial \ell(x, \theta)}{\partial x} > 0 \},
\]
\[
\{ \theta \in \mathbb{R} : \theta < 0 \} = \{ \theta \in \mathbb{R} : \frac{\partial \ell(x, \theta)}{\partial x} < 0 \}.
\]

Hereafter, we shall refer to the function $\ell(x, \theta)$ as a profit function if $\theta$ is positive, and as a loss function if $\theta$ is negative. This profit function becomes a von Neumann - Morgenstern utility function if, given a positive $\theta$, it is concave for every $x \in \mathcal{X}$. It is worth noting that the term “profit function” has specific meaning in microeconomics; and here we have borrowed this term simply because of the natural congruence in the functional form, not because of the microeconomic interpretations.

Before stating the large deviations approximation of the supremum windfall probability, we first make some basic assumptions. Note that we shall provide some examples of the functions satisfying these assumptions after we state Theorem 1.

**Assumption 2.1 (Convexity).** The second-order derivative, $\frac{\partial^2 \ell(x, \theta)}{\partial \theta^2}$, of $\ell(x, \theta)$ is positive on $\{ \theta : \theta \in \mathbb{R}^+ \}$ for every $x \in \mathcal{X}$.

**Assumption 2.2 (Pointwise Convergence).** $\limsup_{\delta \searrow 0} \frac{\ell(x, \theta + \frac{\lambda}{\delta})}{\ell(x, \frac{\theta}{\delta})\ell(x, \frac{\lambda}{\delta})} = 1$ uniformly for every $x \in \mathcal{X}$. That is, $\ell(x, \frac{\theta + \lambda}{\delta}) \triangleq \ell(x, \frac{\theta}{\delta})\ell(x, \frac{\lambda}{\delta})$. 

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Assumption 2.3 (Existence of Feasible Portfolios). The set \( \mathcal{A} \subseteq \mathbb{R}^n \) contains feasible portfolios such that a unique optimal portfolio \( \tilde{\alpha} \) exists in the interior of \( \mathcal{A} \) for all \( \theta > 0 \).

Assumption 2.4 (Asymptotic Stability). In view of Assumption 2.3, let us define

\[
\ell^*(x, \theta) = \limsup_{n \to \infty} \frac{1}{n} \log \ell(x, n\theta), \quad \forall x \in \mathcal{X}.
\]

\[
\Lambda(\theta) = \limsup_{n \to \infty} \frac{1}{n} \sup_{\alpha \in \mathcal{A}} \log E\mathbb{P}[\ell(S_n^{(\alpha)}, n\theta)] = \limsup_{n \to \infty} \frac{1}{n} \log E\mathbb{P}[\ell(S_n^{(\tilde{\alpha})}, n\theta)].
\]

Let \( \bar{\theta} = \inf\{\theta > 0 : \Lambda(\theta) = \infty\} \), then, for each \( x \in B_r(r) \), there exists a sufficiently large \( \theta_r \in (0, \bar{\theta}) \) such that \( \ell^*(x, \theta) = \ell^*(x, \theta_r) + \tau(\epsilon, \theta_r) \) for all \( \theta \in [\theta_r, \bar{\theta}] \), where \( \ell^*(x, \theta_r) = \frac{\partial \ell^*(x, \theta)}{\partial \theta} \rvert_{\theta = \theta_r} \). And the remainder \( \tau(\epsilon, \theta_r) \) is infinitesimal (i.e., \( \lim_{\theta_r \to \infty, \epsilon \to 0} |\tau(\epsilon, \theta_r)| = 0 \)). [It is worth mentioning that, as we shall see below, this Taylor approximation allows us to apply the tools of Convex Analysis to the rate function.]

By Assumption 2.1, the function \( \Lambda(\theta) \) must be a first-order differentiable convex function. We define the following Fenchel-Legendre transform of \( \Lambda(\theta) \):

\[
\Lambda^*(s) = \sup_{\theta \in [\theta_r, \bar{\theta}]} [\theta \ell^*(s, \theta_r) - \Lambda(\theta)].
\]

By the definition of the Fenchel-Legendre transform, the rate function \( \Lambda^*(s) \) is a convex function. Following den Hollander (2000, Definition V.2), it is not hard to prove that

\[
\Lambda^*(s) = \begin{cases} 
\hat{\theta}_s \ell^*(s, \theta_r) - \Lambda(\theta) & \text{if } s \in \{s : \Lambda'(s, \theta_r) < \Lambda'(\theta)\}, \\
0 & \text{if } s \in \{s : \Lambda'(s, \theta_r) > \Lambda'(\theta)\},
\end{cases}
\]

(2.1)

where \( \hat{\theta}_s = \theta(s, \theta_r) \) is a unique solution [assuming that it always exists] to \( \ell^*(s, \theta_r) = \Lambda'(\theta) \), where \( \Lambda'(\theta) = \frac{\partial \Lambda(\theta)}{\partial \theta} \), such that \( \hat{\theta}_s \in [\theta_r, \bar{\theta}] \). (Note that the point \( \hat{\theta}_s \) is also called as an exposing point to the hyperplane \( \{\theta : \ell^*(s, \theta_r) = \Lambda'(\theta)\} \) associated with the half-spaces \( \{\theta : \ell^*(s, \theta_r) > \Lambda'(\theta)\} \) and \( \{\theta : \ell^*(s, \theta_r) < \Lambda'(\theta)\} \).)

We need the following lemma for further developments.
Lemma 1. The following inequality:

\[ \ell^*(r, \theta_r)\widehat{\theta}_r - \Lambda^*(r) > \ell^*(s, \theta_r)\widehat{\theta}_r - \Lambda^*(s), \]

holds for every \( s \in B_{r}(r) \).

Proof. Since \( \ell^*(s, \theta_r)\widehat{\theta}_r - \Lambda(\theta_r) < \Lambda^*(s) \), we have

\[
\leftrightarrow \Lambda(\widehat{\theta}_r) > \Lambda(\widehat{\theta}_s) + \ell^*(s, \theta_r)\widehat{\theta}_s - \ell^*(s, \theta_r)\widehat{\theta}_r
\]

\[
\leftrightarrow \ell^*(r, \theta_r)\widehat{\theta}_r - (\ell^*(r, \theta_r)\widehat{\theta}_r - \Lambda(\theta_r)) > \ell^*(s, \theta_r)\widehat{\theta}_r - \Lambda^*(s)
\]

\[
\leftrightarrow \ell^*(r, \theta_r)\widehat{\theta}_r - \Lambda^*(r) > \ell^*(s, \theta_r)\widehat{\theta}_r - \Lambda^*(s).
\]

The first theorem of this paper is stated below.

Theorem 1. If Assumptions 2.1-2.4 hold, then the supremum windfall probability can be approximated as follows:

\[
\lim_{n \to \infty} \sup_{n, \alpha \in \mathcal{A}} \frac{1}{n} \log P(S_n^{(\alpha)} \geq r) = -\Lambda^*(r) - \tau(0, \theta_r). \tag{2.2}
\]

Theorem 1 postulates that an investor who holds a sufficiently large number of assets can always find an optimal combination of these assets by maximizing an expected endogenous profit function so as to ensure that the windfall probability is maximum. Moreover, although investors may have different optimal portfolios associated with their own profit functions, they can always achieve the same maximum windfall probability by increasing the size of their portfolio. This is a vindication of the diversification tenet that diversification will eliminate all the idiosyncratic risks, and everyone will benefit from diversification. As such, the only remaining risk is the market [nondiversifiable]
risk, which every investor has to share. That is, whatever the choice of profit function, it is essential that the individual investors’ optimal portfolios guarantee the same windfall probability.

**Proof of Theorem 1.** We shall prove the following bounds:

- **Upper Bound:** Under Assumption 2.1, the Tchebyshev inequality yields

  \[ P(S_n^{(\alpha)} \geq r) \leq \frac{E_{\mathbb{P}}[\ell(S_n^{(\alpha)}, n\theta)]}{\ell(r, n\theta)} \]

  \[ \Leftrightarrow \log P(S_n^{(\alpha)} \geq r) \leq \log E_{\mathbb{P}}[\ell(S_n^{(\alpha)}, n\theta)] - \log \ell(r, n\theta) \]

  \[ \Leftrightarrow \sup_{\alpha \in A} \log P(S_n^{(\alpha)} \geq r) \leq \log \sup_{\alpha \in A} E_{\mathbb{P}}[\ell(S_n^{(\alpha)}, n\theta)] - \log \ell(r, n\theta) \]

  \[ \Leftrightarrow \limsup_{n \to \infty} \frac{1}{n} \sup_{\alpha \in A} \log P(S_n^{(\alpha)} \geq r) \leq \limsup_{n \to \infty} \frac{1}{n} \log \sup_{\alpha \in A} E_{\mathbb{P}}[\ell(S_n^{(\alpha)}, n\theta)] - \limsup_{n \to \infty} \frac{1}{n} \log \ell(r, n\theta), \]

  where the optimal portfolio, \( \tilde{\alpha} \), exists in view of Assumption 2.3. Moreover, by Assumption 2.4, we have

  \[ \limsup_{n \to \infty} \frac{1}{n} \sup_{\alpha \in A} \log P(S_n^{(\alpha)} \geq r) \leq \Lambda(\theta) - \ell^*(r, \theta). \]

  Hence, it follows that

  \[ \limsup_{n \to \infty} \frac{1}{n} \sup_{\alpha \in A} \log P(S_n^{(\alpha)} \geq r) \leq - \sup_{\theta \in [\theta, \theta]} [\theta \ell^*(r, \theta_r) - \Lambda(\theta)] - r(0, \theta_r) = \Lambda^*(r) - r(0, \theta_r). \] (2.3)

- **Lower Bound:** Given a benchmark return, \( r \), satisfying the constraint \( \Lambda'(\theta_r) \leq \ell^*(r, \theta_r) < \Lambda'(\bar{\theta}) \), let us define a conjugate joint probability measure, \( \mathbb{P}^* \), on the product probability space \((\Omega, \otimes_{i=1}^n \mathcal{F}_{i=1}^n, \mathbb{P})\), where \((\Omega, \mathcal{F}_i)\) denotes the sample space of \( X_i \), as follows:

  \[ \frac{d\mathbb{P}^*}{d\mathbb{P}} = \frac{\ell(S_n^{(\hat{\alpha})}, n\hat{\theta}_r)}{E_{\mathbb{P}}[\ell(S_n^{(\hat{\alpha})}, n\hat{\theta}_r)]}. \] (2.4)
Then, choosing \( r_n \) such that \( s \in B_{\epsilon}(r_n) \) implies \( s \in \{ s : \Lambda'(\theta_r) \leq \ell'(s, \theta_r) < \Lambda'(\hat{\theta}) \} \), we have

\[
P(S_n^\alpha > r_{n} - \epsilon) = \int 1(S_n^\alpha > r_{n} - \epsilon) \frac{E_{\mathbb{P}}[\ell(S_n^\alpha, \hat{\theta}_r)]}{\ell(S_n^\alpha, \hat{\theta}_r)} d\mathbb{P}^* \\
\geq \frac{E_{\mathbb{P}}[\ell(S_n^\alpha, \hat{\theta}_r)]}{\ell(r_{n} + \epsilon, \hat{\theta}_r)} \int 1(S_n^\alpha \in (r_{n} - \epsilon, r_{n} + \epsilon)) d\mathbb{P}^*,
\]

thus

\[
\log P(S_n^\alpha > r_{n} - \epsilon) \geq -\{ \log \ell(r_{n} + \epsilon, \hat{\theta}_r) - \log E_{\mathbb{P}}[\ell(S_n^\alpha, \hat{\theta}_r)] \} \right) + P^*(S_n^\alpha \in (r_{n} - \epsilon, r_{n} + \epsilon)).
\]

This gives

\[
\limsup_{n \to \infty} \frac{1}{n} \sup_{\alpha \in A} \log P(S_n^\alpha \geq r) \geq \liminf_{n \to \infty} \frac{1}{n} \log P(S_n^\alpha > r_{n} - \epsilon) \\
\geq -\left\{ \limsup_{n \to \infty} \frac{1}{n} \log \ell(r_{n} + \epsilon, \hat{\theta}_r) \\
- \limsup_{n \to \infty} \frac{1}{n} \log E_{\mathbb{P}}[\ell(S_n^\alpha, \hat{\theta}_r)] \right\} \\
+ \liminf_{n \to \infty} \frac{1}{n} \log P^*(S_n^\alpha \in (r_{n} - \epsilon, r_{n} + \epsilon)).
\]

Hence, we have

\[
\limsup_{n \to \infty} \frac{1}{n} \sup_{\alpha \in A} \log P(S_n^\alpha \geq r) \geq -\{ \ell^*(r+\epsilon, \hat{\theta}_r) - \Lambda(\hat{\theta}_r) \} + \liminf_{n \to \infty} \frac{1}{n} \log P^*(S_n^\alpha \in (r_{n} - \epsilon, r_{n} + \epsilon)).
\]

(2.5)

Now, in order to prove the lower bound we shall prove that

\[
\liminf_{n \to \infty} \frac{1}{n} \log P^*(S_n^\alpha \in (r_{n} - \epsilon, r_{n} + \epsilon)) = 0,
\]

which is equivalent to

\[
\limsup_{n \to \infty} \frac{1}{n} \log P^*(S_n^\alpha \not\in (r_{n} - \epsilon, r_{n} + \epsilon)) < 0.
\]

(2.6)
Since \( P^*(S_n^{(\bar{a})} \not\in (r_n - \epsilon, r_n + \epsilon)) \leq (P^*(S_n^{(\bar{a})} \geq r_n + \epsilon), P^*(S_n^{(\bar{a})} \leq r_n - \epsilon))^+ \), in order to prove Eq. (2.6), we need to show that

\[
\limsup_{n \to \infty} \frac{1}{n} \log P^*(S_n^{(\bar{a})} \geq r_n + \epsilon) < 0, \quad (2.7)
\]

\[
\limsup_{n \to \infty} \frac{1}{n} \log P^*(S_n^{(\bar{a})} \leq r_n - \epsilon) < 0. \quad (2.8)
\]

We first prove Eq. (2.7). In view of Assumption 2.1, an application of the Tchebyshev inequality yields, for a \( \lambda \in (0, \hat{\theta} - \hat{\theta}_r) \),

\[
P^*(S_n^{(\bar{a})} \geq r_n + \epsilon) \leq \frac{E_{\bar{P}^*}[\ell(S_n^{(\bar{a})}, n\lambda)]}{\ell(r_n + \epsilon, n\lambda)} \]

\[
\Leftrightarrow \limsup_{n \to \infty} \frac{1}{n} \log P^*(S_n^{(\bar{a})} \geq r_n + \epsilon) \leq -\left[ \limsup_{n \to \infty} \frac{1}{n} \log \ell(r_n + \epsilon, n\lambda) \right. - \left. \limsup_{n \to \infty} \frac{1}{n} \log E_{\bar{P}^*}[\ell(S_n^{(\bar{a})}, n\lambda)] \right] = -[\ell^*(r + \epsilon, \lambda) - \tilde{\Lambda}(\lambda)], \quad (2.9)
\]

where, in view of Assumption 2.4, \( \ell^*(r + \epsilon, \lambda) = \limsup_{n \to \infty} \frac{1}{n} \log \ell(r_n + \epsilon, n\lambda) \) and \( \tilde{\Lambda}(\lambda) = \limsup_{n \to \infty} \frac{1}{n} \log E_{\bar{P}^*}[\ell(S_n^{(\bar{a})}, n\lambda)] \). By Assumption 2.2, setting \( \delta = 1/n \), we obtain

\[
\tilde{\Lambda}(\lambda) = \limsup_{n \to \infty} \frac{1}{n} \log \bb{E}_{\bar{P}^*}\left[\ell(S_n^{(\bar{a})}, n\lambda)\ell(S_n^{(\bar{a})}, n\hat{\theta}_r)\right] - \limsup_{n \to \infty} \frac{1}{n} \log \bb{E}_P[\ell(S_n^{(\bar{a})}, n\hat{\theta}_r)]
\]

\[
= \Lambda(\lambda + \hat{\theta}_r) - \Lambda(\hat{\theta}_r).
\]

Moreover, one can immediately verify that Assumption 2.2 implies that \( \ell^*(x, \theta) = -\ell^*(x, -\theta) \) and \( \ell^*(x, \theta + \lambda) = \ell^*(x, \theta) + \ell^*(\lambda) \) for all \( x \in X \). Thus, Eq. (2.9) can be rewritten as

\[
\limsup_{n \to \infty} \frac{1}{n} \log P^*(S_n^{(\bar{a})} \geq r_n + \epsilon) \leq -[\ell^*(r + \epsilon, \lambda + \hat{\theta}_r) - \Lambda(\lambda + \hat{\theta}_r)] + [\ell^*(r + \epsilon, \hat{\theta}_r) - \Lambda(\hat{\theta}_r)].
\]
By Assumption 2.4, we have

\[
\limsup_{n \to \infty} \frac{1}{n} \log P^* \left( S_n^\alpha \geq r_n + \epsilon \right) \leq -[\ell' \left( r + \epsilon, \theta_r \right) (\lambda + \hat{\theta}_r) - \Lambda(\lambda + \hat{\theta}_r)] - \varphi(\epsilon, \theta_r) \\
+ \left[ \ell' \left( r, \hat{\theta}_r \right) \hat{\theta}_r - \Lambda(\hat{\theta}_r) \right] - \ell'(r, \hat{\theta}_r) \hat{\theta}_r + \ell'(r + \epsilon, \hat{\theta}_r) \hat{\theta}_r + \varphi(\epsilon, \theta_r).
\]

Hence, it follows that

\[
\limsup_{n \to \infty} \frac{1}{n} \log P^* \left( S_n^\alpha \geq r_n + \epsilon \right) \leq - \sup_{\xi \in [\tilde{\theta}, \tilde{\vartheta}]} \left[ \ell' \left( r + \epsilon, \theta_r \right) \xi - \Lambda(\xi) \right] + \Lambda^*(r) - \ell'(r, \hat{\theta}_r) \hat{\theta}_r \\
+ \ell'(r + \epsilon, \hat{\theta}_r) \hat{\theta}_r \\
= -\Lambda^*(r + \epsilon) + \Lambda^*(r) - \ell'(r, \hat{\theta}_r) \hat{\theta}_r + \ell'(r + \epsilon, \hat{\theta}_r) \hat{\theta}_r < 0,
\]

where the last inequality follows from Lemma 1 by setting \( s = r + \epsilon \). More importantly, note that \( \Lambda^*(r + \epsilon) = \sup_{\xi \in [\tilde{\theta}, \tilde{\vartheta}]} \left[ \ell' \left( r + \epsilon, \theta_r \right) \xi - \Lambda(\xi) \right] \) because the exposing point \( \xi(r) \) is an nondecreasing function of \( r \). Therefore, Eq. (2.7) has been proved.

We now proceed to the proof of Eq. (2.8). In view of Assumption 2.1, an application of the Tchebyshev inequality yields, for \( \lambda \in [\tilde{\theta} - \hat{\theta}_r, 0) \),

\[
P^* \left( S_n^\alpha \leq r_n - \epsilon \right) = P^* \left( \ell(S_n^\alpha, \lambda) \geq \ell(r_n - \epsilon, \lambda) \right) \leq \frac{E_{\mathbb{P}}[\ell(S_n^\alpha, n \lambda)]}{\ell(r_n - \epsilon, n \lambda)}.
\]

By Assumption 2.4, we have

\[
\limsup_{n \to \infty} \frac{1}{n} \log P^* \left( S_n^\alpha \leq r_n - \epsilon \right) \leq - \left[ \limsup_{n \to \infty} \frac{1}{n} \log \ell(r_n - \epsilon, n \lambda) - \lim_{n \to \infty} \frac{1}{n} \log E_{\mathbb{P}}[\ell(S_n^\alpha, n \lambda)] \right] \\
= [\ell^*(r - \epsilon, \lambda) - \bar{\Lambda}(\lambda)],
\]

where \( \ell^*(r - \epsilon, \lambda) = \limsup_{n \to \infty} \frac{1}{n} \log \ell(r_n - \epsilon, n \lambda) \) and \( \bar{\Lambda}(\lambda) = \lim_{n \to \infty} \frac{1}{n} \log E_{\mathbb{P}}[\ell(S_n^\alpha, n \lambda)] \).
Similarly, by Assumption 2.2, setting $\delta = 1/n$, we obtain

$$\Lambda(\lambda) = \Lambda(\lambda + \hat{\theta}_r) - \Lambda(\hat{\theta}_r),$$

thus, Eq. (2.10) can be rewritten as

$$\limsup_{n \to \infty} \frac{1}{n} \log P^\ast(S_n^{(\alpha)} \leq r_n - \epsilon) \leq -[\ell^\ast(r - \epsilon, \lambda + \hat{\theta}_r) - \Lambda(\lambda + \hat{\theta}_r)]$$

$$+ [\ell^\ast(r - \epsilon, \theta_r) - \Lambda(\theta_r)]$$

$$+ \ell^\ast(r - \epsilon, \hat{\theta}_r) \hat{\theta}_r - \ell^* (r, \hat{\theta}_r) \hat{\theta}_r,$$

where the last equality follows from Assumption 2.4. Therefore, we obtain

$$\limsup_{n \to \infty} \frac{1}{n} \log P^\ast(S_n^{(\alpha)} \leq r_n - \epsilon) \leq - \underset{\xi \in [\theta_r, \hat{\theta}_r]}{\sup} \left[ \ell^\ast(r - \epsilon, \xi) - \Lambda(\xi) \right] + \Lambda^\ast(r) + \ell^\ast(r - \epsilon, \hat{\theta}_r) \hat{\theta}_r$$

$$- \ell^\ast(r, \hat{\theta}_r) \hat{\theta}_r$$

$$= -\Lambda^\ast(r - \epsilon) + \Lambda^\ast(r) + \ell^\ast(r - \epsilon, \hat{\theta}_r) \hat{\theta}_r - \ell^\ast(r, \hat{\theta}_r) \hat{\theta}_r < 0,$$

where the last inequality follows from Lemma 1 by setting $s = r - \epsilon$. Eq. (2.8) has been proved.

Returning to Eq. (2.5), let $\epsilon$ approach 0, in view of Assumption 2.4 we get the following lower bound:

$$\limsup_{n \to \infty} \frac{1}{n} \sup_{\alpha \in A} \log P(S_n^{(\alpha)} \geq r) \geq -[\ell^\ast(r, \theta_r) \hat{\theta}_r - \Lambda(\hat{\theta}_r)] - v(0, \theta_r).$$

(2.11)

The theorem follows from Eqs (2.3) and (2.11). QED.

**Examples** of functions that satisfy Assumptions 2.1-2.4 are as follows:

- the exponential function $e^{\theta x}$: $\mathcal{X} = \mathbb{R}$, $\theta_\ast = 0$, $\ell^\ast(x, \theta) = \theta x$, where $\theta \in (0, \hat{\theta})$, and $v(\epsilon, \theta_r) = 0$

- the power function $x^\theta$: $\mathcal{X} = \mathbb{R}^+$, $\theta_\ast = 0$, $\ell^\ast(x, \theta) = \theta \log(x)$, and $v(\epsilon, \theta_r) = 0$
the power exponential function $x^\theta e^{\theta x}$: $\mathcal{X} = \mathbb{R}_+^*$, $\theta_* = 0$, $\ell^*(x, \theta) = \theta(\log(x)+x)$, and $\tau(\epsilon, \theta_r) = 0$

- $(x^2 - 2x + 2)^\theta$: $\mathcal{X} = \mathbb{R}$, $\theta_* = 0$, $\ell^*(x, \theta) = \theta \log(x^2 - 2x + 2)$, and $\tau(\epsilon, \theta_r) = 0$

- $e^{\theta x} + \frac{e^x}{\theta}$: $\mathcal{X} = \mathbb{R}$, $\theta_* = 0$, $\ell^*(x, \theta) = \limsup_{n \to \infty} \frac{1}{n} \log \left( e^{n \theta x} + \frac{e^x}{(n \theta)^x} \right) = \theta x$, and $\tau(\epsilon, \theta_r) = 0$.

Assumption 2.2 can immediately be verified by noting that $\lim_{\delta \searrow 0} \frac{e^{x \delta}}{(x \delta)^\frac{1}{2}} = 0$, $\forall x \in \mathcal{X}$.

Now, in order to state the large deviations approximation of the infimum shortfall probability, we shall need some other assumptions.

Assumption 2.1' (Convexity). The second-order derivative, $\frac{\partial^2 \ell(x, \theta)}{\partial \theta^2}$, of $\ell(x, \theta)$ is positive on $\{\theta : \theta \in \mathbb{R}^-\}$ for every $x \in \mathcal{X}$.

Assumption 2.3' (Existence of Feasible Portfolios). The set $\mathcal{A} \subseteq \mathbb{R}_n$ contains feasible portfolios such that a unique optimal portfolio $\tilde{\alpha}$ exists in the interior of $\mathcal{A}$ for all $\theta < 0$.

Assumption 2.4' (Asymptotic Stability). In view of Assumption 2.3, let us define

$$
\ell^*(x, \theta) = \liminf_{n \to \infty} \frac{1}{n} \log \ell(x, n\theta), \forall x \in \mathcal{X}.
$$

$$
\Lambda(\theta) = \liminf_{n \to \infty} \frac{1}{n} \inf_{\alpha \in \mathcal{A}} \log E_P[\ell(S_n^{(\alpha)}, n\theta)] = \liminf_{n \to \infty} \frac{1}{n} \log E_P[\ell(S_n^{(\tilde{\alpha})}, n\theta)].
$$

Let $\theta = \inf\{\theta < 0 : \Lambda(\theta) = -\infty\}$, then, for each $x \in B_r(r)$, there exists a sufficiently small $\theta_r \in (\theta, 0)$ such that $\ell^*(x, \theta) = \ell^*(x, \theta_r)\theta + \tau(\epsilon, \theta_r)$ for all $\theta \in (\theta, \theta_r)$, where $\ell^*(x, \theta_r) = \frac{\partial \ell^*(x, \theta)}{\partial \theta} |_{\theta = \theta_r}$. And the remainder $\tau(\epsilon, \theta_r)$ is infinitesimal (i.e., $\lim_{\theta_r \to \infty, \epsilon \searrow 0} |\tau(\epsilon, \theta_r)| = 0$).

We define the following Fenchel-Legendre transform of $\Lambda(\theta)$:

$$
\Lambda^*(s) = \sup_{\theta \in \theta_r} [\theta \ell^*(s, \theta_r) - \Lambda(\theta)].
$$

By the definition of the Fenchel-Legendre transform, the rate function $\Lambda^*(s)$ is a convex function.
Following Dembo and Zeitouni (1998, Lemma 2.3.9), it is not hard to prove that

$$
\Lambda^*(s) = \begin{cases} 
\hat{\theta}_s \ell^*(s, \theta_r) - \Lambda(\hat{\theta}_s) & \text{if } s \in \{s : \Lambda'(\theta) < \ell^*(s, \theta_r) \leq \Lambda'(\theta_r)\}, \\
0 & \text{if } s \in \{s : \Lambda'(\theta_r) < \ell^*(s, \theta_r)\},
\end{cases}
$$

(2.12)

where $\hat{\theta}_s = \theta(s, \theta_r)$ is a unique solution [assuming that it always exists] to $\ell^*(s, \theta_r) = \Lambda'(\theta)$, where $\Lambda'(\theta) = \frac{\partial \Lambda(\theta)}{\partial \theta}$, such that $\hat{\theta}_s \in (\theta, \theta_r]$. (Note that the point $\hat{\theta}_s$ is also called as an exposing point to the hyperplane $\{\theta : \ell^*(s, \theta_r) = \Lambda'(\theta)\}$ associated with the half-spaces $\{\theta : \ell^*(s, \theta_r) > \Lambda'(\theta)\}$ and $\{\theta : \ell^*(s, \theta_r) < \Lambda'(\theta)\}$.)

The second theorem of this paper is stated below.

**Theorem 2.** If Assumptions 2.1', 2.2, 2.3', and 2.4' hold, the the infimum shortfall probability can be approximated as follows:

$$
\liminf_{n \to \infty} \frac{1}{n} \inf_{\alpha \in A} \log P(S_n^{(\alpha)} \leq r) = -\Lambda^*(r) - \tau(0, \theta_r).
$$

(2.13)

Theorem 2 states that an investor who holds a sufficiently large number of risky assets can always find an optimal combination of these assets by minimizing an expected endogenous loss function so as to ensure that the shortfall probability is minimum. Furthermore, an average investor can do best by simply diversifying among all classes of risky assets and then minimizing this investor’s expected endogenous loss function. Hence, diversification apparently benefits most investors endowed with certain endogenous convex loss functions.

**Proof of Theorem 2.** Since this theorem is dual to Theorem 1, the proof can be done with the same method used to prove the latter. QED.

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3 Specific Results

If \( \ell(x, \theta) \) is an exponential function (i.e., \( \ell(x, \theta) = e^{\theta x} \)), then Assumptions 2.1, 2.2, and 2.1' are obviously satisfied; Assumption 2.3 implies Assumption 2.4 with \( \tau(e, \theta_r) = 0 \) (i.e., the Taylor approximation is exact); and Assumption 2.3' implies Assumption 2.4' with \( \tau(e, \theta_r) = 0 \). The large deviations estimation of the optimal windfall and shortfall probabilities can be derived in the same spirit as the Gärtner-Ellis theorem (see, e.g., den Hollander (2000, Chap. 5)). We now state the following corollary:

**Corollary 1.** If Assumption 2.3 holds, then the supremum windfall probability can be approximated as follows:

\[
\limsup_{n \to \infty} \frac{1}{n} \sup_{\alpha \in \mathcal{A}} P(S_n^{(\alpha)} \geq r) = -\Lambda^*(r), \tag{3.1}
\]

where \( \Lambda^*(r) = \sup_{\theta \in (0, \bar{\theta})} [r\theta - \Lambda(\theta)] \) with \( \Lambda(\theta) = \limsup_{n \to \infty} \frac{1}{n} \sup_{\alpha \in \mathcal{A}} \log E_{\mathbb{P}}[e^{n\theta S_n^{(\alpha)}}], \) for a \( r \in \mathbb{R} : \Lambda'(0) < r < \Lambda'(\bar{\theta}) \).

Furthermore, if Assumption 2.3' holds, then the infimum shortfall probability can be approximated as follows:

\[
\liminf_{n \to \infty} \frac{1}{n} \inf_{\alpha \in \mathcal{A}} P(S_n^{(\alpha)} \leq r) = -\Lambda^*(r), \tag{3.2}
\]

where \( \Lambda^*(r) = \sup_{\theta \in (0, \bar{\theta})} [r\theta - \Lambda(\theta)] \) with \( \Lambda(\theta) = \liminf_{n \to \infty} \frac{1}{n} \inf_{\alpha \in \mathcal{A}} \log E_{\mathbb{P}}[e^{n\theta S_n^{(\alpha)}}], \) for a \( r \in \mathbb{R} : \Lambda'(\bar{\theta}) < r < \Lambda'(0) \).

Although the proof of the above corollary implicitly follows from that of Theorem 1, but for the sake of a lucid exposition of the large deviations theory, we shall provide the proof of Eq. (3.1), which is analogous to the proof of Theorem 1. The proof of Eq. (3.2) can be done in the same way.

**Proof of Corollary 1.** We shall prove the upper and lower bounds.

- **Upper Bound:** The Tchebyshev inequality gives

\[
P(S_n^{(\alpha)} \geq r) = E_{\mathbb{P}}[1(S_n^{(\alpha)} \geq r)] \leq \frac{E_{\mathbb{P}}[e^{n\theta S_n^{(\alpha)}}]}{e^{nr}}, \quad \forall \ \theta \in (0, \bar{\theta}).
\]
Linearizing the RHS of the above inequality, by Assumption 2.3 we obtain

\[
\frac{1}{n} \sup_{\alpha \in A} \log P(S_n^{(\alpha)} \geq r) \leq -\{\theta r - \frac{1}{n} \sup_{\alpha \in A} \log E_\mathbb{P}[e^{n\theta S_n^{(\alpha)}}]\}
\]

\[
\Leftrightarrow \limsup_{n \to \infty} \frac{1}{n} \sup_{\alpha \in A} \log P(S_n^{(\alpha)} \geq r) \leq -\{\theta r - \limsup_{n \to \infty} \frac{1}{n} \log E_\mathbb{P}[e^{n\theta S_n^{(\alpha)}}]\}.
\]

It follows that

\[
\limsup_{n \to \infty} \frac{1}{n} \sup_{\alpha \in A} \log P(S_n^{(\alpha)} \geq r) \leq -\sup_{\theta \in (0, \Lambda')} [\theta r - \Lambda(\theta)]
\]

\[
= -\Lambda^*(r), \quad (3.3)
\]

where \(\Lambda(\theta) = \lim_{n \to \infty} \frac{1}{n} \log E_\mathbb{P}[e^{n\theta S_n^{(\alpha)}}]\) and \(r \in \{r \in \mathbb{R} : \Lambda'(0) < r < \Lambda'(')\}\).

- **Lower Bound:** Let us define a conjugate joint probability measure, \(\mathbb{P}^*\), on the product probability space \((\Omega, \otimes_{i=1}^n \mathcal{F}_i, \mathbb{P})\) as follows:

\[
\frac{d\mathbb{P}^*}{d\mathbb{P}} = e^{n\hat{\theta}_r S_n^{(\alpha)} - \log E_\mathbb{P}[e^{n\hat{\theta}_r S_n^{(\alpha)}}]},
\]

where \(\hat{\theta}_r\) is a unique exposing point [assuming that it exists] to the hyperplane \{\(\theta > 0 : \Lambda'(\theta) = r\)\}.

Pick up a sequence, \(r_n\), such that \(\lim_{n \to \infty} r_n = r\) and \([r_n - \epsilon, r_n + \epsilon] \in \{r \in \mathbb{R} : \Lambda'(0) < r < \Lambda'(\bar{\theta})\}\), we obtain

\[
P(S_n^{(\alpha)} > r_n - \epsilon) = \int 1(S_n^{(\alpha)} > r_n - \epsilon)e^{\log E_\mathbb{P}[e^{n\hat{\theta}_r S_n^{(\alpha)}}]-n\hat{\theta}_r S_n^{(\alpha)}} d\mathbb{P}^*
\]

\[
\geq \int 1(S_n^{(\alpha)} \in (r_n - \epsilon, r_n + \epsilon))e^{\log E_\mathbb{P}[e^{n\hat{\theta}_r S_n^{(\alpha)}}]-n\hat{\theta}_r S_n^{(\alpha)}} d\mathbb{P}^*
\]

\[
\geq e^{\log E_\mathbb{P}[e^{n\hat{\theta}_r S_n^{(\alpha)}}]-n\hat{\theta}_r (r_n + \epsilon)} P^*(S_n^{(\alpha)} \in (r_n - \epsilon, r_n + \epsilon)).
\]
Hence, it is straightforward to show that

\[
\limsup_{n \to \infty} \frac{1}{n} \sup_{\alpha \in A} \log P(S_n^{(\alpha)} \geq r_n) \geq \liminf_{n \to \infty} \frac{1}{n} \log P(S_n^{(\tilde{\alpha})} > r_n - \epsilon) \\
\geq -\left(\tilde{\theta}_r (r + \epsilon) - \Lambda(\tilde{\theta}_r)\right) \\
+ \liminf_{n \to \infty} \frac{1}{n} \log P^*(S_n^{(\tilde{\alpha})} \in (r_n - \epsilon, r_n + \epsilon)) \tag{3.4}
\]

In view of Eq. (3.4), in order to show the lower bound, we need to prove that

\[
\liminf_{n \to \infty} \frac{1}{n} \log P^*(S_n^{(\tilde{\alpha})} \in (r_n - \epsilon, r_n + \epsilon)) = 0.
\]

This is equivalent to show

\[
\limsup_{n \to \infty} \frac{1}{n} \log P^*(S_n^{(\tilde{\alpha})} \not\in (r_n - \epsilon, r_n + \epsilon)) < 0.
\]

Hence, it is necessary to show that

\[
\limsup_{n \to \infty} \frac{1}{n} \log P^*(S_n^{(\tilde{\alpha})} \geq r_n + \epsilon) < 0 \tag{3.5}
\]

and

\[
\limsup_{n \to \infty} \frac{1}{n} \log P^*(S_n^{(\tilde{\alpha})} \leq r_n - \epsilon) < 0. \tag{3.6}
\]

To prove Eq. (3.5), an application of the Tchebyshev inequality gives

\[
P^*(S_n^{(\tilde{\alpha})} \geq r_n + \epsilon) \leq e^{-n\lambda(r_n + \epsilon) - \log E_{\tilde{P}^*}[e^{n\lambda S_n^{(\tilde{\alpha})}}]}
\]

\[
\iff \limsup_{n \to \infty} \frac{1}{n} \log P^*(S_n^{(\tilde{\alpha})} \geq r_n + \epsilon) \leq -\{(r_n + \epsilon)\lambda - \limsup_{n \to \infty} \frac{1}{n} \log E_{\tilde{P}^*}[e^{n\lambda S_n^{(\tilde{\alpha})}}]\}
\]

\[
= -\{(r + \epsilon)\lambda - \tilde{\Lambda}(\lambda)\},
\]

where \(\tilde{\Lambda}(\lambda) = \limsup_{n \to \infty} \frac{1}{n} \log E_{\tilde{P}^*}[e^{n\lambda S_n^{(\tilde{\alpha})}}]\) for every \(\lambda \in (0, \bar{\theta} - \tilde{\theta}_r)\).
Moreover, since $\tilde{\Lambda}(\lambda) = \Lambda(\lambda + \hat{\theta}_r) - \Lambda(\hat{\theta}_r)$ we obtain

$$\limsup_{n \to \infty} \frac{1}{n} \log P^\ast(\alpha_n \geq r_n + \epsilon) \leq -(r + \epsilon)(\lambda + \hat{\theta}_r) - \Lambda(\lambda + \hat{\theta}_r) + \rho r \epsilon$$

$$\leq - \sup_{\xi \in (\hat{\theta}_r, \mathfrak{F})} [(r + \epsilon) \xi - \Lambda(\xi)] + \Lambda^*(r) + \hat{\theta}_r \epsilon$$

$$= - \Lambda^*(r + \epsilon) + \Lambda^*(r) + \hat{\theta}_r \epsilon$$

$$= -[\hat{\theta}_r r - \Lambda^*(r)] + [\hat{\theta}_r(r + \epsilon) - \Lambda^*(r + \epsilon)]$$

$$< 0,$$

where the last inequality follows from Lemma 1. Hence, we have proved Eq. (3.5). And Eq. (3.6) can be proved in the same way.

In view of Eq. (3.4), we have

$$\limsup_{n \to \infty} \frac{1}{n} \sup_{\alpha \in A} \log P(S_n(\alpha) \geq r) \geq -(\hat{\theta}_r(r + \epsilon) - \Lambda(\hat{\theta}_r)).$$

By letting $\epsilon$ go to 0, we obtain the following lower bound:

$$\limsup_{n \to \infty} \frac{1}{n} \sup_{\alpha \in A} \log P(S_n(\alpha) \geq r) \geq -\Lambda^*(r).$$

(3.7)

Eqs (3.3) and (3.7) implies Eq. (3.1). QED.

As an application of Corollary 1, we state the following proposition:

**Proposition 1.** Suppose that $\{X_1, \ldots, X_n\}'$ is a multivariate normal random vector, $\mathcal{N}(\mu', \Sigma)$, we have

$$\liminf_{n \to \infty} \frac{1}{n} \inf_{\alpha \in \mathbb{R}^n} \log P(S_n(\alpha) \leq r) = -\Lambda^*(r),$$

where

$$\Xi' \alpha = 1.$$
where

$$\Lambda^*(r) = r\hat{\theta}_r - \Lambda(\hat{\theta}_r) \text{ for some } r \in (-\infty, \Lambda'(0)), \quad (3.8)$$

$$\Lambda(\theta) = 2 \left\{ \frac{1}{4} \theta^2 \frac{C}{B^2} + \frac{1}{2} \theta \left( \frac{A}{B} + 2 \frac{AC}{B^2} - 2 \frac{D}{B} \right) + \frac{A^2}{B} + \frac{A^2C}{B^2} - 2 \frac{AD}{B} + F - E \right\}, \quad (3.9)$$

$$\hat{\theta}_r = \frac{B^2}{C} \left[ r - \left( \frac{A}{B} + 2 \frac{AC}{B^2} - 2 \frac{D}{B} \right) \right], \quad (3.10)$$

and

$$\lim_{n \to \infty} \frac{\mu' (\Sigma + \Sigma')^{-1} 1}{n} = A,$$

$$\lim_{n \to \infty} \frac{1' (\Sigma + \Sigma')^{-1} 1}{n} = B,$$

$$\lim_{n \to \infty} \frac{1' (\Sigma + \Sigma')^{-1} \Sigma (\Sigma + \Sigma')^{-1} 1}{n} = C,$$

$$\lim_{n \to \infty} \frac{1' (\Sigma + \Sigma')^{-1} \Sigma (\Sigma + \Sigma')^{-1} \mu}{n} = D,$$

$$\lim_{n \to \infty} \frac{\mu' (\Sigma + \Sigma')^{-1} \mu}{n} = E,$$

$$\lim_{n \to \infty} \frac{\mu' (\Sigma + \Sigma')^{-1} \Sigma (\Sigma + \Sigma')^{-1} \mu}{n} = F. \quad (3.11)$$

**Proof of Proposition 1.** Note that the set of feasible portfolios is $\mathcal{A} = \{ \alpha \in \mathbb{R}^n : 1' \alpha = 1 \}$, $\mathbb{P}$ is the multivariate normal distribution, and

$$\log E[e^{n\theta S_{\alpha}^n}] = n\theta \left( \alpha' \mu + \frac{1}{2} n\theta \alpha' \Sigma \alpha \right).$$

Since $\frac{\partial^2 \log E[e^{n\theta S_{\alpha}^n}]}{\partial \alpha \partial \alpha'} = \frac{n^2 \theta^2}{2} (\Sigma + \Sigma') > 0$, Assumption 2.3' is true. First, an application of Eq. (3.2) in Corollary 1 and some tedious algebra manipulation yield the following optimal portfolio vector:

$$\bar{\alpha}(\theta) = \frac{2}{\theta^2 n^2} (\Sigma + \Sigma')^{-1} (\eta 1 - n\theta \mu),$$
where

\[ \eta = \frac{1}{n} \left[ \frac{1}{\Lambda'(\Sigma + \Sigma')^{-1}1} + 2n^2 - n\theta \right] \left( \Sigma + \Sigma' \right)^{-1} \mu \].

Next, we can derive the rate function \( \Lambda^*(r) \) by initially substituting \( \tilde{\alpha}(\theta) \) into

\[ \Lambda(\theta) = \lim_{n \to \infty} \frac{1}{n} \log E[e^{n\sigma S_n}] \].

Unfortunately, the only known result is the following cumbersome formula:

\[
\Lambda(\theta) = \limsup_{n \to \infty} \frac{1}{n} \left[ \frac{1}{4n^2} \theta^2 \left( \frac{1}{\Sigma + \Sigma'} - n\theta \right) \left( \Sigma + \Sigma' \right)^{-1} \right] + \frac{1}{2} \theta \left( \frac{1}{\Sigma + \Sigma'} - n\theta \right) \left( \Sigma + \Sigma' \right)^{-1} \mu
\]

\[+ \frac{1}{\Sigma + \Sigma'} \left[ \frac{1}{\Sigma + \Sigma'} - n\theta \right] \left( \Sigma + \Sigma' \right)^{-1} \mu \]

\[= 2 \left[ \frac{1}{4} \theta^2 \frac{C}{B^2} + \frac{1}{2} \theta \left( \frac{A}{B} + 2 \frac{AC}{B^2} - 2 \frac{D}{B} \right) + \frac{A^2}{B^2} - 2 \frac{AD}{B} + F - E \right] \]

where the last equality follows from Eq. (3.11). Thus Eq. (3.9) has been proved. Finally, solving the equation \( \Lambda'(\theta) = r \), we can obtain the exposing point \( \hat{\theta} \), given in Eq. (3.10). QED.

\[\square\]

4 Conclusion

In this paper, we present the large deviations approximations for the windfall and shortfall probabilities of the [one-period-ahead] return of a diversified optimal portfolio. Moreover, we show that, in a sufficiently large portfolio, an optimal invested portfolio, which yields the maximum windfall
probability or the minimum shortfall probability, can also be obtained by maximizing an endogenous profit function or minimizing an endogenous loss function, respectively. The results in this paper suggest that, whatever the choice of investment vehicle, it is essential to have a sufficient diversified portfolio. This is because investing in only one or two assets is extremely risky and entirely inappropriate for the majority of investors. For instance, as far as hedge funds are concerned, holding funds of hedge funds has now become popular.

Our analytical framework, although not taking into consideration dynamic strategies, proves tractable in solving the problems posed above and illustrates the potential insights offered by this type of approach. Future research shall focus on studying dynamic investment strategies for diversified portfolios. We are aware that this is a quite challenging problem because of two reasons: First, we need to include some potentially important features of the data such as serial dependence and heteroscedasticity. Second, handling those features of the data by using well-known mathematical apparatuses such as stochastic calculus is nontrivial in the high dimension.
References


