The Strategic Use of Ambiguity in Games

Frank Riedel    Linda Sass

1Institute for Mathematical Economics
   Bielefeld University,
   ORFE, Princeton University

2Institute for Mathematical Economics
   Bielefeld University,
   Paris School of Economics

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Outline

1. Uncertainty vs. Risk
2. Strategic Uncertainty in Games
3. Ellsberg Games
4. An Example: Peace Negotiation
5. Two–Player Games
Uncertainty versus Risk

Motivation

- Knightian uncertainty versus (objective) risk
- Objective probability versus no probabilities, just uncertain outcomes
- Classical approach (de Finetti, Savage, Anscombe-Aumann): even under uncertainty, betting behavior allows to infer subjective probability measure $P$
- Large decision-theoretic literature on ambiguity aversion
- Many applications for single agent problems or large competitive economies
- Fewer investigations in strategic environments
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Uncertainty in Games

Classic Game Theory
- Players use pure strategies ...
- and mixed strategies = roulette wheels
- evaluate payoffs according to expected utility

Our Approach: Ellsberg Urns as Strategies in Uncertainty–Averse Environments
- imagine a player is allowed to use an Ellsberg urn
- ... credibly, and commit to use it
- for example, through a trustworthy laboratory
- what are the consequences for noncooperative games?
Back to von Neumann and Morgenstern

A Game

Let $N = \{1, \ldots, n\}$ be the set of players. Each player has a finite strategy set $S_i, i = 1, \ldots, N$. Players’ payoffs are given by functions

$$u_i : S \to \mathbb{R} \quad (i \in N).$$

Randomization

- von Neumann and Morgenstern allow players to use probability vectors $p_i$ over $S_i$.
- and impose expected utility (linearity in payoffs over probabilities).
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Justification in zero–sum games (von Neumann, Morgenstern)

- convexity of strategy sets
- you want to use mixed strategies to conceal your behavior
- Stackelberg version argument
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**Further Literature**

- **Bade**, GEB 2010, Ambiguous Act Equilibria in two Player Games
- Uncertainty about Beliefs about other players’ actions:
  - **Dow, Werlang**, JET 1994
  - **Lo**, JET 1996
  - **Marinacci**, GEB 2000
  - **Klibanoff**, 1996
  - **Ryan**, ET 2002
  - **Eichberger, Kelsey**, GEB 2000, JET 2002
  - **Eichberger, Kelsey, Schipper**, OEP 2009
  - **Mukerji, Shin**, ATE 2002
- to be continued . . .
Ellsberg Urns as Strategies

“I play $Up$ if a red ball is drawn from this urn that contains 100 red and blue balls in unknown proportions.”

Ellsberg urns

- Player $i$ chooses $(\Omega, \mathcal{F}, \mathcal{P})$
- $\Omega \neq \emptyset$, $\mathcal{F}$ σ-field, $\mathcal{P}$ set of probability measures on the measurable space $(\Omega, \mathcal{F})$
- Ellsberg strategy is such a model plus a measurable act $f_i : \Omega \rightarrow \Delta S_i$
- $\Delta S_i$: objective mixed strategies as before
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Ellsberg Urns as Strategies: Discussion

Remark

- Bade introduces "ambiguous acts" as measurable mappings $f_i : \Omega \to \Delta S_i$ and ambiguity-averse preferences over such acts and consistency of beliefs.

- we have "objective ambiguity" as characterized by Gajdos, Hayashi, Tallon, Vergnaud, JET 2008 and Giraud, Raphael, 2011.
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Ellsberg Game

- as von Neumann and Morgenstern, we need to decide how players evaluate Ellsberg urns
- All players are ambiguity–averse and use a pessimistic approach
- payoff of player $i$ for profile $(f_1, \ldots, f_N)$ (and Ellsberg urns $(\Omega_j, \mathcal{F}_j, \mathcal{P}_j)$)

$$U_i(f) = \min_{P_1 \in \mathcal{P}_1, \ldots, P_n \in \mathcal{P}_n} \int_{\Omega_1} \cdots \int_{\Omega_n} u_i(f_1(\omega_1), \ldots, f_n(\omega_n)) \, dP_n \cdots dP_1.$$
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Ellsberg Equilibrium

An *Ellsberg equilibrium* is a profile of Ellsberg urns 
\[(\Omega^1, \mathcal{F}^1, \mathcal{P}^1), \ldots, (\Omega^n, \mathcal{F}^n, \mathcal{P}^n)\] and acts \(f^* = (f^*_1, \ldots, f^*_n)\) such that no player has an incentive to deviate, i.e. for all players \(i \in N\) and all Ellsberg urns \((\Omega_i, \mathcal{F}_i, \mathcal{P}_i)\), and all acts \(f_i\) for player \(i\) we have

\[U_i(f^*) \geq U_i(f_i, f^* _i).\]
Ellsberg Equilibrium: Reduced Form

As in correlated equilibrium, only the laws of the acts $f_i$ on $\Delta S_i$ matter; one can thus go directly to sets of probability measures on $\Delta S_i$

**Definition**

A *reduced form Ellsberg equilibrium* is a profile of sets of probability measures $\mathcal{P}_i^* \subseteq \Delta S_i$, such that for all players $i \in N$ and all sets of probability measures $\mathcal{P}_i$ on $S_i$ we have

$$\min_{P_1 \in \mathcal{P}_1^*, \ldots, P_n \in \mathcal{P}_n^*} \int_S u_i(s) dP_1 \ldots dP_n \geq \min_{P_i \in \mathcal{P}_i, P_{-i} \in \mathcal{P}_{-i}^*} \int_S u_i(s_i, s_{-i}) dP_1 \ldots dP_n$$
General Remarks

- One does not improve one’s own payoff by introducing more ambiguity (as players are ambiguity-averse).
- Consequence: if all other players play classic Nash, the Nash equilibrium strategy is a best reply as well.
- Ellsberg equilibrium is a coarsening of the concept of Nash equilibrium.
- In particular, Ellsberg equilibria exist.
- Can one get interesting equilibria outside the set of Nash equilibria?
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Greenberg’s Example

- Unique Nash Equilibrium: A mixes uniformly, B plays War, C mixes uniformly
- War occurs with probability 1
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Greenberg’s Example in Normal Form

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<tr>
<th></th>
<th>war</th>
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<tbody>
<tr>
<td>war</td>
<td>0,9,1</td>
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<td>peace</td>
<td>9,0,0</td>
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<tr>
<td>peace</td>
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Punish A

Punish B
Suppose C plays the Ellsberg strategy \([0, 1]\), i.e. all probabilities between 0 and 1 possible for “Punish A”

- if A plays War, minimal expected payoff 0
- \((\text{peace, peace, } [0, 1])\) is an Ellsberg equilibrium
Greenberg’s Example: Ellsberg Equilibrium

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Greenberg’s Example: Conclusion

- there exist Ellsberg equilibria that lie outside the union of the support of all Nash equilibria
- empirically testable conclusion
- ambiguity has a strategic use as a threat in negotiations
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Bade 2010: the support of Ellsberg equilibria is contained in the union of supports of Nash equilibria

so “indistinguishable”? We do not think so. Interesting effects:

- nonlinear payoffs
- immunization against strategic ambiguity
- equilibria “easier” to play than classical mixed strategy equilibria
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Full ambiguity is not an equilibrium

- Suppose player 2 uses an Ellsberg strategy that allows for any $0 \leq Q \leq 1$ for HEAD.
- Player 1 has then a unique best reply.
- Play HEAD with probability $1/3$.
- If she does so, payoff always
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  \frac{1}{3} \cdot (3Q - (1 - Q)) + \frac{2}{3} \cdot (-q + (1 - q)) = \frac{1}{3},
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  independent of $Q$.
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Ellsberg Equilibria
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  independent of \(Q\)
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Suppose Player 2 plays the Ellsberg strategy \([q_0, q_1]\) with \(q_0 < \frac{1}{3} < q_1\), say \(q_0 = \frac{1}{4}, q_1 = \frac{2}{3}\)

Then the expected payoff from playing \(p\) for HEAD is

\[
\begin{align*}
&\text{plot}\left(\min\left(\frac{3\cdot p}{4} - \frac{p}{4} - \frac{(1-p)\cdot 1}{4} + \frac{(1-p)\cdot 3}{4}, \frac{3\cdot p}{3} - \frac{p}{3} - \frac{(1-p)\cdot 2}{3} + \frac{(1-p)\cdot 1}{3}\right), p = 0..1\right);
\end{align*}
\]
Ellsberg Equilibria II

How do the Ellsberg equilibria look like?

- player 1 plays HEAD with probability $P \in [1/2, P_1]$, $P_1 \geq 1/2$
- player 2 plays HEAD with probability $Q \in [1/3, Q_1]$, $Q_1 \leq 1/2$
- equilibrium payoffs are the same as in Nash equilibrium

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consistent with empirical findings of Goeree and Holt, AER 2001
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- consistent with empirical findings of Goeree and Holt, AER 2001
Ellsberg Equilibria II

How do the Ellsberg equilibria look like?

- player 1 plays HEAD with probability $P \in [1/2, P_1]$, $P_1 \geq 1/2$
- player 2 plays HEAD with probability $Q \in [1/3, Q_1]$, $Q_1 \leq 1/2$
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Consider the competitive two–person $2 \times 2$ game with payoff matrix

<table>
<thead>
<tr>
<th></th>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U$</td>
<td>$L$</td>
<td>$R$</td>
</tr>
<tr>
<td></td>
<td>$a, d$</td>
<td>$b, e$</td>
</tr>
<tr>
<td>$D$</td>
<td>$b, e$</td>
<td>$c, f$</td>
</tr>
</tbody>
</table>

such that

$$a, c > b \text{ and } d, f < e.$$
Ellsberg Equilibria III

Theorem

Let \([(P^*, 1 - P^*), (Q^*, 1 - Q^*)]\) denote the unique Nash equilibrium. Then the Ellsberg equilibria of the game are the following: For \(P^* > Q^*\) all Ellsberg equilibria are of the form

\[
([P^*, P_1], [Q^*, Q_1]) \text{ for } P^* \leq P_1 \leq 1, \ Q^* \leq Q_1 \leq P^*;
\]

for \(P^* < Q^*\) all Ellsberg equilibria are of the form

\[
([P_0, P^*], [Q_0, Q^*]) \text{ for } 0 \leq P_0 \leq P^*, \ P^* \leq Q_0 \leq Q^*;
\]

and for \(P^* = Q^*\) all Ellsberg equilibria are of the form

\[
(Q^*, [Q_0, Q_1]) \text{ where } Q_0 \leq Q^* \leq Q_1
\]

and

\[
([P_0, P_1], P^*) \text{ where } P_0 \leq P^* \leq P_1.
\]
Goeree–Holt Experiment

<table>
<thead>
<tr>
<th></th>
<th>HEAD</th>
<th>TAIL</th>
</tr>
</thead>
<tbody>
<tr>
<td>HEAD</td>
<td>320,40</td>
<td>40,80</td>
</tr>
<tr>
<td>TAIL</td>
<td>40,80</td>
<td>80,40</td>
</tr>
</tbody>
</table>

- **observation:** most row players play HEAD, column players close to Nash
- **Nash equilibrium:** \( ((1/2, 1/2), (1/8, 7/8)) \)
- **Ellsberg equilibrium:**

\[
([1/2, \bar{p}], [1/8, \bar{q}]) \text{ with } 1/2 \leq \bar{p} \leq 1 \text{ and } 1/8 \leq \bar{q} \leq 1/2.
\]
Further Topics

- Subjective Equilibria
- Selfconfirming Equilibrium
- Games in Extensive Form
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Summary

- going back to the foundations of game theory
- coarsening of the original formulation
- players allowed to play Ellsberg urns
- evaluate uncertainty pessimistically
- coarsening of Nash
- new, interesting phenomena, ambiguity as a threat, immunization, simpler to play