Bayesian Nash equilibria: a general perspective

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Abstract. Games with incomplete information were modelled by Harsanyi so as to have payoffs that are essentially estimates of random payoffs. For such estimates Harsanyi proposed to use the (conditional) expectation of the random payoffs. Here we propose a very general formulation of Bayesian games, based on an interim model, in a very precise, measure theoretical way. This general formulation not only allows for other estimators, but it also brings out and formalizes the fundamental concept of strategy-induced conditional priors. We formulate very general existence results for mixed Nash equilibria. As particular cases, these contain the existence results of Milgrom-Weber and others.

Keywords: games with incomplete information, large games, Nash equilibrium, private information, public information, non-strategic information.

1 Introduction

We present a general formulation of Bayesian games that goes considerably beyond the usual formulation that started with Harsanyi [20] (e.g., see [19]). The underlying idea is that Harsanyi’s formulation forces the players to make pointwise “estimates” of random payoff values to create deterministic payoff functions for the purpose of further game-theoretical treatment. However, this is done entirely by means of certain conditional expectations, which may not always correctly capture the stochastic behavior of those variables.

Because of this, the present study takes a more general look at Bayesian Nash equilibria in games with incomplete information. We propose a model that contains and improves the model for games with incomplete information of [20, 24]. Similar to the paper by Milgrom and Weber [24], we formulate existence results for both mixed and pure Bayesian Nash equilibria. These do not only contain those of [24] and [4] as specializations to a special version of the model, but even yield some improvements of those existence results themselves (for instance, the inclusion of public information). The central objects of study of our model are payoff functionals and the formal notion of strategy-induced conditional priors. The former have the interpretation of interim payoffs, and appear actually as collections of payoff functions, each having a different domain, because of our inclusion of public information. The latter ones go hand in hand with Milgrom-Weber’s condition of absolutely continuous information and can be said to form the true conditional Bayesian externalities for the players in a sense that is made precise in (5).

Similar to existence results for games in internal-external form, our mixed equilibrium existence results rely on the use of the topology of narrow convergence for transition probabilities, which has considerable unifying power. Appendix A contains a survey on this topology, which extends...
the classical weak convergence topology for probability measures as can be found for instance in [14]. Our presentation also benefits from the tensor product notion for transition probabilities and its continuity properties, as introduced in [4]. The prime emphasis is on the existence of mixed equilibria, which come forward as certain fixed point vectors of transition probabilities. The existence of pure equilibria will follow in a second, routine stage, where additional nonatomicity or convexity/quasiconcavity conditions enable the use of well-known purification methods. This will be the subject of a more expanded version of this paper (cf. [11]).

2 A general model for Bayesian games

In this section we propose a very general interim model for a so-called general Bayesian game $\Delta$ that has the following abstract form

$$\Delta := (I, \Omega_0, \{ (\Omega_i, S_i, U_i) \}_{i \in I}, \eta).$$

Here $I$ is the set of all players and $\Omega_0$ is the set of all states of nature, commonly observable. Also, each $\Omega_i$ is the set of all possible manifestations (types) of player $i \in I$, only privately observable, and $S_i$ is the set of all possible actions for player $i$ (subsequent type-contingent restrictions will also be discussed). Further, $U_i$ stands for player $i$’s interim payoff functional, which is a collection of functions of player $i$’s type and action and also of the strategy profile along $i$’s opponents, either in mixed or pure strategies. Finally, $\eta$ stands for the joint Bayesian prior distribution of both public and private outcomes.

We now introduce some technical details of the model. The set $I$ of all players is at most countable, as is $\Omega_0$, the set of all possible states of nature. From now on we shall write $\Omega_0 = \{ \omega_{0k} : k \in K \}$, where the index set $K \subset \mathbb{N}$ is either finite or countable. The realized state of nature is supposed to be observable by all the players. For each player $i$ the set $\Omega_i$ of all his/her possible types is equipped with a $\sigma$-algebra $F_i$. The joint common prior distribution of publicly and privately observable outcomes is formally given by a probability measure $\eta$ on the product set $\Omega_0 \times \Omega$, where $\Omega := \Pi_{i \in I} \Omega_i$ (below we also use the usual but slightly abusive game-theoretical notation such as $\Omega^{-1} := \Pi_{j \in I \setminus I_i} \Omega_j$, etc.). The set $\Omega$ is is equipped with the product $\sigma$-algebra $\mathcal{F} := \otimes_{i \in I} \mathcal{F}_i$ and $\Omega_0 \times \Omega$ has the $\sigma$-algebra $\mathcal{F}_0 \otimes \mathcal{F}$. Here $\mathcal{F}_0 := 2^{\Omega_0}$ denotes the power set of the discrete space $\Omega_0$. Let $\alpha_k := \eta(\{ \omega_{0k} \} \times \Omega)$; we shall assume that $\alpha_k > 0$ for every $k \in K$. Each player $i \in I$ is only informed about the the realized state of nature $\omega_0 \in \Omega_0$, communicated to all players, as well as his/her realized own type $\omega_i \in \Omega_i$, which is private information.

Further, the action space $S_i$ of each player $i$ is supposed to be a metrizable Suslin space (see Appendix A). Below we shall also use $S := \Pi_{i \in I} S_i$, $S^{-i} := \Pi_{j \in I \setminus I_i} S_j$, etc. Contingent on the publicly observed state of nature $\omega_0$ and his/her privately observed type $\omega_i$, each player $i$ is constrained to choose his/her action from a certain given subset $A_i(\omega_0, \omega_i)$ of $S_i$. We suppose that these subsets are as follows: for every $i \in I$ and $k \in K$

(A1) $A^i_0(\omega_i) := A_i(\omega_{0k}, \omega_i)$ is nonempty and compact for every $\omega_i \in \Omega_i$,

(A2) the set $\text{gph} A^i_0 := \{(\omega_i, s_i) \in \Omega_i \times S_i : s_i \in A^i_0(\omega_i)\}$ is $\mathcal{F}_i \otimes \mathcal{B}(S_i)$-measurable.

Here $\mathcal{B}(S_i)$ stands for the Borel $\sigma$-algebra on $S_i$. Let $\mathcal{R}( (\Omega_0 \times \Omega_i, \mathcal{F}_0 \otimes \mathcal{F}_i); S_i)$ be the set of all transition probabilities with respect to $(\Omega_0 \times \Omega_i, \mathcal{F}_0 \otimes \mathcal{F}_i)$ and $(S_i, \mathcal{B}(S_i))$ and let $\mathcal{R}( (\Omega, \mathcal{F}); S_i)$ be the set of all transition probabilities with respect to $(\Omega, \mathcal{F})$ and $(S_i, \mathcal{B}(S_i))$; see Appendix A(ii). Let $\eta_i$ be the marginal probability on $(\Omega_0 \times \Omega_i, \mathcal{F}_0 \otimes \mathcal{F}_i)$ that is obtained by setting $\eta_i := \eta(\cdot \times \Omega^{-i})$. A mixed strategy for player $i \in I$, based on both common and private information, is a transition probability $\delta_i \in \mathcal{R}( (\Omega_0 \times \Omega_i, \mathcal{F}_0 \otimes \mathcal{F}_i); S_i)$ such that

$$\delta_i(\omega_0, \omega_i)(A_i(\omega_0, \omega_i)) = 1 \text{ for } \eta_i\text{-a.e. } (\omega_0, \omega_i) \text{ in } \Omega_0 \times \Omega_i.$$

The set of all mixed strategies for player $i$ will be denoted by $\mathcal{R}_i$. Then $\mathcal{R}_i := \Pi_{i \in I} \mathcal{R}_i$ is the set of all mixed strategy profiles and $\mathcal{R}^{-i} := \Pi_{j \in I \setminus I_i} \mathcal{R}_j$ is the set of all mixed strategy profiles along $i$’s
opponents. For each $k \in K$ let $\mathcal{R}_k$ be the set of all transition probabilities $\gamma_i \in \mathcal{R}(\Omega_i, \mathcal{F}_i, \eta_k^i); S_i)$ such that

$$\gamma_i(\omega_i)(A_k^i(\omega_i)) = 1$$

for $\eta^i_k$-a.e. $\omega_i$ in $\Omega_i$.

Here the conditional probability measure $\eta_k^i$ on $(\Omega_i, \mathcal{F}_i)$ is defined by

$$\eta_k^i := \eta(\{\omega_{i<k}\} \times \cdots \Omega_{i-1})/\alpha_k$$

on $(\Omega_i, \mathcal{F}_i)$. It is a marginal of the conditional probability measure $\eta^k$ on $(\Omega, \otimes_{i \in I} \mathcal{F}_i)$, which is defined by

$$\eta^k := \eta(\{\omega_{i<k}\} \times \cdots \Omega_i)/\alpha_k.$$ 

For later reference, we also define a related conditional probability measure that is a marginal of $\eta^k$: let $\eta_{k-i}^i$ be the probability measure on $(\Omega_{i-1}, \otimes_{j \not\in I_i} \mathcal{F}_j)$ defined by

$$\eta_{k-i}^i := \eta(\{\omega_{i<k}\} \times \Omega_i)/\alpha_k.$$

Now it is easy to see that $\mathcal{R}_i$ is the set of all $\delta_i \in \mathcal{R}(\Omega \times \Omega_i, \mathcal{F}_0 \otimes \mathcal{F}_i); S_i)$ such that

$$\delta_i(\omega_{i<k}, \omega_i) \in \mathcal{R}_k^i$$

for every $k \in K$. (1)

We equip each $\mathcal{R}_k^i$ with the relative narrow topology that it inherits from its superset $\mathcal{R}(\Omega_i, \mathcal{F}_i, \eta_k^i); S_i)$. Note that this notation indicates that $\eta^k$ is taken on $\Omega_i$ as the underlying measure for the narrow topology; see Appendix A(v) for the definition of this topology. The Cartesian products $\mathcal{R}^k := \Pi_{i \in I} \mathcal{R}_k^i$ and $\mathcal{R}_{-i}^k := \Pi_{j \not\in I_i} \mathcal{R}_j^k$ will be equipped with the corresponding product topologies. It is important to see that a rational player $i$, after observing $\omega_{i<k}$, will only deal with the profile $(\delta_j(\omega_{i<k},\))_{j \in I_i} \in \mathcal{R}_{-i}^k$ along his/her opponents.

We can now make precise what we mean by player $i$’s interim payoff functional in this Bayesian game. This is a collection $U_i := \{U_k^i\}_{k \in K}$ of functions $U_k^i : \text{gph} A_k^i \times \mathcal{R}_{-i}^k \to \mathbb{R}, k \in K$, with the following interpretation: player $i$ observes the realized state of nature $\omega_{i<k} \in \Omega$, along with the other players, as well as the realization of his/her private type $\omega_i \in \Omega_i$. Then the public outcome $\omega_{i<k}$ determines that player $i$ will evaluate his/her payoff via the function $U_k^i$. Namely, by taking action $s_i \in A_i(\omega_{i<k}, \omega_i)$ under the mixed strategy profile $(\delta_j)_{j \in I_i} \in \mathcal{R}_{-i}$ player $i$’s payoff is the real number $U_i(\omega_{i<k}, \omega_i, s_i, (\delta_j(\omega_{i<k},\))_{j \in I_i})$. We suppose that these payoff functionals have the following properties, where the product narrow topology on $\mathcal{R}_{-i}^k$ plays an important role: for every $i \in I$ and $k \in K$

$$(U1) \ U_k^i(\omega_i, \cdot, \cdot) : A_k^i(\omega_i) \times \mathcal{R}_{-i}^k \to \mathbb{R}$$

is upper semicontinuous for every $\omega_i \in \Omega_i$,

$$(U2) \ (\omega_i, s_i) \mapsto U_k^i(\omega_i, s_i, (\gamma_j)_{j \in I_i})$$

is $\mathcal{F}_i \otimes \mathcal{B}(S_i) \cap \text{gph} A_k^i$-measurable for every $(\gamma_j)_{j \in I_i} \in \mathcal{R}_{-i}$,

$$(U3) \text{for every } \omega_i \in \Omega_i \text{ the function } (\gamma)_{j \in I_i} \mapsto \sup_{s_i \in A_i^k(\omega_i)} U_k^i(\omega_i, s_i, (\gamma)_{j \in I_i})$$

is narrowly lower semicontinuous on $\mathcal{R}_{-i}^k$.

This completes the description of the game $\Delta$ and its conditions. A mixed Nash equilibrium for $\Delta$ is a mixed strategy profile $(\delta_j^*_{i})_{j \in I} \in \mathcal{R}$ such that for every $i \in I$ and $k \in K$

$$\delta^*_{i}(\omega_{i<k}, \omega_i)(\argmax_{s_i \in A_i^k(\omega_i)} U_k^i(\omega_i, s_i, (\delta_{j}^*(\omega_{i<k},\))_{j \in I_i})) = 1$$

for $\eta^k$-a.e. $\omega_i$ in $\Omega_i$.

Developments below will show that in special cases such a mixed Nash equilibrium is a Bayesian Nash equilibrium in the sense of Harsanyi [20]. The main result of this section is as follows.

**Theorem 2.1** Under Assumptions (A1)-(A2) and (U1)-(U3) there exists a mixed Nash equilibrium for the game $\Delta$.

The generality of this result can already be measured from the fact that if the game $\Delta$ has only one player and if there is only one state of nature (i.e., $I = \{1\}$ and $K = \{1\}$), then we obtain the main mixed Nash equilibrium existence result of [10], i.e., its Theorem 2.1.2, of which several applications were given in [5, 6, 10] as corollaries (for instance, the main existence results of [23, 24, 27]).
that substitution $\Delta$ undergoes an interpretational metamorphosis: namely, it can also be seen as a
game with a measure space of players, where $(\Omega_1, \mathcal{F}_1)$, the only type space, serves as its space of
players and $U_i(\omega_{0i}, \omega_{i\cdot})$ is interpreted as the payoff function of player $\omega_{i\cdot} \in \Omega_1$. Of course, this
is a minor variation on the fact that in the original model $\Omega_1$ contains all possible manifestations
of player 1. Conversely, the proof of Theorem 2.1 will be made it clear that it can also be derived
from Theorem 2.1.2 of [10]: see section 4.

3 Specializations: dependence on strategy-induced conditional priors

In this section we consider a specialization of the game $\Delta$, where the payoff functional $U_i = \{U^k_i\}_{k \in K}$
of each player $i$ has a special form; this form depends on what we call player $i$’s strategy-induced conditional prior. We shall establish this fundamental notion for Bayesian games and discuss its continuity and measurability properties in considerable generality. Although one can maintain the possibility of $I$ being countable [12, section 2], we shall make the simplifying assumption that the set of players is finite from now on.

First, recall the product tensor notion for transition probabilities, which is the natural extension of the concept of a product of probability measures. Fix $i \in I$ and $k \in K$ and let $\gamma_j, j \in I \setminus i$, be transition probabilities in $\mathcal{R}((\Omega_i, \mathcal{F}_i); S_i)$. The tensor product $\gamma$ of these is uniquely determined by taking product measures pointwise:

$$\gamma(\omega_{-i}) := \times_{j \in I \setminus i} \gamma_j(\omega_j).$$

It is easy to check that the above defines a transition probability $\gamma$ with respect to $(\Omega_{-i}, \otimes_{j \in I \setminus i} \mathcal{F}_j)$ and $(S_{-i}, \mathcal{B}(S_{-i}))$ (cf. [4]). Usually we shall denote this transition probability $\gamma$ by $\otimes_{j \in I \setminus i} \gamma_j$. In the above tensor product we can take $(\gamma_j)_{j \in I \setminus i} \in \mathcal{R}^k_{-i}$ as a special choice, because we can extend each $\gamma_j$ in $\mathcal{R}_j$ canonically to a transition probability with respect to $(\Omega_j, \mathcal{F}_j)$ and $(S_j, \mathcal{B}(S_j))$, which will be frequently done in this paper [explicitly, let $\tilde{\gamma}_j(\omega_j)(B_j) := \gamma_j(\omega_j)(B_j \cap A^k_j(\omega_j))$] and identify the extension $\tilde{\gamma}_j$ with the original transition probability $\gamma_j$ — for measures this is standard practice. Thus, the tensor product $\otimes_{j \in I \setminus i} \tilde{\gamma}_j$ is well-defined for any $(\tilde{\gamma}_j)_{j \in I \setminus i} \in \mathcal{R}^k_{-i}$. Then we see from the above definition that

$$(\otimes_{j \in I \setminus i} \tilde{\gamma}_j)(A^k_{-i}(\omega_{-i})) = 1 \text{ for } \tilde{\eta}_{-i}^k \text{-a.e. } \omega_{-i} \text{ in } \Omega_{-i}. \tag{2}$$

Here $A^k_{-i}(\omega_{-i}) := \Pi_{j \in I \setminus i} A^k_j(\omega_j)$ and $\tilde{\eta}^k$ stands for the product measure $\times_{i \in I} \eta_i^k$, i.e., the product measure formed by all the marginals of the probability measure $\eta_i^k$. Of course, using (1) we can extend these definitions to mixed strategy profiles along player $i$’s opponents. Indeed, if $(\delta_j)_{j \in I \setminus i}$ belongs to $\mathcal{R}_{-i}$, then $(\delta_j(\omega_{0k}, \cdot))_{j \in I \setminus i}$ belongs to $\mathcal{R}^k_{-i}$ for each $k \in K$. So the tensor product $\delta := \otimes_{j \in I \setminus i} \delta_j$ can simply be defined by setting

$$\delta(\omega_{0k}, \cdot) := \otimes_{j \in I \setminus i} \delta^k_j$$

for every $k \in K$, where we use the following shorthand notation

$$\delta^k_j := \delta_j(\omega_{0k}, \cdot) \in \mathcal{R}^k_j. \tag{3}$$

This clearly defines a transition probability $\delta =: \otimes_{j \in I \setminus i} \delta_j$ in $\mathcal{R}(\Omega_0 \times \Omega_{-i}; S_{-i})$ with the following support property

$$\delta(\omega_{0k}, \omega_{-i})(A^k_{-i}(\omega_{-i})) = 1 \text{ for } \tilde{\eta}^k_{-i} \text{-a.e. } \omega_{-i} \text{ in } \Omega_{-i}.$$

A fundamental condition about absolutely continuous information, given by Milgrom and Weber [24] to improve over [1], is as follows.

(a) For every $k \in K$ the measure $\eta^k$ is absolutely continuous with respect $\tilde{\eta}^k$.

This condition is not only sufficient for what follows below, but it is also essential for the existence results that follow, in view of a counterexample of Simon [28]. Let $\psi^k$ be a fixed version of the
Radon-Nikodym derivative in condition (aci). Fix any player \( i \in I \). Then from the \( \tilde{\eta}^k \)-integrability of \( \psi^k \) it follows that for \( \eta^k_{\omega_i} \)-a.e. \( \omega_i \in \Omega \) (say apart from a null set \( N_{\omega_i} \)) the function \( \psi^k(\omega_i, \cdot) \) is integrable over \( \Omega_{-i} \) with respect to the product measure \( \tilde{\eta}_{\omega_i}^k \( = \times_{j \in I \setminus \{i\}} \eta^k_j \) \) and with an integral value

\[
\iota(\omega_i) := \int_{\Omega_{-i}} \psi^k(\omega_i, \cdot) d\tilde{\eta}_{\omega_i}^k.
\]

Indeed, we can note that \( \int_{E} \iota d\tilde{\eta}_{\omega_i}^k = \int_{E \times \Omega_{-i}} \psi^k d\tilde{\eta}_{\omega_i}^k \) for every \( E \in \mathcal{F}_i \), which gives \( \int_{E} \iota d\tilde{\eta}_{\omega_i}^k = \tilde{\eta}_{\omega_i}^k(E_i) \). By suitably modifying \( \psi^k \) outside \( \Omega_i \in \Gamma(\Omega_i \setminus N_i) \) (we make it equal to one there), we see that for every \( \omega_i \in \Omega_i \), the \( \omega_i \)-section \( \psi^k(\omega_i, \cdot) : \Omega_{-i} \to \mathbb{R} \) of \( \psi^k \) acts as a conditional density in the following way. For any \( E \in \otimes_{j \in I \setminus I} \mathcal{F}_j \)

\[
q^k_{\omega_i}(E) := \int_E \psi^k(\omega_i, \omega_{-i}) \tilde{\eta}_{\omega_i}^k(d\omega_{-i})
\]

defines for player \( i \), under the prior \( \eta_i \), (a version of) the conditional distribution of his/her opponents’ types, given his/her own type \( \omega_i \) and given that \( \omega_{-i} \) is the realized state of nature. Indeed, for any \( E_i \in \mathcal{F}_i \) the defining property of \( q^k \) gives \( \int_{E} q^k_{\omega_i}(E) \eta^k_i(d\omega_i) = \int_{E \times E} \psi^k d\tilde{\eta}_{\omega_i}^k = \eta^k_i(E_i \times E) \). Here it is important to observe that in particular \( q^k_{\omega_i}(\Omega_{-i}) = \iota(\omega_i) = 1 \) by the above. From the definition of \( q^k \) we see that \( q^k(\cdot)(E) \) is measurable for every \( E \in \otimes_{j \in I \setminus I} \mathcal{F}_j \) (apply Appendix A(iv)) and \( \sigma \)-additivity of \( q^k_{\omega_i} \) is classical. So \( \omega_i \mapsto q^k_{\omega_i} \) is a transition probability with respect to \( (\Omega_i, \mathcal{F}_i) \) and \( (\Omega_{-i}, \otimes_{j \in I \setminus I} \mathcal{F}_j) \). Thus, if player \( i \) supposes that his/her opponents will use the strategy profile \( (\delta_j)_{j \in I \setminus I} \), then, still given his/her own public and private information realized by \( (\omega_{i\delta}, \omega_i) \), he/she is confronted with a probability measure, called player \( i \)'s (opponents') strategy-induced conditional prior and denoted by \( \pi^k_{\omega_i}(\delta^*_j : j \in I \setminus I) \). It is a probability measure on the graph \( \text{gph } A^k_i \) over opponents’ types and actions which is determined by

\[
\pi^k_{\omega_i}(\gamma_j : j \in I \setminus I)(E \times B) := \int_E \psi^k(\omega_i, \omega_{-i})(\otimes_{j \in I \setminus I} \gamma_j)(\omega_{-i})(B) \tilde{\eta}_{\omega_i}^k(d\omega_{-i}) \tag{4}
\]

for any \( E \in \otimes_{j \in I \setminus I} \mathcal{F}_j, B \in \mathcal{B}(S_{-i}) \) and \( (\gamma_j)_{j \in I \setminus I} \in \mathcal{R}^k_{-i} \). Here \( \mathcal{R}^k_{-i} \) is the set of all \( (\omega_{-i}, s_{-i}) \in \Omega_{-i} \times S_{-i} \) such that \( s_{-i} \in A^k_i(\omega_i) \). By notation explained in Appendix A(iv) this product measure can also be expressed more succinctly as

\[
\pi^k_{\omega_i}(\gamma_j) := q^k_{\omega_i}(\omega_{-i} \otimes (\otimes_{j \in I \setminus I} \gamma_j)
\]

in view of the above definition of the probability measure \( q^k_{\omega_i} \). Note once again that we tacitly identify \( \gamma_j \) with its extension \( \gamma_j \) defined above, as is standard practice in measure theory. Observe that the marginal on \( \Omega_{-i} \) of each \( \pi^k_{\omega_i}(\gamma_j) \) in the above definition (4) is the probability measure \( q^k_{\omega_{-i}} \) (set \( B = S_{-i} \) in (4)). Of course, this is also evident from the succinct expression given above.

Let \( \text{Prob}_{q^k_{\omega_i}}(\text{gph } A^k_i) \) be the set of all \( \nu \in \text{Prob}_{\tilde{q}^k_{\omega_i}}(\text{gph } A^k_i) \) having \( q^k_{\omega_i} \) as their marginal on \( \Omega_{-i} \) (the same notation is used in Appendix A(vii)). Then (4) defines a mapping

\[
(\omega_i, (\gamma_j)_{j \in I \setminus I}) \mapsto \pi^k_{\omega_i}(\gamma_j : j \in I \setminus I) : \Omega_i \times \mathcal{R}^k_{-i} \to \text{Prob}_{q^k_{\omega_i}}(\text{gph } A^k_i).
\]

We now equip \( \text{Prob}(\Omega_{-i} \times S_{-i}) \), as well as its subset \( \text{Prob}_{q^k_{\omega_i}}(\text{gph } A^k_i) \), with the \( w\alpha \)-topology. We also need the following condition: for every \( i \in I \)

\((cg)\) the \( \sigma \)-algebra \( \mathcal{F}_i \) is countably generated.

Under this condition, it follows from Proposition 2.3 in [9] that \( \text{Prob}_{q^k_{\omega_i}}(\text{gph } A^k_i) \) is \( w\alpha \)-metrizable. Thus, that same set is also \( w\alpha \)-compact by Theorem 2.4 in [9]. We already announce that, because of these properties of \( \text{Prob}_{q^k_{\omega_i}}(\text{gph } A^k_i) \), the measurability part of our proof of Theorem 3.1 will actually show that for any \( (\gamma_j)_{j \in I \setminus I} \in \mathcal{R}^k_{-i} \) the mapping \( \omega_i \mapsto \pi^k_{\omega_i}(\gamma_j : j \in I \setminus I) \) is a transition probability when \( \text{Prob}_{q^k_{\omega_i}}(\text{gph } A^k_i) \) is equipped with the Borel \( \sigma \)-algebra \( \mathcal{B}(\text{Prob}_{q^k_{\omega_i}}(\text{gph } A^k_i)) \) for this \( w\alpha \)-topology.

We can now stipulate the nature of the payoff functionals that characterizes this section. For each player \( i \in I \) let \( \{V^k_i\}_{k \in K} \) be a collection of functions \( V^k_i : A^k_i \times \text{Prob}_{q^k_{\omega_i}}(\text{gph } A^k_i) \to \mathbb{R} \).
We suppose that these functions have the following properties, where the $ws$-topology is used, as mentioned above: for every $i \in I$ and $k \in K$

(V1) $V^k_i(\omega_i, \cdot, \cdot) : A^k_i(\omega_i) \times \text{Prob}_{\nu_{\omega_i}}(\text{gph} A^k_i) \to \mathbb{R}$ is upper semicontinuous for every $\omega_i \in \Omega_i$,

(V2) $(\omega_i, s_i, \nu) \mapsto V^k_i(\omega_i, s_i, \nu)$ is measurable with respect to the product $\sigma$-algebra $((\mathcal{F}_i \otimes \mathcal{B}(S_i)) \cap \text{gph} A^k_i) \otimes \mathcal{B}(\text{Prob}_{\nu_{\omega_i}}(\text{gph} A^k_i))$.

(V3) for every $\omega_i \in \Omega_i$, the function $\nu \mapsto \sup_{s_i \in A^k_i(\omega_i)} V^k_i(\omega_i, s_i, \nu)$ is $ws$-lower semicontinuous on $\text{Prob}_{\nu_{\omega_i}}(\text{gph} A^k_i)$.

The special form required for the payoff functionals is the following one: for every $i \in I$ and $k \in K$

$$U^k_i(\omega_i, s_i, (\gamma_j)_{j \in I \setminus i}) = V^k_i(\omega_i, s_i, \pi^k_{\omega_i, (\gamma_j)_{j \in I \setminus i}})$$

for all $\omega_i \in \Omega_i$, $s_i \in A^k_i(\omega_i)$ and $(\gamma_j)_{j \in I \setminus i} \in \mathcal{R}^k_{-i}$. (5)

Of course, this form gives

$$U^k_i(\omega_i, s_i, (\delta^k_j)_{j \in I \setminus i}) = V^k_i(\omega_i, s_i, \pi^k_{\omega_i, (\delta^k_j)_{j \in I \setminus i}})$$

This means that player $i$, upon learning the state of nature $\omega_0k$ and his/her own private type $\omega_i$, appraises the consequences of taking an action $s_i$ exclusively on the basis of his/her strategy-induced conditional prior $\pi^k_{\omega_i, (\delta^k_j)_{j \in I \setminus i}}$, which therefore functions as his/her conditional Bayesian externality.

Observe that conditions (V1) and (V3) certainly hold if for every $i, k$ and $\omega_i \in \Omega_i$ the function $V^k_i(\omega_i, \cdot, \cdot)$ is continuous on $A^k_i(\omega_i) \times \text{Prob}_{\nu_{\omega_i}}(\text{gph} A^k_i)$. In this situation the essential condition for the payoffs $U^k_i(\omega_i, \cdot, \cdot)$ is that they are continuous in the own player’s actions and his/her strategy-induced conditional prior.

**Theorem 3.1** Suppose that the payoff functionals have the form requested in (5). Then under Assumptions (A1)-(A2), (cg), (V1)-(V3) and (aci) there exists a mixed Nash equilibrium for the game $\Delta$.

**Proof.** Fix any $i \in I$ and $\omega_i \in \Omega_i$. We claim that the mapping

$$(\gamma_j)_{j \in I \setminus i} \mapsto \pi^k_{\omega_i, (\gamma_j)_{j \in I \setminus i}} : \mathcal{R}^k_{-i} \to \text{Prob}_{\nu_{\omega_i}}(\text{gph} A^k_i) \subset \text{Prob}(\Omega_{-i} \times S_{-i})$$

is continuous. By definition of the $ws$-topology (see Appendix A(vii)) this amounts to proving that for any $E \in \bigotimes_{j \in I \setminus i} \mathcal{F}_j$ and any continuous bounded function $c : S_{-i} \to \mathbb{R}$ the mapping

$$(\gamma_j)_{j \in I \setminus i} \mapsto \int_E c(s_{-i}) \pi^k_{\omega_i, (\gamma_j)_{j \in I \setminus i}}(d(\omega_{-i}, s_{-i}))$$

is continuous for the product narrow topology on $\mathcal{R}^k_{-i}$. Now by (4)

$$\int_E c(s_{-i}) \pi^k_{\omega_i, (\gamma_j)_{j \in I \setminus i}}(d(\omega_{-i}, s_{-i})) = \int_E \psi^k(\omega_i, s_{-i}) \int_{S_{-i}} c(s_{-i})(\delta_{j \in I \setminus i} (\omega_{-i})(ds_{-i})) \eta^k_{-i}(d\omega_{-i}).$$

For our fixed $\omega_i$ the function $(\omega_{-i}, s_{-i}) \mapsto \psi^k(\omega_i, s_{-i}) c(s_{-i})$ is a Carathéodory integrand on $\Omega_{-i} \times S_{-i}$ with respect to $\eta^k_{-i}$ (see Appendix A(iii)), because the section $\psi^k(\omega_i, \cdot)$ is $\eta^k_{-i}$-integrable as we saw above. Also, by Theorem 2.5 of [4], condition (aci) implies the continuity of the tensor mapping

$$(\gamma_j)_{j \in I \setminus i} \mapsto \bigotimes_{j \in I \setminus i} \gamma_j : \mathcal{R}^k_{-i} \to \mathcal{R}(\Omega_{-i}, \bigotimes_{j \in I \setminus i} \mathcal{F}_j, \eta^k_{-i}; S_{-i}).$$

So the claim follows from applying Theorem 2.2(b) of [4] (see also Appendix A(v)) and the fact that compositions of continuous mappings are continuous. In turn, the fact that the claim is true implies immediately that conditions (U1) and (U3) for the payoff functionals $U_i$ follow from (V1) and (V3).
Next, fix any \(i \in I\) and \(\{(\gamma_j)_{j \in I, i}\} \in \mathcal{R}^{k_i}\). We claim that the mapping

\[
\omega_i \mapsto \pi^k_{\omega_i,(\gamma_j)_{j \in I, i}} : \Omega_i \to \text{Prob}_{\Omega_i} (\text{gph } A^k_{i,\omega_i}) \subseteq \text{Prob}_{\Omega_i} (\Omega_i \times S_{-i})
\]

is measurable. To prove this claim, the definition of the ws-topology requires us to show that for any \(E \in \otimes_{j \in I \setminus I} \mathcal{F}_j\) and any continuous bounded function \(c : S_{-i} \to \mathbb{R}\) the mapping

\[
\omega_i \mapsto \int_E c(s_{-i}) \pi^k_{\omega_i,(\gamma_j)_{j \in I, i}} (d(\omega_{-i}, s_{-i}))
\]

is \(\mathcal{F}_i\)-measurable. This follows immediately from Appendix A(iv). So the second claim also holds. The validity of condition (U2) now follows by (V2), because compositions of measurable functions remain measurable. We conclude that all conditions of Theorem 2.1 hold; the desired existence result follows immediately from applying that theorem. QED

We now specialize the original payoff functional even further. Let \(A^k(\omega) := \Pi_{i \in I} A^k_i(\omega_i)\). Then the graph of the multifunction \(A^k\) is given by

\[
\text{gph } A^k := \{ (\omega, s) : \omega \in \Omega, s \in A^k(\omega) \}.
\]

Let \(\{g^k_n\}_{n=1}^m\) be a collection of \((\otimes_{j \in I} \mathcal{F}_j) \otimes \mathcal{B}(S)\) \cap \text{gph } A^k\)-measurable functions \(g^k_n : \text{gph } A^k \to \mathbb{R}\) such that

\[
\begin{align*}
(1) & \ g^k_n(\omega, \cdot) \text{ is continuous on } A^k(\omega) \text{ for every } \omega \in \Omega, \\
(2) & \ \omega \mapsto \sup_{s \in A^k(\omega)} |g^k_n(\omega, s)| \text{ is } k\text{-integrable.}
\end{align*}
\]

For each player \(i \in I\) the mapping \(\nu \mapsto \int_{gph A^k_i} g^k_n(\omega_i, \omega_{-i}, s_i, s_{-i}) \nu (d(\omega_{-i}, s_{-i}))\) maps \(\text{Prob}_{\Omega_i} (\text{gph } A^k_{i,\omega_i})\) into \(\mathbb{R}^m\); let \(Y^k_i\) be the range of that mapping. Let \(\{W^k_i\}_{k \in K}\) be a collection of functions \(W^k_i : \text{gph } A^k \times Y^k_i \to \mathbb{R}\) with the following properties: for every \(k \in K\) and \(i \in I\)

\[
\begin{align*}
(W1) & \ W^k_i(\omega_i, \cdot, \cdot) : A^k_i(\omega_i) \times Y^k_i \to \mathbb{R} \text{ is upper semicontinuous for every } \omega_i \in \Omega_i, \\
(W2) & \ (\omega_i, s_i, y) \mapsto W^k_i(\omega_i, s_i, y) \in (\mathcal{F}_i \otimes \mathcal{B}(S_i) \cap \text{gph } A^k_i) \otimes \mathcal{B}(Y^k)\)-measurable, \\
(W3) & \text{for every } \omega_i \in \Omega_i \text{ the function } y \mapsto \sup_{s_i \in A^k_i(\omega_i)} W^k_i(\omega_i, s_i, y) \text{ is lower semicontinuous on } Y^k_i.
\end{align*}
\]

The following special form is required for the functionals \(V^k_i\) used in Theorem 3.1: for every \(i \in I\) and \(k \in K\)

\[
V^k_i(\omega_i, s_i, \nu) = W^k_i(\omega_i, s_i, \left( \int_{gph A^k_i} g^k_n(\omega_i, \omega_{-i}, s_i, s_{-i}) \nu (d(\omega_{-i}, s_{-i})) \right)_{n=1}^m ) ,
\]

which means that the original payoff functionals obtain the following form:

\[
U^k_i(\omega_i, s_i, (\delta^k_j)_{j \in I, i}) = W^k_i(\omega_i, s_i, \left( \int_{gph A^k_{i,j}} g^k_n(\omega_i, \omega_{-i}, s_i, s_{-i}) \pi^k_{\omega_i,(\delta^k_j)_{j \in I, i}} (d(\omega_{-i}, s_{-i})) \right)_{n=1}^m ) .
\]

**Corollary 3.1** Suppose that the payoff functionals have the form requested in (5) via (6). Then under Assumptions (A1)-(A2), (g1)-(g2), (eg), (W1)-(W3) and (aci) there exists a mixed Nash equilibrium for the game \(\Delta\).

By adopting the trick of [15, p. 78], the condition (eg) can actually be dropped ex post from this result. See [6, Remark 4.2] for the details.

**Proof.** Fix any \(k \in K\), \(i \in I\), \(n = 1, \ldots, m\) and \(\omega_i \in \Omega_i\). We claim that the mapping

\[
(s_i, \nu) \mapsto \int_{gph A^k_{i,j}} g^k_n(\omega_i, \omega_{-i}, s_i, s_{-i}) \nu (d(\omega_{-i}, s_{-i})) : A^k_i(\omega_i) \times \text{Prob}_{\Omega_i} (\text{gph } A^k_{i,\omega_i}) \to \mathbb{R}
\]

is measurable.
is continuous. To do this, recall from the above observations that \( \text{Prob}_{q_κ} (\text{gph} \, A^k_i) \) is metrizable for the \( ws \)-topology. So it is enough to prove sequential continuity. Let \( \{s_{i,p}\}_p \) converge to \( s_{i,0} \) in \( A^k_i(ω_i) \) and let \( \{ν_p\}_p \) \( ws \)-converge to \( ν_0 \) in \( \text{Prob}_{q_κ} (\text{gph} \, A^k_i) \). Then we must prove \( β_p \to β_0 \), where

\[
β_p := \int_{\text{gph} \, A^k_i} g^k_i(ω_i, ω_{-i}, s_{i,p}, s_{-i})ν_p(d(ω_{-i}, s_{-i})).
\]

First, observe that by Appendix A(vii) the \( ws \)-topology on \( \text{Prob}_{q_κ} (Ω_{-i} × S_{-i}) \) is homeomorphic to \( R((Ω_{-i}, ⊗_{j}∈I_{i}, F_j, q^k_i(ω_i); S_{-i}) \). So we may apply the last part of Appendix A(v); for \( p ∈ \mathbb{N} \) we define

\[
\tilde{g}(ω_{-i}, p, s_{-i}) := \begin{cases} 
g^k_i(ω_i, ω_{-i}, s_{i,p}, s_{-i}) & \text{if } s_{-i} ∈ A^k_i(ω_{-i}) 
+∞ & \text{otherwise}
\end{cases}
\]

and for \( p = ∞ \) we repeat this definition for \( \tilde{g}(ω_{-i}, ∞, s_{-i}) \) by replacing \( s_{i,p} \) by \( s_{i,0} \). Then we obtain \( \liminf_p β_p ≥ β_0 \). By repeating the above proof for \( g^k_n \) replaced by \( −g^k_n \), we also obtain \( \limsup_p β_p ≤ β_0 \). This proves the desired continuity and it immediately follows from (W1) and (W3) that (V1) and (V3) are fulfilled. Next, we prove that the mapping

\[
Φ : (ω_i, s_i, ν) ↦ \int_{\text{gph} \, A^k_i} g^k_i(ω_i, ω_{-i}, s_i, s_{-i})ν(d(ω_{-i}, s_{-i}))
\]

is measurable with respect to \( (F_i, ⊕ B(S_i)) \cap \text{gph} \, A^k_i \) \( ⊕ B(\text{Prob}_{q_κ} (\text{gph} \, A^k_i)) \). From the above continuity result it follows that for every \( (ω_i, s_i) ∈ A^k_i \) the function \( Φ(ω_i, s_i, ·) \) is continuous on the separable metrizable space \( \text{Prob}_{q_κ} (\text{gph} \, A^k_i) \) (recall our earlier conclusion that the latter space is metrizable and compact for the \( ws \)-topology). Also, by Appendix A(iv) it is evident that for fixed \( ν ∈ \text{Prob}_{q_κ} (\text{gph} \, A^k_i) \) the mapping \( Φ(·, ·, ν) \) is product measurable. So by an application of [15, III.14] it follows that \( Φ \) is product measurable. Then (W2) implies that (V2) holds. So the existence result now follows by an application of Theorem 3.1.

Both existence result of Milgram and Weber [24, Theorem 1] and its generalization by the present author in [4, Theorem 3.1] follow – in equivalent, interim form – from the above corollary. Namely, take \( m = 1 \) and let \( W^k_i(ω_i, s_i, y) := y \). Then

\[
V^k_i(ω_i, s_i, ν) = \int_{\text{gph} \, A^k_i} g^k_i(ω_i, ω_{-i}, s_i, s_{-i})ν(d(ω_{-i}, s_{-i}))
\]

causes (see (5))

\[
U^k_i(ω_i, s_i, (δ^k_j)_{j} ∈ I_{i,1}) = \int_{\text{gph} \, A^k_i} ψ^k_i(ω_i, ω_{-i}) \left[ \int_{A_{-i}(ω_{-i})} (\otimes_{j} \delta^k_j)(ω_{-i})(d(ω_{-i})g^k_i(ω_i, ω_{-i}, s_i, s_{-i})η^k_{-i}(dω_{-i}),
\]

which is precisely what is needed for the above interim form. In the special situation where \( m = 1 \), such as the one just discussed, condition (g1) can be relaxed as follows:

1. \( g^k_1(ω_1, ·) \) is upper semicontinuous on \( A^k(ω) \) for every \( ω ∈ Ω \).
2. \( g^k_3(ω) \) for every \( ω ∈ Ω \) the function \( s_{-i} ↦ \sup_{s_i ∈ A^k_i(ω)} g^k_i(ω_i, ω_{-i}, s_i, s_{-i}) \) is lower semicontinuous.
3. \( g^k_2(ω, s) \mid [g^k_i(ω, s)] \) is \( η^k \)-integrable.

We observe that in situations with \( n = 1 \), such as the one just discussed, condition (g^k_4) can be relaxed into an upper semicontinuity condition, but then a flanking condition is needed: for every \( i ∈ I, k ∈ K \) and \( ω ∈ Ω \) the function \( s_{-i} ↦ \sup_{s_i ∈ A^k_i(ω)} g^k_i(ω_i, ω_{-i}, s_i, s_{-i}) \) must be lower semicontinuous. This leads to another improvement of the existence results in [4, 24], but we shall leave the details for the reader.
4 Proof of Theorem 2.1

We shall obtain Theorem 2.1 by reformulating the game $\Delta$ as a game $\Gamma$ that has a measure space of players. We shall then apply Theorem 2.1.2 of [10] to $\Gamma$. Let $T := \Omega_0 \times (\cup_{i \in I} \Omega_i)$ be the set of “new players” of $\Gamma$. Here we suppose without loss of generality that the sets $\Omega_i$, $i \in I$, are mutually disjoint (otherwise, we could work with the disjoint sets $\Omega_i \times \{i\}$). Of course, we can also suppose without loss of generality that the elements $\omega_k$ of $\Omega_0$ are all different. For every $i \in I$ and $k \in K$ define $T_{k,i} := \{\omega_k\} \times \Omega_i$; then $T = \cup_{k \in K} \cup_{i \in I} T_{k,i}$, so $\{T_{k,i}\}_{k,i}$ forms an at most countable partition of $T$. Let $\mathcal{C}$ be the $\sigma$-algebra on $\cup_{i \in I} \Omega_i$ consisting of all sets $E \subset \cup_{i \in I} \Omega_i$ for which $E \cap \Omega_i$ belongs to $\mathcal{F}_i$ for every $i \in I$. We equip $T$ with the product $\sigma$-algebra $T := \mathcal{F}_0 \otimes \mathcal{C}$ and with the measure $\mu$ determined by

$$
\mu(\{\omega_k\} \times E) := \sum_{i \in I} 2^{-i} \eta(\{\omega_k\} \times (E \cap \Omega_i) \times \Omega_i).
$$

(7)

This gives $\mu(\{\omega_k\} \times (\cup_{j} \Omega_j)) \leq \alpha_k$, so the measure $\mu$ is finite. Observe also that for every $j \in I$, $B_0 \subset \Omega_0$ and $E \subset \Omega_j$ with $E \in \mathcal{F}_j$ definition (7) gives $\mu(B_0 \times E) = 2^{-j} \eta(B_0 \times E \times \Omega_j) = 2^{-j} \eta(B_0 \times E)$, so $2^{-j} \eta_j$ is the trace of $\mu$ on $\Omega_0 \times \Omega_j$. Further, we set $Z := \cup_{i \in I} S_i$; being an at most countable union of metrizable Suslin spaces $Z$ is itself also metrizable Suslin. We also define $\Sigma(t) := A_i(\omega_{0k}, \omega_i)$ if $t = (\omega_{0k}, \omega_i) \in T_{k,i}, k \in K, i \in I$.

By the $T$-measurability of the sets $T_{k,i}$ and Assumption (A2) we easily reach

**Conclusion 1:** The graph of $\Sigma$ is $T \otimes \mathcal{B}(Z)$-measurable, and the following is obvious from Assumption (A1):

**Conclusion 2:** All sets $S_i := \Sigma(t) \subset S, t \in T$, are compact and nonempty.

So $\Sigma$ meets Assumption 2.1.2 in [10]. Next, observe that [10] works with the set $\mathcal{R}_\Sigma$ of all transition probabilities $\delta$ with respect to $(T, T)$ and $(Z, \mathcal{B}(Z))$ such that $\delta(t)(S_i) = 1$ for $\mu$-a.e. $t \in T$. In the present situation to any given $\delta \in \mathcal{R}_\Sigma$ there corresponds a unique mixed strategy profile $(\delta_i)_{i \in I} \in \mathcal{R}$; namely, we can define each $\delta_i$ as a restriction:

$$
\delta_i(\omega_i) := \delta_i(\omega_{0k}, \omega_i) := \delta(t) \text{ for } i \in I, t := (\omega_{0k}, \omega_i) \in T_{k,i}, k \in K
$$

(9)

and then $\delta_i$ is easily seen to belong to $\mathcal{R}_i$. Recall that $2^{-i} \eta_i$ was shown to be the trace measure of $\mu$ on $\Omega_0 \times \Omega_i$. By $\delta(t)(S_i) = 1$ for $\mu$-a.e. $t \in T$ that gives $\delta_i(\omega_{00}, \omega_i)(A(\omega_{00}, \omega_i)) = 1$ for $\eta_i$-a.e. $(\omega_{00}, \omega_i)$ in $\Omega_0 \times \Omega_i$; so $\delta_i \in \mathcal{R}_i$. Conversely, given any mixed strategy profile $(\delta_i)_{i \in I} \in \mathcal{R}$, we can define a unique $\delta \in \mathcal{R}_\Sigma$ by concatenation:

$$
\delta(t) := \delta_i(\omega_{0k}, \omega_i) \text{ if } t = (\omega_{0k}, \omega_i) \in T_{k,i}, k \in K.
$$

(10)

Here again the role of $2^{-i} \eta_i$ being the trace measure of $\mu$ on $\Omega_0 \times \Omega_i$ is used to conclude that $\delta$ belongs to $\mathcal{R}_\Sigma$.

The above establishes that $\pi : \delta \mapsto (\delta_i)_{i \in I}$ is a bijection from $\mathcal{R}_\Sigma$ onto $\mathcal{R}$. In fact, we have much more: for the narrow topology and product narrow topology respectively

**Conclusion 3:** $\delta \mapsto (\delta_i)_{i \in I}$ is a homeomorphism from $\mathcal{R}_\Sigma$ onto $\mathcal{R}$.

That is to say, both $\pi$ and its inverse are continuous for the given topologies on $\mathcal{R}_\Sigma$ and $\mathcal{R}$. This is because the product topology on $\mathcal{R}$ is, by definition, the initial topology with respect to the coordinate projections from $\mathcal{R}$ into $\mathcal{R}_i$, $i \in I$, and because by (9) each such coordinate projection amounts to restriction to the set $\Omega_0 \times \Omega_i = \cup_k T_{k,i} \subset T$. As the latter set is $T$-measurable, such restriction to a measurable subset of $T$ maintains continuity with respect to the narrow topology. Conversely, a similar argument holds for the inverse mapping $\pi^{-1} : (\delta_i)_{i \in I} \mapsto \delta$.

Next, we define the payoff structure for the new game $\Gamma$. Let $t \in T$; then $t = (\omega_{0k}, \omega_i) \in T_{k,i}$ for some unique pair $(k, i) \in K \times I$, which causes $S_i := \Sigma(t)$ to be $A_i(\omega_{0k}, \omega_i)$; for any $s := a_i \in A_i(\omega_{0k}, \omega_i)$; we now define

$$
U_i(s, \delta) := U^i_k(\omega_i, a_i, (\delta^k_j)_j).
$$
Here the strategy profile \((\delta^k_j)_{j \in I}\) corresponds to \(\delta\), as explained in (9). By conclusion 3 it follows from (U1) and (U3) that Assumption 2.1.4 of [10] is fulfilled. Also, by (U2) we have that Assumption 2.1.5 of [10] holds, when we eliminate social interaction just as in [10, (2.1)]. This also means that Assumption 2.1.3 of [10] holds trivially (see Remark 2.1.1 in [10]). Finally, Assumption 2.1 of [10] can be handled as in Remark 2.1.1(iii) of [10]. We are thus in a position to apply Theorem 2.1.1 of [10]. This yields the existence of a Nash equilibrium profile \(\delta_* \in \mathcal{RS}^\prime\) for the game \(\Gamma\), i.e., with

\[
\delta_*(t)(\arg\max_{s \in S} U_t(s, \delta_*)) = 1 \text{ for } \mu\text{-a.e. } t \text{ in } I.
\]

Then \((\delta^*_{i})_{i \in I} := \pi^{-1}(\delta_*)\) has all the properties that we wanted to prove.

\[\text{A The narrow topology for transition probabilities: highlights}\]

In this section we recapitulate some important results for the narrow topology for transition probabilities (alias Young measures). For more material on Young measure theory the reader is referred to [4, 7, 9]; general background material about measure theory, topology and multifunctions can be found in [14, 15, 16, 25].

Let \((T, T, \mu)\) be a finite measure space and let \(X\) be a metrizable Suslin space. Recall from [17] that a topological space \(X\) is said to be Polish if there exists a compatible metric on \(X\) for which \(X\) is separable and complete. A Hausdorff space \(X\) is said to be Suslin if it is the surjective continuous image of some Polish space (i.e., if there exists a Polish space \(X'\) and a continuous mapping from \(X'\) onto \(X\)). So a space \(X\) is metrizable Suslin if there exists a compatible metric for which \(X\) is Suslin. By Prob\((X)\) we denote the set of all probability measures on \(X\), where \(X\) is equipped with its Borel \(\sigma\)-algebra \(\mathcal{B}(X)\). Likewise, if \((Y, \mathcal{Y})\) is an abstract measurable space then Prob\((Y)\) denotes the set of all probability measures on \((Y, \mathcal{Y})\).

In this setup we recall the following key notions and results:

(i) The classical weak topology on Prob\((X)\) is the coarsest topology on Prob\((X)\) for which the mapping \(\nu \mapsto \int_X c \, d\nu\) is continuous for every bounded and continuous function \(c : X \to \mathbb{R}\); cf. [14, 17].

(ii) A transition probability from \((T, T)\) into a measurable space \((Y, \mathcal{Y})\) is a function \(\delta : T \to \text{Prob}(Y)\) such that \(t \mapsto \delta(t)(B)\) is \(T\)-measurable for every \(B \in \mathcal{Y}\) [25, III.2]. The set of all such transition probabilities is denoted by \(\mathcal{R}((T, T); Y)\) or by \(\mathcal{R}((T, T, \mu); Y)\) if it is also necessary to stipulate the given measure \(\mu\) on \((T, T)\). If \((Y, \mathcal{Y}) = (X, \mathcal{B}(X))\) then by [13, Proposition 7.25] one has for any \(\delta : T \to \text{Prob}(X)\) that \(\delta \in \mathcal{R}((T, T); X)\) if and only if \(\delta\) is measurable with respect to \(T\) and the Borel \(\sigma\)-algebra on Prob\((X)\) that corresponds to the above-mentioned weak topology.

(iii) A normal integrand on \(T \times X\) is a \(T \otimes \mathcal{B}(X)\)-measurable function \(g : T \times X \to \mathbb{R}\) such that \(g(t, \cdot)\) is lower semi-continuous on \(X\) for every \(t \in T\). The function \(g\) is said to be \(\mu\)-integrably bounded from below if there exists a \(\mu\)-integrable \(f : T \to \mathbb{R}\) such that \(g(t, x) \geq f(t)\) for all \((t, x) \in T \times X\). The set of all normal integrands on \(T \times X\) that are \(\mu\)-integrably bounded from below is denoted by \(\mathcal{G}^{\text{lb}}((T, \mu); X)\). The same function \(g\) is said to be a Carathéodory integrand on \(T \times X\) with respect to \(\mu\) if both \(g\) and \(-g\) belong to \(\mathcal{G}^{\text{lb}}((T, \mu); X)\).

(iv) By the theory involving Fubini's theorem [25, Proposition III.2.1] one has that for every \(T \otimes \mathcal{B}(X)\)-measurable function \(g : T \times X \to (-\infty, +\infty]\) that is \(\mu\)-integrably bounded from below the integral

\[
I_g(\delta) := \int_T \int_X g(t, x) \delta(t)(dx) |\mu(dt|
\]

is well-defined in \((-\infty, +\infty]\); in particular, [25, Proposition III.2.1] shows that \(t \mapsto \int_X g(t, x) \delta(t)(dx)\) is a \(T\)-measurable function. Moreover, that same result also shows that every \(\delta \in \mathcal{R}((T, T); X)\) induces a finite product measure on \((T \times X, T \otimes \mathcal{B}(X))\), which we denote by \(\mu \otimes \delta\). We then have

\[
I_g(\delta) = \int_{T \times X} g \, d(\mu \otimes \delta)
\]

for all such \(g\).
(v) The narrow topology on $\mathcal{R}((T, T, \mu); X)$ is the coarsest topology for which all mappings $\delta \mapsto \int_A [\int_X c(x)\delta(t)(dx)]\mu(dt)$ are continuous for every $A \in \mathcal{T}$ and every bounded continuous function $c : X \to \mathbb{R}$. This generalizes the usual weak topology on $\text{Prob}(X)$, discussed in (i) above, as is seen by taking $T$ to be a singleton. Several useful and equivalent characterizations of the narrow topology are available; see [4, Theorem 2.2] or similar results in [7, 8]. For instance, the narrow topology on $\mathcal{R}((T, T, \mu); X)$ can equivalently be defined as the coarsest topology for which all functionals $\delta \mapsto I_g(\delta)$, $g \in \mathcal{G}^{bb}(T, T, \mu; X)$, are lower semi-continuous. Alternatively, it is also the coarsest topology for which the same functionals are continuous for all Carathéodory integrands $g$ on $T \times X$ with respect to $\mu$. Another equivalent of this kind is as follows [8, Theorem 4.13]. Denote by $\bar{N} := \mathbb{N} \cup \{\infty\}$ the usual Alexandrov compactification of the set of natural numbers (this is a compact metric space). Then $\{\delta_k\}_k := \{\delta_k\}_{k \in \bar{N}}$ converges narrowly to $\delta_0$ in $\mathcal{R}((T, T, \mu); X)$ if and only if
\[
\liminf_k \int_T \int_X \tilde{g}(t, k, x)\delta_k(t)(dx)\mu(dt) \geq \int_T \int_X \tilde{g}(t, \infty, x)\delta_0(t)(dx)\mu(dt)
\]
for every $\tilde{g} \in \mathcal{G}^{bb}(T, T, \mu; \bar{N} \times X)$.

(vi) The existence of a semi-metric $\rho$ for the narrow topology on $\mathcal{R}((T, T, \mu); X)$ allows the use of sequential topological arguments; such a semi-metric exists when the measure space $(T, T, \mu)$ is separable [8, Theorem 4.6].

(vii) The $ws$-topology on $\text{Prob}(T \times X)$ is is the coarsest topology for which all mappings $\pi \mapsto \int_{T \times X} c(x)\pi(dt, x)$ are continuous for every $A \in \mathcal{T}$ and every bounded continuous function $c : X \to \mathbb{R}$. Suppose that $\mu(T) = 1$. Then we have
\[
\{\mu \otimes \delta : \delta \in \mathcal{R}((T, T, \mu); X)\} = \text{Prob}_\mu(T \times X),
\]
where $\text{Prob}_\mu(T \times X)$ is defined as the set of all $\pi \in \text{Prob}(T \times X)$ whose marginal on $T$ is equal to $\mu$. This identity depends upon an important result about disintegration: for every $\pi \in \text{Prob}(T \times X)$ there exists $\delta_\pi \in \mathcal{R}((T, T, \mu); X)$, unique modulo null sets, such that $\pi = \mu \otimes \delta_\pi$. The $ws$-topology on $\text{Prob}_\mu(T \times X) \subset \text{Prob}(T \times X)$ is homeomorphic with the narrow topology on the quotient of $\mathcal{R}((T, T, \mu); X)$, taken with respect to the standard a.e.-equivalence relation. We refer to [9] for details.

References


