Content Articles in Economics

In this section, the *Journal of Economic Education* publishes articles concerned with substantive issues, new ideas, and research findings in economics that may influence or can be incorporated into the teaching of economics.

HIRSCHEL KASPER, Section Editor

On the Geometry of Constant Returns

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Abstract: Constant returns to scale, always a simplifying assumption, is often also much more: many important results depend critically on the very special properties of this class of production function. The author provides a unified set of simple proofs for most of the crucial analytical properties of constant returns production and their implications for firm costs. He uses only familiar diagrams and high school geometry, and the proofs are written to be easily understood by college sophomores.

Key words: constant returns, geometric proof
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Students are typically introduced to constant returns production in intermediate microeconomics. Thereafter—in courses on international trade, industrial organization, public finance, and other applied fields—they will need to have mastered the special properties of constant returns production if they are to understand fully the many telling illustrations, useful examples, and sometimes central results that ultimately depend upon them.

In the theory of international trade, the Heckscher-Ohlin Theorem, along with its principal corollaries and extensions, including the Factor Price Equalization Theorem, depends crucially on the special properties of constant returns production. So, too, do the Non-Substitution Theorem in general equilibrium and the Product Exhaustion Theorem in the theory of distribution. In the theory of economic growth, constant returns is an indispensable assumption in neoclassical (Solow) growth models, as well as many newer two-sector endogenous growth models. Ubiquitous in theory and applications, constant returns is usually far from a benign assumption—instead, it is usually at the very heart of the chain of reasoning and the eventual result.

When production displays constant returns to scale, firm cost functions take sim-

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ple and analytically convenient forms. With two factors and fixed factor prices, short-run average cost is U-shaped and short-run marginal cost is upward sloping; in the long run, average and marginal costs are constant and equal to one another. This combination of conventional short-run and simple long-run cost behavior accounts for much of the popularity of constant returns with theorists and teachers.

Because it typically precedes the specialized field courses, the intermediate micro class is a good place for students to become acquainted with the unique properties of constant returns. Yet many instructors choose not to employ calculus at this level of the curriculum, and so may feel they must simply assert without proof what the student will need to know. Fortunately, this need not be the case. In this article, simple proofs are provided for most of the crucial analytical properties of constant returns production and their implications for firm costs. Only familiar diagrams and high school geometry are used, and the proofs are written to be easily understood by college sophomores. Technical jargon that will normally be unfamiliar to students is minimized so that they may read and fully comprehend the arguments.

GEOMETRY

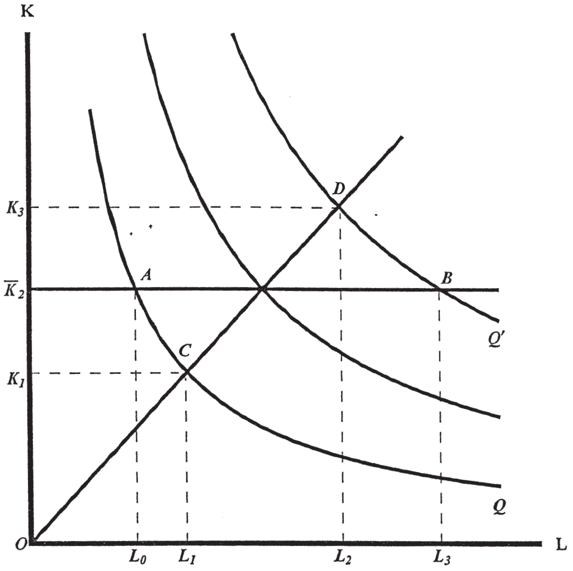
Consider a typical production function, f, summarizing efficient possibilities for combining two inputs to produce a single output. Let $Q = f(L,K)$ denote the greatest quantity of output that can be produced during some period of time if the firm uses amounts L and K of two factors I call labor and capital, respectively. I assume throughout that $f(0,0) = 0$ and that this production function is strictly increasing in nonnegative L and K, so that (1) no output is possible without positive amounts of some input and (2) the marginal product of labor, $MPL \equiv \Delta Q/\Delta L$, and the marginal product of capital, MPK $\equiv \Delta Q/\Delta K$, are finite and strictly positive everywhere.

A production function such as this can be represented by its isoquant map. Each isoquant is a level curve of the production function drawn in the (L,K) plane, giving all combinations of the inputs capable of producing a common level of output. Under the assumptions so far, some isoquant will pass through every point in the (L,K) plane, isoquants will not cross, they will be negatively sloped, and isoquants denoting greater levels of output will lie farther from the origin in a northeasterly direction.

With any production function, it is useful to distinguish between two broad classes of properties: its returns to variable proportions and its returns to scale. When economists consider returns to variable proportions, they ask how output behaves as more or less of a variable factor is combined with a given amount of some fixed factor, and this is most relevant to the firm's decision making in the short run. When they consider returns to scale, they hold constant the proportions in which factors are combined and ask how output behaves as the scale with which both are employed is increased or decreased together. Returns to scale are thus most relevant to the firm's long-run production decisions because only in the long run is the firm free to vary all the factors it uses.

To understand this distinction better, consider the production function represented in Figure 1. If, in the short run, the firm must employ fixed capital of $\bar{K}_2$,

FIGURE 1
Returns to Scale and Variable Proportions



it can increase output from Q to Q' only by increasing labor from L_0 to L_3, varying the proportions in which the two factors are used. Returns to variable proportions may thus be usefully thought of as describing how output behaves as one moves out a *horizontal* through the isoquant map, such as the horizontal $\overline{K}_2AB$. By contrast, consider the input combination (L_1,K_1) producing output level Q at point C. If both capital and labor are doubled, tripled, or scaled by any common factor, t, the proportions in which they are combined remain unchanged, that is, capital per worker remains constant at $K_1/L_1 = tK_1/tL_1$ for all $t > 0$, but the production point moves in or out the ray OCD. Thus, one may think of returns to scale as describing how output behaves as one moves out through the isoquant map along a *ray* from the origin such as OCD.

In general, returns to scale may be increasing, decreasing, or constant, as output increases more than in proportion, less than in proportion, or exactly in proportion to any change in the scale of input use. My focus in this article is exclusively upon constant returns to scale. Moreover, I restrict attention even further to only those constant returns production functions whose isoquants have the familiar convex-away-from-the-origin shape most commonly encountered in theory and applications. For future reference, assumptions and precise definitions follow.

Definition: Constant Returns Production

Let the production function $f(L,K)$ be strictly increasing in L and K, and let its isoquants be strictly convex away from the origin.[1] Then $f(L,K)$ has the property of constant returns to scale (globally) if, for all scalars $t > 0$ and all nonnegative input combinations (L,K),

$$f(tL,tK) = tf(L,K).$$

When the production function displays constant returns to scale, doubling both inputs always doubles output; indeed, scaling both inputs by any common factor, $t > 0$, scales output by exactly that same factor, t. Familiar examples of this sort of production function include the Cobb-Douglas form, $Q = AL^{a}Kl^{-a}$, where $A > 0$ and $0 < a < 1$, and the CES form, $Q = A(L^r + K^r)^{1/r}$ where $A > 0$ and $0 \neq r < 1$.

Constant returns has significant—and unique—structural implications for the isoquant map. In the following sections, these are established in a series of propositions, focusing first on the implications of constant returns for the spacing of isoquants looking out a ray from the origin, then looking out a horizontal. In each of these propositions the production function is assumed to satisfy the conditions of the preceding definition of constant returns production.

Looking Out a Ray

I begin with a most basic property of the isoquant map under constant returns to scale and show that the level of output produced by any combination of inputs will always be proportional to the distance from the origin of the corresponding point in the (L,K) plane. Because a ray is any straight line emanating from the origin, this property may be expressed as follows.

Proposition 1: Output is proportional to distance out any ray.

Proof*:* Choose any ray from the origin, *OR*, in the isoquant map depicted in Figure 2. Let (L_1,K_1) be the coordinates of the point where *OR* intersects the isoquant producing one unit of output (the "unit-isoquant"). Let

$$\alpha \equiv \sqrt{L_1^2 + K_1^2} \qquad (1)$$

denote the distance along *OR* from the origin to the unit isoquant.

Pick any level of output, Q. The point A marks the intersection of the Q-level isoquant and the ray *OR*. Obviously, Q units of output is Q times as much as one unit of output. Under constant returns, if L_1 and K_1 together produce one unit then, keeping factor proportions the same, it will take just Q times as much of each to produce Q units of output. Thus, the coordinates of A must be (QL_1,QK_1), as indicated.

Now compute the distance *OA* to the Q-unit isoquant along *OR* and compare it to the distance to the unit isoquant along that same ray. Using Pythagoras' Theoras, one has

$$\begin{aligned} OA &= \sqrt{(QL_1{}^2 + (QK_1)^2} \\ &= \sqrt{(L_1^2 + K_1^2)Q^2} \\ &= Q\sqrt{(L_1^2 + K_1^2)}. \end{aligned}$$

> Substituting from equation (1) into the last line, then solving for Q, one gets
>
> $$Q = \frac{1}{\alpha} OA.$$
>
> This says the output produced at any point along a ray from the origin will always be strictly proportional to the distance of that point from the origin.

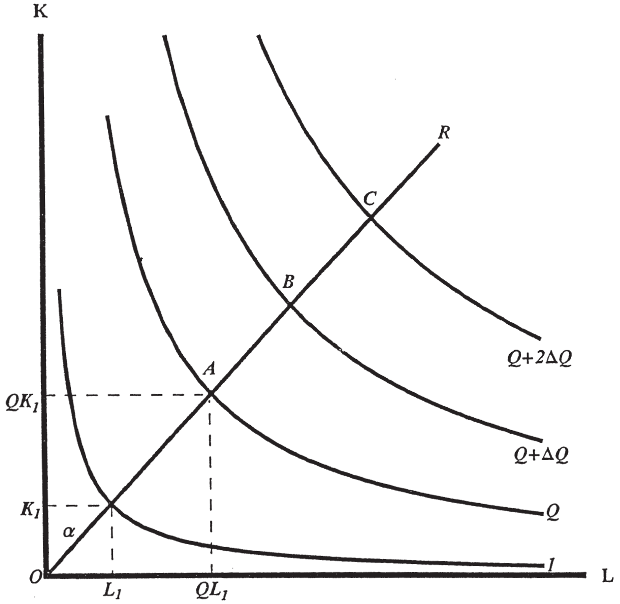
When output is proportional to distance from the origin, it is easy to see that as long as one remains along the same ray, cutting that distance in half will put one on an isoquant producing only half as much output. Doubling or tripling that distance will put one on an isoquant producing twice and three times as much output, respectively, and so on.

In the next proposition, consider the distance along a common ray between isoquants giving, successively, equal increments in output.

Proposition 2: *Successive isoquants giving equal increments in output are equally spaced along any ray.*

> **Proof**: Choose any output, Q, and locate its isoquant in Figure 2. Pick an increment in output of any size, $\Delta Q > 0$, and locate the isoquants giving $Q + \Delta Q$ and $Q + 2\Delta Q$ units of output so that the increment in output between them is the same and equal to ΔQ. It needs to be shown that as I look out any ray, OR, the distances AB and BC are equal.

FIGURE 2
Spacing of Isoquants Along a Ray



The previous result can be used to prove this proposition. Because output is proportional to distance out the ray *OR*, we know from Proposition 1 that

$$OC = \alpha(Q + 2\Delta Q) \quad \text{and}$$
$$OB = \alpha(Q + \Delta Q) \quad \text{and}$$
$$OA = \alpha Q.$$

Subtracting the second from the first and the third from the second gives,

$$OC - OB = \alpha(Q + 2\,\Delta Q - Q - \Delta Q) = \alpha\Delta Q \quad \text{and}$$
$$OB - OA = \alpha(Q + \Delta Q - Q) = \alpha\Delta Q,$$

so $OC - OB = OB - OA$. But $BC = OC - OB$ and $AB = OB - OA$, so $AB = BC$, and the proof is complete.

Propositions 1 and 2 provide useful geometric groundwork for what lies ahead, but the next proposition is the first with direct economic importance. Students will have learned that the (absolute value of the) slope of an isoquant at any point in the (L,K) plane is called the *marginal rate of technical substitution* or MRTS. The MRTS measures (locally) the rate at which labor can be substituted for capital with no change in the level of output produced and is therefore important in many decisions the firm must make. In general, the MRTS will depend on both *L* and *K* separately—if either, or both, are changed, one generally expects the MRTS to change as well. Under constant returns, however, the MRTS is completely *independent of scale* and depends only on factor proportions—whether producing 1 unit or 1 million, as long as the firm uses the same amount of capital per worker, and its possibilities for substituting one for the other remain unchanged.

The implications of this for the isoquant map are sweeping. First recall that only by moving along a ray from the origin in the (L,K) plane will capital per worker remain constant as scale is varied. If the MRTS depends only on factor proportions, not on scale, then the slope of every isoquant as it crosses a common ray from the origin must always be the same. Although the MRTS will be different along different rays, changing the overall scale of production by moving in or out any given ray will have no effect at all on the MRTS. Thus, under constant returns, isoquants must all be parallel as one looks out any ray. This important implication of constant returns production can now be established.

Proposition 3: *Isoquants are radially parallel.*

Proof: In Figure 3, isoquants for arbitrary levels of output, *Q* and Q′, are identified, and an arbitrary ray from the origin, *OAA′*, is chosen. Ultimately, it must be shown that the slope of the tangent to the isoquant at *A* is equal to the slope of the tangent to the isoquant at *A′*.

The approach is somewhat indirect. To preview, first another ray, *OB′*, and the chords *A′B′* and *AB*, are constructed. The chord *AB* is then shown to be parallel to the chord *A′B′*, and then a limiting argument is made to complete the proof.

To begin, define $\lambda \equiv Q'/Q$ so that $Q' \equiv \lambda Q$. Under constant returns, to achieve the λ-fold increase in output between A and A' requires that the amount of labor and capital at A' be exactly λ times as much as at A. Similarly, the coordinates of B' must be λ times as great as the coordinates at B'. The coordinates of these points are therefore marked accordingly in Figure 3.

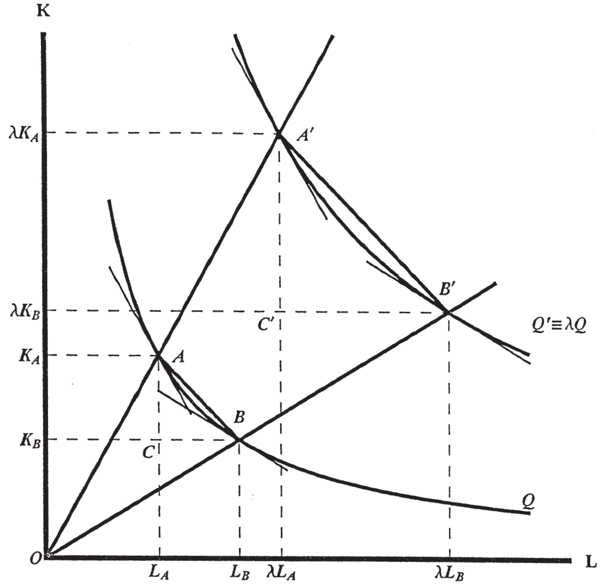
It is easy to see that the slope of chord AB is negative one times the ratio AC/CB, or

$$\text{Slope of the chord } AB = \frac{-(K_A - K_B)}{L_B - L_A}. \tag{2}$$

Similarly, the slope of the chord $A'B'$ is negative one times the ratio $A'C'/C'B'$, or,

$$\begin{aligned}\text{Slope of the chord } A'B' &= \frac{-(\lambda K_A - \lambda K_B)}{\lambda L_B - \lambda L_A} \\ &= \frac{-\lambda(K_A - K_B)}{\lambda(L_B - L_A)} \\ &= \frac{-(K_A - K_B)}{L_B - L_A}.\end{aligned} \tag{3}$$

FIGURE 3
Radially Parallel Isoquants



The right-hand sides of equation (2) and and the last line of equation (3) are the same, so the slope of $A'B'$ is equal to the slope of AB.

Now, the slope of $A'B'$ approximates the slope of the tangent at A', and the slope of AB approximates the slope of the tangent at A. To complete the argument, imagine picking the ray OBB' closer and closer to the ray OAA'. The slopes of $A'B'$ and AB remain equal to one another as in equations (2) and (3). At the same time, the slope of $A'B'$ becomes a better and better approximation to the slope of the tangent at A', and the slope of AB becomes a better and better approximation to the slope of the tangent at A. In the limit, as the ray OBB' swings toward OAA', the slope of $A'B'$ converges to the slope of the isoquant at A', and the slope of AB converges to the slope of the isoquant at A. But because the chords remain parallel as they approach their respective limits, those limits must be equal also, so the slope of the tangent at A' must equal the slope of the tangent at A, and the proof is complete.

Notice the high degree of generality in the proof. The two isoquants were picked arbitrarily and so was the ray OAA'. Therefore, one can be sure that the slopes of any two isoquants will be equal along any common ray from the origin under constant returns to scale.

Looking Out a Horizontal

In the short run, a firm must operate with fixed amounts of some factor. Just how output behaves as more of the variable factor is combined with that fixed factor has a direct and important impact on the cost of output in the short run. In the typical textbook example of production in the short run, the total product curve first rises at an increasing rate then rises at a decreasing rate as output expands—that is, the production function first displays increasing marginal returns and then diminishing marginal returns to the variable factor.

Yet this will not be the behavior of marginal returns when production exhibits constant returns to scale. It is a little-emphasized, and perhaps not widely appreciated property of constant returns that, as long as isoquants are strictly convex, both the marginal product of labor and the marginal product of capital are everywhere diminishing—that is, there is no region in which marginal returns to either factor are either increasing or constant.

For what follows, recall that the marginal product of a factor (*MPL* or *MPK*) is the increment in output produced by a one-unit increase in the amount of that factor used, holding the amount of the other factor constant. There are diminishing marginal returns to a factor whenever successive one-unit increments in that factor produce smaller and smaller corresponding increments in output. Of course, this is equivalent to saying that to produce equal increments in output requires successively larger and larger increments of the factor. It is this latter interpretation that is most useful in our proof.

Proposition 4: Marginal returns to both factors (MPL and MPK) are everywhere diminishing.

Proof: The arguments required to establish these claims with respect to the separate factors labor and capital are completely identical, so I only give the arguments for labor here, leaving the reader to supply them for the other factor, capital.

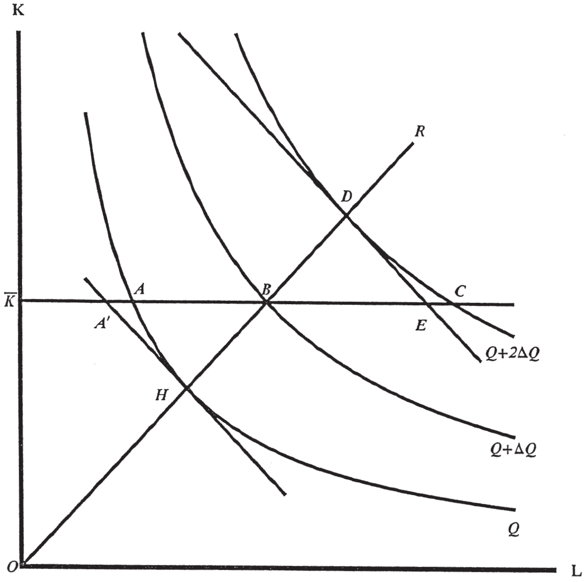
First, in Figure 4, choose an arbitrary level of output, Q, and an arbitrary increment, ΔQ. Let capital be fixed at $\bar{K}$. To prove this proposition, we must show that under constant returns to scale, the horizontal distance between these isoquants giving equal increments in output becomes larger moving out that horizontal. In Figure 4, we need to show that $AB < BC$.

First construct the ray *OHBDR* and lines tangent to the isoquants at H and at D. Now look at the triangles $A'BH$ and EBD. Clearly, angle $A'BH$ equals angle EBD, as these are opposite angles formed by intersecting straight lines. By Proposition 2, $BH = BD$ because these are distances along a common ray between isoquants giving equal increments in output. By Proposition 3, the slope of the isoquant at D must be equal to the slope of the isoquant at H, so side $A'H$ is parallel to side ED. Therefore, as *OHBDR* cuts those two parallel lines, we must have angle BHA' equal to angle BDE.

I have now established that two corresponding angles and the sides including them are equal in triangle $A'BH$ and triangle EBD. By angle-side-angle, these two triangles must therefore be congruent. From this, one concludes that $A'B = BE$, because these are corresponding sides of congruent triangles.

Now notice that with isoquants convex away from the origin, we must also have $AB < A'B$ and $BE < BC$. Putting this all together, one gets

FIGURE 4
Diminishing Returns to a Fixed Factor



$$AB < A'B = BE < BC,$$

so $AB < BC$, and the proof is complete.

Notice once again the very general nature of this result: because our choices of output level, increment in output, and level of the fixed factor all were arbitrary, we can be confident that these results apply in every region of the technology.

PRODUCTION AND COSTS

Properties of the production function have their greatest influence on firm behavior through the impact they have on costs, both short run and long run. The very special properties of constant returns production studied in the previous section have stark implications for firm costs.

To begin with the short run, recall that when factor prices are fixed, wherever the production function displays increasing, decreasing, or constant marginal returns to the variable factor, the firm will experience, respectively, decreasing, constant and increasing short-run marginal costs in the corresponding regions of output. In the typical textbook illustration of this result, the short-run marginal cost curve first declines at low levels of output, then at higher levels of output, in regions of the technology that begin to display diminishing marginal returns, the marginal cost curve begins to rise.

Although U-shaped short-run marginal cost curves may be typical in textbooks, this is definitely not how they should look under constant returns to scale. Given what was established in Proposition 4, it follows directly that, instead, short-run marginal cost will be everywhere upward sloping whenever production displays constant returns to scale. Illustrated in Figure 5, this is important enough to mention in the form of a proposition.

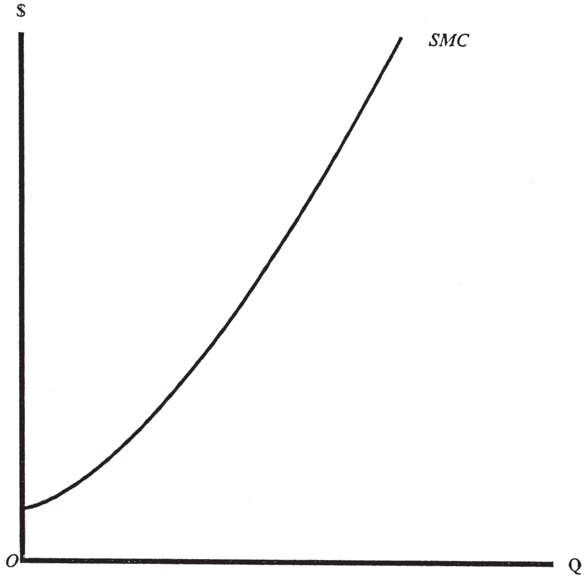
Proposition 5: Short-run marginal cost is everywhere increasing.

Proof: Suppose capital is fixed in the short run. Then output can increase only if the amount of labor the firm uses is increased. If the (fixed) wage of labor is $w > 0$, then short-run marginal cost at any level of output, defined as the rate of change of short-run total cost at that level of output, will equal the wage cost of the additional labor necessary to produce an incremental unit of output. Thus

$$\begin{aligned} SMC &= \frac{\Delta STC}{\Delta Q} \\ &= w\Delta L \,/\, \Delta Q \\ &= w\frac{1}{\Delta Q \,/\, \Delta L} \\ &= w \,/\, MPL. \end{aligned}$$

Here, the first line is the definition of *SMC*, the second line follows from our argument preceding the display, and the third line is a simple rearrange-

FIGURE 5
Increasing Short-run Marginal Cost



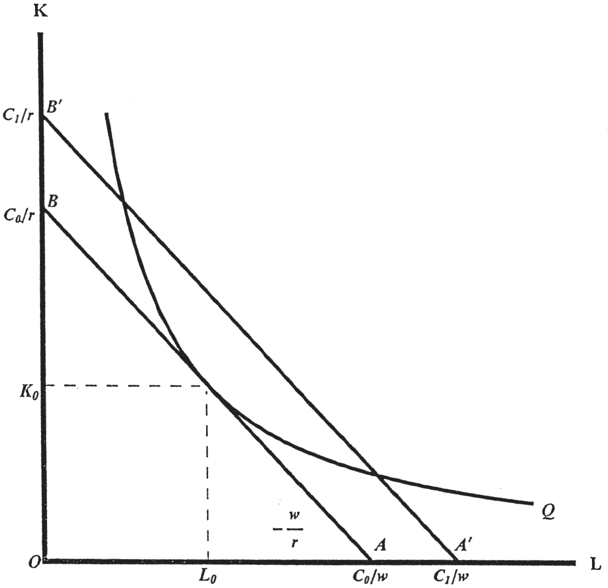
ment of the one preceding it. Note, however, that the denominator in that third line is the rate of change in output as the amount of labor is changed, holding capital constant. That, of course, is just the definition of the marginal product of labor, *MPL*, so the last line results.

In all of this, I have done nothing more than present the well-known relationship between short-run marginal cost and the marginal product of labor that holds for any production function. Now suppose that production exhibits constant returns to scale. Then by Proposition 4 the marginal product of labor always diminishes as labor and output are increased. It is easy to see in this display that *SMC* must then be always increasing as output rises.[2]

In the long run, no factor is fixed, and the profit-maximizing firm chooses amounts of both labor and capital to minimize the cost of producing any level of output. The long-run total cost of output is then simply the cost of the cost-minimizing combination of inputs capable of producing that given level of output.

In the common textbook illustration, the solution to the firm's cost-minimization problem is illustrated by the familiar tangency between the relevant isoquant and the lowest isocost curve the firm can achieve while still producing the level of output in question, as illustrated in Figure 6.[3] There, all input combinations capable of producing Q units of output lie along the Q-level isoquant. Facing fixed factor prices $w > 0$ and $r > 0$, suppose that all input combinations costing the firm C_0 dollars lie along the isocost curve $B'A'$, with constant slope $-w/r$; and

FIGURE 6
Isocost Curves and Cost Minimization



that all input combinations costing $C_1 > C_0$ dollars lie along the isocost curve $B'A'$, also with constant slope $-w/r$. Then, in Figure 6, the input combination (L_0,K_0) both produces output Q and achieves the lowest possible isocost curve, solving the firm's cost-minimization problem for output level Q. The cost of that input combination is therefore the long-run total cost of output Q.

A very close relationship exists between scale properties of the technology and the behavior of long-run average cost (*LAC*), and students find these relationships quite intuitive. Under increasing and decreasing returns, the familiar *LAC* curve will be upward sloping and downward sloping, respectively, whereas under constant returns to scale the *LAC* curve will be everywhere horizontal in the output-cost plane. Because our focus in this article is on constant returns, we will content ourselves with establishing only the relation between constant returns and long-run average costs. The cases of increasing and decreasing returns can be established by adapting (though not by simply mimicking) the proof to be given below, and we will leave that as a challenge for the interested reader. For now, we have the following important result.

Proposition 6: Long-run average cost is constant.

Proof: Suppose the production function depicted in Figure 7 has constant returns to scale. There, the unit isoquant, giving input combinations capable of producing one unit of output is identified as is an arbitrary isoquant giving all input combinations capable of producing an output level of Q units.

Suppose that at factor prices $w > 0$ and $r > 0$ the cost of one unit of output is minimized by the input combination (L_1,K_1) at point *A*. One may then express the long-run total cost (*LTC*) of one unit of output as

$$LTC(1) = wL_1 + rK_1. \tag{4}$$

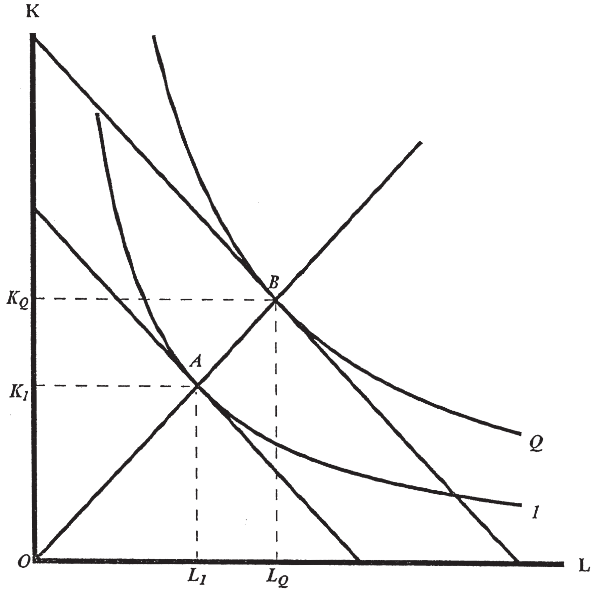
To find the input combination that minimizes the cost of *Q* units of output, it is necessary to find the point where an isocost curve parallel to the one through *A* is just tangent to the *Q*-level isoquant. According to Proposition 3, isoquants are radially parallel, so points of equal slope on any two isoquants are always to be found along the same ray from the origin.[4] Hence, the input combination (L_Q,K_Q), along the ray *OAB*, must minimize the cost of *Q*.

Now recall that, according to Proposition 1, obtaining a *Q*-fold increase in output by moving out the same ray requires exactly a *Q*-fold increase in each factor. Therefore, $L_Q = QL_1$ and $K_Q = QK_1$. After some algebra and using equation (4), one obtains

$$\begin{aligned} LTC(\mathrm{Q}) &= wL_Q + rK_Q \\ &= wQL_1 + rQK_1 \\ &= Q(wL_1 + rK_1) \\ &= QLTC(1). \end{aligned} \tag{5}$$

Equation (5) tells us that the long-run total cost of any output level will

FIGURE 7
Demonstration of Constant Costs



be proportional to the cost of the very first unit produced: 10 units will cost 10 times as much, 100 units 100 times as much, and so on.

Now long-run average cost is long-run total cost divided by output. Thus, $LAC(Q) = LTC(Q)/Q$ will be the average cost of Q units of output, and $LAC(1) = LTC(1)/1$ will be the average cost of the first unit of output. Dividing both sides of equation (5) by Q and substituting from these definitions, I have

$$LAC(Q) = LAC(1).$$

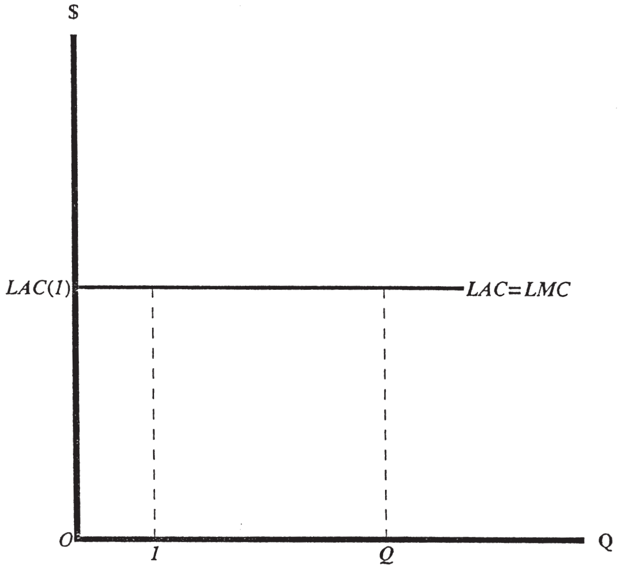
In other words, the long-run average cost of *any* output level, Q, is always the same and is equal to the cost of the first unit produced.

To visualize better what Proposition 6 established, I consider the long-run average cost curve in Figure 8. In the preceding proof, I picked an arbitrary level of output, Q, and then showed that $LAC(Q) = LAC(1)$, so that the long-run average cost curve must be horizontal, as depicted in Figure 8.

In that same figure, notice that the horizontal curve bears the label *LAC*, for long-run average cost, and the label *LMC,* for long-run marginal cost. Recall that, according to the well-known relationship between averages and marginals, when the average is constant, the marginal is also constant—and must be equal to the average.[5] It therefore follows immediately from Proposition 6 that under constant returns to scale, long-run marginal cost, *LMC*, is constant, too, and is equal to

FIGURE 8

Long-run Average Cost and Long-run Marginal Cost Are Constant and Equal



long-run average cost. To help remember the important connection between *LAC* and *LMC*, I conclude by recording, without further proof, this important corollary to the previous proposition.

Proposition 7: Long-run marginal cost is constant and equal to long-run average cost.

CONCLUSION

Simple geometry has been used to establish several crucial properties of production and cost under constant returns to scale, both in the short run and in the long run. Yet even the list of topics examined here is far from exhaustive—one can discover more about constant returns production. Soper (1967), for example, using little more geometry than was used here, presented a cogent and very accessible geometric proof of Euler's Theorem, so central to the famous product exhaustion theorem of competitive economics.

Many times, relaxing the assumption of constant returns can dramatically alter the conclusions obtained in complex arguments. Cannon (2000, 294), for one, in reviewing the theory of economic growth, warned that there, as elsewhere, "small deviations from constant returns lead to dramatically different results." Yet hunting for instances of this can be great sport when studying economic theory. I hope that, armed with a deeper understanding of its unique properties, the reader will be better equipped to judge just how sensitive a result is to the assumption of constant returns when invoked and so be better equipped to gauge its true scope and significance.

Finally, it should be noted that many of the results presented here in the context of constant returns production have direct analogies in the theory of consumer demand under homogeneous (or, indeed, homothetic) utility. By simply reinterpreting, and occasionally extending, the principles established here, the reader should be able to explore, alone, those closely related neighborhoods of economic theory.

NOTES

1. A production function having the properties I have assumed is said to be strictly increasing, strictly quasiconcave, and homogeneous of degree one in its arguments. (Note I specifically exclude here linear production functions whose isoquants will be parallel straight lines.) See, for example, Jehle and Reny (2001) for more detail on the mathematical properties of such functions.
2. Another way to see this is to focus on the second line of the equations at the bottom of p. 50. If under constant returns, *MPL* is everywhere diminishing, then its reciprocal, $\Delta L/\Delta Q$, is everywhere increasing. This means that obtaining equal increments in output, ΔQ, requires that ever-increasing increments in labor, ΔL, be applied, as noted in the proof of Proposition 4. At a constant wage, w, this makes those equal increments in output increasingly costly for the firm to produce.
3. When the firm faces fixed factor prices $w > 0$ and $r > 0$, the C-dollar isocost curve is the locus of points in the (L,K) plane satisfying $C = wL + rK$. Rearranging, this implies that along that isocost curve, $K = (C/r) - (w/r)L$. When graphed in the (L,K) plane, this will be a straight line with slope $-w/r$, vertical intercept C/r, and horizontal intercept C/w.
4. Hence, under constant returns, the output expansion path is always a ray from the origin.
5. Marginals measure change in their associated "total." If the average is rising, the margin, or that being added to the total, must be above the average, pulling the average up. If the average is falling, the margin must be below the average, pulling the average down. If the average is

unchanging, then the margin must be neither above nor below the average—it must equal the average. Mathematically, let $T(x)$ be any total measure (e.g., total cost, total product, total profit). Then let $M(x)$ be the associated marginal measure and $A(x)$ the associated average measure. By definition, $M(x) \equiv T'(x)$ and $A(x) \equiv T(x)/x$, so we can write $T(x) \equiv xA(x)$. Differentiating both sides of this identity with respect to x (remembering to use the chain rule on the rhs), substituting from the definition of $M(x)$, and rearranging, the slope of the average curve at any point, $x > 0$, can be expressed as follows:

$$A'(x) = \frac{M(x) - A(x)}{x}.$$

Note that for any $x > 0$, $A'(x) > 0$ [$A'(x) < 0$] if and only if $M(x) > A(x)$ [$M(x) > A(x)$]. Similarly, $A'(x) = 0$ (i.e., the average curve is flat) if and only if $M(x) = A(x)$ (i.e., the marginal and average curves coincide).

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