Field Theory for Four Fundamental Interactions

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V. Summary
I. Laws of Gravity, Dark Matter and Dark Energy

General relativity (Albert Einstein, 1915):

- **The principle of equivalence** says that the space-time is a 4D Riemannian manifold \((M, g_{\mu\nu})\) with \(g_{\mu\nu}\) being regarded as gravitational potentials.

- **The principle of general relativity** requires that the law of gravity be covariant under general coordinate transformations, and dictates the Einstein-Hilbert action:

\[
L_{EH}(\{g_{\mu\nu}\}) = \int_M \left( R + \frac{8\pi G}{c^4} S \right) \sqrt{-g} dx.
\]

- **The Einstein field equations** are the Euler-Lagrangian equations of \(L_{EH}\):

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{c^4} T_{\mu\nu}, \quad \nabla^\mu T_{\mu\nu} = 0
\]
**Dark Matter:** Theoretically for galactic rotating stars, one should have

\[
\frac{v_{\text{theoretical}}^2}{r} = \frac{M_{\text{observed}}}{r^2} \cdot G.
\]

In 1970s, Vera Rubin observed the discrepancy between the observed velocity and theoretical velocity

\[v_{\text{observed}} > v_{\text{theoretical}},\]

which implies that

\[\Delta M = (\text{the mass corresponding to } v_{\text{observed}}) - M_{\text{observed}} > 0,\]

which is called *dark matter*.

**Dark Energy** was introduced to explain the observations since the 1990s by indicating that the universe is expanding at an accelerating rate. The 2011 Nobel Prize in Physics was awarded to Saul Perlmutter, Brian P. Schmidt and Adam G. Riess for their observational discovery on the accelerating expansion of the universe.
New Gravitational Field Equations (Ma-Wang, 2012):

The presence of dark matter and dark energy implies that the energy-momentum tensor of visible matter $T_{\mu\nu}$ may no longer be conserved:

$$\text{div } T_{\mu\nu} \neq 0$$

Then we can show that

$$T_{\mu\nu} = \tilde{T}_{\mu\nu} - \frac{c^4}{8\pi G} \nabla_\mu \Phi_\nu,$$

$$\text{div } \tilde{T}_{\mu\nu} = 0.$$

Hence, we derive a new set of gravitational field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{8\pi G}{c^4} T_{\mu\nu} - \nabla_\mu \Phi_\nu,$$

(3)

$$\nabla^\mu \left[ \frac{8\pi G}{c^4} T_{\mu\nu} + \nabla_\mu \Phi_\nu \right] = 0.$$
• Equivalently, (3) are the variation of $L_{EH}$ under div-free constraints representing energy-momentum conservation:

\begin{align*}
(4) \quad \frac{d}{d\lambda} \bigg|_{\lambda=0} L_{EH}(g_{\mu\nu} + \lambda X_{\mu\nu}) = 0 \quad \forall X = \{X_{\mu\nu}\} \text{ with } \nabla^\mu X_{\mu\nu} = 0.
\end{align*}

This observation leads us to postulate a general principle, called principle of interaction dynamics (PID) (Ma-Wang, 2012).

• The new term $\nabla_\mu \Phi_\nu$ cannot be derived 1) from any existing $f(R)$ theories, and 2) from any scalar field theories. In other words, the term $\nabla_\mu \Phi_\nu$ does not correspond to any Lagrangian action density, and is the direct consequence of PID.
• The field equations (3) establish a natural **duality**:

\[
\text{spin}-2 \text{ graviton field } \{g_{\mu\nu}\} \longleftrightarrow \text{ spin}-1 \text{ dual vector graviton field } \{\Phi_\mu\}
\]

• A spherically symmetric central matter field with mass \(M\) and radius \(r_0\) exerts a force on an object with mass \(m\).

\[
F = -m \nabla g_{00} = -mMG \left[ \frac{1}{r^2} + \frac{k_0}{r} - k_1 r \right]
\]

where \(k_0 = 4 \times 10^{-18} km^{-1}\), \(k_1 = 10^{-57} km^{-3}\).

• Gravity can display both attractive and repulsive behaviors.

It is the nonlinear interaction between this particle field \(\Phi_\mu\) and the graviton leads to a **unified theory of dark matter and dark energy** and explains the acceleration of expanding universe.
Non Well-Posedness of the Einstein Equations: The Einstein equations contain 10 equations with 6 unknowns \( \{g_{\mu\nu}\} \), and may lead to ill-posedness in general. With the introduction of \( \Psi_\mu \), the new gravitational field equations contain exactly 10 equations, resolving this difficulty.

Example: The metric of central gravitational field takes the form

\[
ds^2 = -c^2 e^u dt^2 + e^v dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad u = u(r, t), v = v(r, t).
\]

Take \( T_{\mu\nu} = \begin{pmatrix} -g_{00}\rho & 0 \\ 0 & 0 \end{pmatrix} \), where \( \rho \) is the energy density, a constant. Then the Einstein equations take the form:

\[
R_{00} = \frac{4\pi G}{c^4} g_{00}\rho, \quad R_{11} = -\frac{4\pi G}{c^4} g_{11}\rho,
\]

\[
R_{22} = -\frac{4\pi G}{c^4} g_{22}\rho, \quad D^\mu T_{\mu\nu} = 0.
\]
It is then easy to verify that \( u \) and \( v \) are independent of \( t \), and the nonzero Ricci tensors are

\[
R_{00} = -e^{\mu - \nu} \left[ \frac{u''}{2} + \frac{u'}{r} + \frac{u'}{4}(u' - v') \right], \quad R_{11} = \frac{u''}{2} - \frac{v'}{r} + \frac{u'}{4}(u' - v'), \\
R_{22} = e^{-\nu} \left[ 1 - e^{\nu} + \frac{r}{2}(u' - v') \right], \quad R_{33} = \sin^2 \theta R_{22}.
\]

(6)

We derive from \( D^\mu T_{\mu\nu} = 0 \) that \( \Gamma^0_{10}T_{00} = \frac{1}{2}u'\rho = 0 \). Hence \( u' = 0 \). Then by (6) we have

\[
R_{00} = 0,
\]

which is a contradiction to the first equation of (5). Therefore the equations (5) have no solutions.
II. Principle of Interaction Dynamics (PID)

- **Principle of Lagrangian Dynamics (PLD)** is a general principle in physics. It says that every conservative physical system is described by a set of state functions \( u = (u_1, \cdots, u_n) \), and there is a Lagrangian action

\[
L(u) = \int \mathcal{L}(u_1, \cdots, u_n) \, dx
\]

such that \( u \) obeys the Euler-Lagrangian equation

\[
\delta L(u) = 0.
\]
The new gravitational field equations satisfy

$$\delta L_{EH}(g_{\mu\nu}) = -\nabla_\mu \Psi_\nu,$$

and violate PLD.

However the new field equations are the variation of $L_{EH}$ under div-free constraints representing energy-momentum conservation:

$$\left. \frac{d}{d\lambda} \right|_{\lambda=0} L_{EH}(g_{\mu\nu} + \lambda X_{\mu\nu}) = 0 \quad \forall X = \{X_{\mu\nu}\} \text{ with } \nabla^\mu X_{\mu\nu} = 0.$$
Principle of Interaction Dynamics (Ma-Wang, 2012). 1. For all four fundamental interactions of Nature, there exists a Lagrangian action

\[ L(\{g_{\mu\nu}\}, A, \psi) = \int_M \mathcal{L}(\{g_{\mu\nu}\}, A, \psi) \sqrt{-g} dx \]

where \( g_{\mu\nu} \) is the gravitational field, \( A \) is a set of vector fields representing the gauge fields, and \( \psi \) is the wave functions of particles.

The action obeys the principle of general relativity, the Lorentz and gauge invariances, and the principle of representation invariance (PRI).

2. The interaction fields \((g_{\mu\nu}, A, \psi)\) satisfy the variation equations of \(L\) with \( \text{div}_A \)-free constraints.
PID-induced Field Equations (Ma-Wang, 2012):

\[
\frac{\delta}{\delta g^{\mu\nu}} L \left( \{g_{ij}\}, A, \psi \right) = (\nabla_\mu + \alpha_b A_\mu^b) \Phi_\nu, \\
\frac{\delta}{\delta A_\mu^a} L \left( \{g_{ij}\}, A, \psi \right) = (\nabla_\mu + \beta_b^{(a)} A_\mu^b) \phi^a, \\
\frac{\delta}{\delta \psi} L \left( \{g_{ij}\}, A, \psi \right) = 0.
\]

- The action is gauge invariant, but the field equations breaks the gauge symmetry
- PID offers a simpler way of introducing Higgs fields from the first principle.
III. Principle of Representation Invariance

An $SU(N)$ gauge theory describes an interacting $N$ particle system:

- $N$ Dirac spinors, representing the $N$ (fermionic) particles:
  \[ \Psi = (\psi_1, \cdots, \psi_N)^T \]

- $K \overset{\text{def}}{=} N^2 - 1$ gauge fields, representing the interacting potentials between these $N$ particles:
  \[ G^a_{\mu} = (G^a_0, G^a_1, G^a_2, G^a_3) \quad \text{for } a = 1, \cdots, K. \]
The Dirac equations for the $N$ fermions are:

$$[i\gamma^\mu D_\mu - m] \Psi = 0, \quad D_\mu = \partial_\mu + igG^a_\mu \tau_a.$$ (8)

Consider the $SU(N)$ gauge transformation

$$\tilde{\Psi} = \Omega \Psi \quad \forall \Omega = e^{i\theta^a \tau_a} \in SU(N),$$ (9)

where $\{\tau_1, \cdots, \tau_K\}$ is a basis of the set of traceless Hermitian matrices with $\lambda^j_{kl}$ being the structure constants.

The covariance of Dirac eqs (8) is equivalent to $\tilde{D}_\mu \tilde{\Psi} = \Omega D_\mu \Psi$, which implies that

$$\tilde{G}^a \tau_a = G^a_\mu \Omega \tau_a \Omega^{-1} + \frac{i}{g}(\partial_\mu \Omega)\Omega^{-1}.$$ (10)
Hence the $SU(N)$ gauge transformation takes the following form:

$$(\tilde{\Psi}, \tilde{G}_\mu^a \tau_a, \tilde{m}) = \left( \Omega \Psi, \ G_\mu^a \Omega \tau_a \Omega^{-1} + \frac{i}{g} (\partial_\mu \Omega) \Omega^{-1}, \ \Omega m \Omega^{-1} \right).$$

**Principle of Gauge Invariance.** The electromagnetic, the weak, and the strong interactions obey gauge invariance:

- the *Dirac* or *Klein-Gordon dynamical equations* involved in the three interactions are gauge covariant, and
- the *actions of the interaction fields* are gauge invariant.

Physically, gauge invariance refers to the conservation of certain quantum property of the underlying interaction. In other words, such quantum property of the $N$ particles cannot be distinguished for the interaction, and consequently, the energy contribution of these $N$ particles associated with the interaction is invariant under the general $SU(N)$ phase (gauge) transformations.
Geometry of $SU(N)$ gauge theory

- spinor bundle: $M \otimes_p (\mathbb{C}^4)^N$

- gauge fields as connections: $D_\mu = \partial_\mu + igG^a_\mu \tau_a$

- field strength as curvature:

\[
\frac{i}{g} [D_\mu, D_\nu] = \frac{i}{g} (\partial_\mu + igG^a_\mu \tau_a)(\partial_\nu + igG^a_\nu \tau_a) - \frac{i}{g} (\partial_\nu + igG^a_\nu \tau_a)(\partial_\mu + igG^a_\mu \tau_a) \\
= (\partial_\mu G^a_\nu - \partial_\nu G^a_\mu) \tau_a - ig [G^a_\mu \tau_a, G^b_\nu \tau_b] \\
= (\partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g\lambda^a_{bc} G^b_\mu G^c_\nu) \tau_a \overset{\text{def}}{=} F^a_{\mu \nu} \tau_a \overset{\text{def}}{=} F_{\mu \nu}
\]

- Gauge invariance and and Lorentz invariance dictate the Yang-Mills action:

\[
L_{YM} = \int_M \left[ -\frac{1}{4} G_{ab} F^a_{\mu \nu} F^{\mu \nu b} + \overline{\Psi} (i \gamma^\mu D_\mu - m) \Psi \right] dx,
\]

\(11\)
Consider the following representation transformation of $SU(N)$:

$$\tau_a = x^b_a \tau_b, \quad X = (x^b_a) \text{ is a nondegenerate complex matrix.}$$

Then it is easy to verify that $\theta^a, A^a_\mu, G_{ab} = \frac{1}{2} \text{Tr}(\tau_a \tau_b^\dagger)$ and $\lambda^c_{ab}$ are $SU(N)$-tensors under the representation transformation (12).

**PRI (Ma-Wang, 2012):** The $SU(N)$ gauge theory must be invariant under the representation transformation (12):

- the Yang-Mills action of the gauge fields is invariant, and
- the corresponding gauge field equations are covariant.
**SU(N) gauge field equations (Ma-Wang, 2012)**

\[
G_{ab} \left[ \partial^\nu F^b_{\nu\mu} - g\lambda^b_{cd}g^{\alpha\beta} F^c_{\alpha\mu} G^d_{\beta} \right] - g\bar{\Psi} \gamma_\mu \tau_a \Psi = \left[ \partial_\mu - \frac{1}{4} k^2 x_\mu + \alpha_b G^b_{\mu} \right] \phi_a,
\]

\[
(i\gamma^\mu D_\mu - m) \Psi = 0
\]

- **Spontaneous gauge symmetry breaking:** The terms \((\partial_\mu + \alpha_b G^b_{\mu}) \phi_a\) on the right-hand side of the gauge field equations break the gauge symmetry, and are induced by the principle of interaction dynamics (PID), first postulated by Ma-Wang (2012).

- PID takes the variation of the Lagrangian action under energy-momentum conservation constraint.

- The term \(\alpha_b G^b_{\mu}\) is the (total) interaction potential, \(S_0\) is the \(SU(N)\) charge potential, and \(F = -g\nabla G_0\) is the force.
By PRI, any linear combination of gauge potentials from two different gauge groups are prohibited by PRI.

For example, the term $\alpha A_\mu + \beta W^3_\mu$ in the electroweak theory violates PRI:

- The physical quantities $W^a_\mu, W^a_{\mu\nu}, \lambda^a_{bc}$ and $G^w_{ab}$ are $SU(2)$-tensors under the $SU(2)$ representation transformation.

- $W^3_\mu$ is simply one particular component of the $SU(2)$ tensor $(W^1_\mu, W^2_\mu, W^3_\mu)$.

- $\alpha A_\mu + \beta W^3_\mu$ will have no meaning if we perform a transformation.

The combination $\alpha A_\mu + \beta W^3_\mu$ in the classical electroweak theory and in the standard model is due to the particular way of coupling the Higgs fields and the gauge fields. Also, the same difficulty appears in the Einstein route of unification by embedding $U(1) \times SU(2) \times SU(3)$ into a larger Lie group.
IV. Unified Field Theory

The unified field model is derived based on the following principles:

• principles of general relativity and Lorentz invariance Einstein (1905, 1915)


• spontaneous symmetry breaking by Y. Nambu 1960, Y. Nambu & G. Jona-Lasinio 1961

• principle of interaction dynamics (PID), Ma-Wang (2012)

• principle of representation invariance (PRI), Ma-Wang (2012)
By PRI, the Lagrangian action coupling the four fundamental interactions is naturally given by

\[
L = \int_M \left[ \mathcal{L}_{EH} + \mathcal{L}_{EM} + \mathcal{L}_W + \mathcal{L}_S + \mathcal{L}_D \right] \sqrt{-g} dx,
\]

\[
\mathcal{L}_{EH} = R + \frac{8\pi G}{c^4} S,
\]

\[
\mathcal{L}_{EM} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu},
\]

\[
\mathcal{L}_W = -\frac{1}{4} G_{ab}^{\nu \mu} W_{\mu \nu}^{a} W^{\mu \nu b},
\]

\[
\mathcal{L}_S = -\frac{1}{4} G_{kl}^{s} S_{\mu \nu}^{k} S^{\mu \nu l},
\]

\[
\mathcal{L}_D = \bar{\Psi} (i\gamma^\mu D_\mu - m) \Psi.
\]
Unified field model (Ma-Wang, 2012):

(13) \[ R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R + \frac{8 \pi G}{c^4} T_{\mu \nu} = \left[ \nabla_\mu + \frac{e \alpha_E}{\hbar c} A_\mu + \frac{g_w \alpha^w_a}{\hbar c} W^a_\mu + \frac{g_s \alpha^s_k}{\hbar c} S^k_\mu \right] \Phi_\nu, \]

(14) \[ \partial^\nu F_{\nu \mu} - e \bar{\psi} \gamma_\mu \psi = \left[ \nabla_\mu + \frac{e \alpha_E}{\hbar c} A_\mu + \frac{g_w \alpha^w_a}{\hbar c} W^a_\mu + \frac{g_s \alpha^s_k}{\hbar c} S^k_\mu \right] \phi^E, \]

(15) \[ G^w_{ab} \left[ \partial^\nu W^b_\nu - \frac{g_w}{\hbar c} \lambda^b_{\alpha \beta} W^c_\alpha W^d_\beta \right] - g_w \bar{L} \gamma_\mu \sigma_a L \]
\[ = \left[ \nabla_\mu - \frac{1}{4} k^2 \alpha^w W^a_\mu + \frac{g_w \alpha^w_a}{\hbar c} W^a_\mu + \frac{g_s \alpha^s_k}{\hbar c} S^k_\mu \right] \phi^w_a, \]

(16) \[ G^s_{kj} \left[ \partial^\nu S^j_\nu - \frac{g_s}{\hbar c} \Lambda^j_{\alpha \beta} S^c_\alpha S^d_\beta \right] - g_s \bar{q} \gamma_\mu \tau_k q \]
\[ = \left[ \nabla_\mu - \frac{1}{4} k^2 \alpha^s S^a_\mu + \frac{g_w \alpha^w_a}{\hbar c} W^a_\mu + \frac{g_s \alpha^s_j}{\hbar c} S^j_\mu \right] \phi^s_k, \]

(17) \[ \left[ i \gamma^\mu D_\mu - \frac{mc}{\hbar} \right] \Psi = 0. \]
Conclusions and Predictions of the Unified Field Model

1. Duality: The unified field model induces a natural duality:

\[
\begin{align*}
\{g_{\mu\nu}\} \quad & \text{(massless graviton)} \quad \longleftrightarrow \quad \Phi_\mu, \\
A_\mu \quad & \text{(photon)} \quad \longleftrightarrow \quad \phi^E, \\
W^a_\mu \quad & \text{(massive bosons } W^\pm \text{ & } Z) \quad \longleftrightarrow \quad \phi^w_a \quad \text{for } a = 1, 2, 3, \\
S^k_\mu \quad & \text{(massless gluons)} \quad \longleftrightarrow \quad \phi^s_k \quad \text{for } k = 1, \cdots, 8.
\end{align*}
\]

2. Decoupling and Unification: An important characteristics is that the unified model can be easily decoupled. Namely, Both PID and PRI can be applied directly to individual interactions.

For gravity alone, we have derived modified Einstein equations, leading to a unified theory for dark matter and dark energy.
3. **Spontaneous symmetry breaking and mass generation mechanism:** We obtained a much simpler mechanism for mass generation and energy creation, completely different from the classical Higgs mechanism. This new mechanism offers new insights on the origin of mass.

4. **Weak and Strong interaction charges and potentials:** The two $SU(2)$ and $SU(3)$ constant vectors $\{\alpha^w_a\}$ and $\{\alpha^s_k\}$, containing 11 parameters, represent the portions distributed to the gauge potentials by the weak charge $g_w$ and strong charge $g_s$. We define, e.g., the total potential $S_\mu$:

$$S_\mu = \alpha^s_k S^k_\mu = \{S_0, \quad S_1, S_2, S_3\}$$

$$= \{\text{strong charge potential, strong rotational potential}\}.$$
5. Strong and weak interaction potentials: For the first time, we derive Layered strong interactions potentials (Ma-Wang, 2012 & 13):

\[
\Phi_s = N g_s(\rho) \left[ \frac{1}{r} - \frac{A}{\rho} (1 + kr)e^{-kr} \right], \quad g_s(\rho) = \left( \frac{\rho_w}{\rho} \right)^3 g_s,
\]

where \( \rho_w \) is the radius of the elementary particle (i.e. the \( w^* \) weakton), \( \rho \) is the particle radius, \( k > 0 \) is a constant with \( k^{-1} \) being the strong interaction attraction radius of this particle, and \( A \) is the strong interaction constant, which depends on the type of particles.

These potentials match very well with experimental data.

Again, strong interaction demonstrates both attraction and repelling behaviors.
Quark confinement: With these potentials, the binding energy of quarks can be estimated as

\[ E_q \sim \left( \frac{\rho_n}{\rho_q} \right)^7 E_n \sim 10^{20} E_n \sim 10^{18} \text{GeV}, \]

where \( E_n \sim 10^{-2} \text{GeV} \) is the binding energy of nucleons.

This clearly explains the quark confinement.

Short-range nature of strong interaction:

With the strong interaction potentials, at the atom/molecule scales, the ratio between strong force and electromagnetic attraction force is

\[ \frac{F_a}{F_e} \sim \begin{cases} 10^{-50} & \text{at the atomic level,} \\ 10^{-56} & \text{at the molecular level.} \end{cases} \]

This demonstrates the short-range nature of strong interaction.
Weak Interaction potential:

\[ W = Ng_w \left( \frac{\rho_w}{\rho} \right)^3 e^{-r/r_0} \left[ \frac{1}{r} - \frac{B}{\rho} \left( 1 + \frac{2r}{r_0} \right) e^{-r/r_0} \right] \]

where \( r_0 = 10^{-16} \), \( N \) is the number of weak charges, \( \rho \) is the radius of the particle, \( \rho_w \) is the radius of the weaktons, and \( B \) is a constant.
IV. Summary

The Unified Field Theory based on PID and PRI gives rise to many physical conclusions consistent with experiments and observations, including e.g.

- the unified theory for dark matter and dark energy,
- the strong and weak interaction potential formulas,
- quark confinement,
- asymptotic freedom,
- nucleon potential,
- first principle based spontaneous symmetry breaking mechanism,
• short-range nature of strong and weak interactions.

These results are all derived based solely on the following five fundamental principles:

• principle of general relativity (Albert Einstein)

• Lorentz invariance (Albert Einstein)

• principle of gauge invariance (Herman Weyl, Chen-Ning Yang & Robert Mills)

• PID (Ma-Wang)

• PRI (Ma-Wang)