Dynamic Transition Theory for Thermohaline Circulation

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Supported in part by NSF and ONR
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I. Motivations and Objectives

The thorough understanding of climate low frequency variability (LFV) is a challenging problem with important practical implications for geophysical efforts to quantify predictability, analyze error growth in dynamical models, and develop efficient forecast methods.

One important source of such LFV is the thermohaline circulation (THC). There have been extensive studies in over the years (Stommel 61; Rooth 82; Welander 86; Salmon 86; Colin de Verdi‘ere 88; Cessi and Young 92; Quon and Ghil 92; Thual and McWilliams 92; Dijkstra and Molemaker 97; Dijkstra and Neelin 99; Dijkstra 00; .....)

Our main objective is to study the dynamic transitions associated with THC.
The main technical approach is the **dynamic transition theory**, that we developed to identify the transition states and to classify them both dynamically and physically.

This theory has led to a number of physical predictions:

- **Equilibrium phase transitions**: Gas-liquid transition (the nature and theory of the Andrews critical point), ferromagnetism (prediction of a new principle of fluctuations), binary systems (existence of second order transitions), superconductivity, and superfluidity (prediction of a new superfluid phase)

- **Classical Fluid Dynamics**: Bénard convection (richness of the transients), Taylor problem, and Taylor-Couette-Poiseuille flows (mechanism of the formation of the Taylor vortices)

- **Geophysical Fluid Dynamics and Climate Dynamics**: thermohaline circulation, ENSO (a new oscillation theory of ENSO), ....
Our philosophy of the theory is to search for the complete set of transition states, represented by a local attractor.

Examples:

\[
\frac{dx_1}{dt} = \lambda x_1 - x_1^3,
\]
\[
\frac{dx_2}{dt} = \lambda x_2 - x_2^3.
\]

The system undergoes a dynamic transition as \( \lambda = 0 \):
II. Dynamic Transitions for the Boussinesq Model

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \Pr (\Delta u - \nabla p) + \Pr \left[ RT - \text{sign}(S_0 - S_1) \tilde{R} S \right] \vec{k} - (u \cdot \nabla) u \\
\frac{\partial T}{\partial t} &= \Delta T + u_3 - (u \cdot \nabla) T, \\
\frac{\partial S}{\partial t} &= \text{Le} \Delta S + \text{sign}(S_0 - S_1) u_3 - (u \cdot \nabla) S \\
\text{div } u &= 0,
\end{align*}
\]

(1)

with Free Slip Boundary Conditions in \( \Omega = (0, L_1) \times (0, L_2) \times (0, 1) \)

\[
\begin{align*}
R &= \frac{\alpha_T g (T_0 - T_1) h^3}{\kappa_T \nu} \quad \text{the thermal Rayleigh number,} \\
\tilde{R} &= \frac{\alpha_S g (S_0 - S_1) h^3}{\kappa_T \nu} \quad \text{the saline Rayleigh number,} \\
\Pr &= \frac{\nu}{\kappa_T}, \quad \text{Le} = \frac{\kappa_S}{\kappa_T} \quad \text{the Prandtl and the Lewis numbers.}
\end{align*}
\]
Let

\[ K = \text{sign}(1 - \text{Le}) \left[ \frac{\text{Le}^2}{1 - \text{Le}} \left( 1 + \frac{1}{\text{Pr}} \right) \sigma_c - \tilde{R} \right], \]

\[ \sigma_c = \min_{(j,k) \in \mathbb{Z}^2, j,k \geq 0, j^2 + k^2 \neq 0, l \geq 1} \frac{\pi^4(j^2L_1^{-2} + k^2L_2^{-2} + 1)^3}{j^2L_1^{-2} + k^2L_2^{-2}} = \frac{\pi^4(j_1^2L_1^{-2} + k_1^2L_2^{-2} + 1)^3}{j_1^2L_1^{-2} + k_1^2L_2^{-2}}, \]

for some integer pair \((j_1, k_1)\) such that \(j_1 \geq 0, k_1 \geq 0, j_1^2 + k_1^2 \neq 0\).
Case $K > 0$: The first dynamic transition of the system occurs as the R-Rayleigh number $\sigma = R - Le^{-1} \tilde{R}$ crosses the critical number $\sigma_c$, leading to multiple equilibria:

- If $b_1 = \sigma_c - \frac{1-Le^2}{Le^3} \tilde{R} > 0$, the transition is of continuous type as shown.

- If $b_1 < 0$, then the transition is of jump type, leading to the existence of metastable stables, saddle-node bifurcations and the hysteresis associated with it.
Case \( K < 0 \): the first transition of the system occurs as the C-Rayleigh number

\[
\eta = \hat{R} - \frac{\Pr + \Le}{\Pr + 1} \tilde{R}
\]

crosses its first critical value

\[
\eta_c = \frac{(\Pr + \Le)(1 + \Le)}{\Pr} \sigma_c,
\]

leading to spatiotemporal oscillations (periodic solutions).

The transition can be either continuous or jump, dictated by the sign of a nondimensional parameter \( b_2 \).
III. Scaling Law

For the classical Bénard convection with free-slip BC, the critical temperature gradient is given by

\[
\Delta T_c \simeq \frac{27 \kappa T \nu \pi^4}{4 g \alpha T h^3} \rightarrow \begin{cases} 
\text{large if } h \to 0, \\
\text{small if } h \to \infty. 
\end{cases}
\]

The resolution of this discrepancy is carried out by adding to the momentum eqs turbulent friction terms as \( F = (\sigma_0 u_1, \sigma_0 u_2, \sigma_1 u_3) \).

Based on our analysis to ensure the independence of \( \Delta T_c \) on the vertical scale \( h \), we propose the following scaling law:

\[
\begin{align*}
\sigma_0 &= C_0 h^2, \\
\sigma_1 &= C_1 h^2 \\
\delta_0 &= C_0 h^4/\nu, \\
\delta_1 &= C_1 h^4/\nu
\end{align*}
\]

with \( C_0 \) and \( C_1 \) independent of \( h \) (nondim form with vert length scaled to 1).
IV. Dynamic Transitions for An Idealized THC Model

\[ h = 4 \times 10^3 \text{m}, \quad L_c = 10^4 \text{m}, \quad \alpha_T = 2.1 \times 10^{-4} \text{ } ^\circ\text{C}^{-1} \]

\[ \text{Pr} = 8, \quad \text{Le} = 10^{-2}, \quad \alpha_S \approx 0.92 \times 10^{-3} (\text{psu})^{-1} \]

\[ \nu = 1.1 \times 10^{-6} \text{ } \text{m}^2\text{s}^{-1}, \quad \kappa_T = 1.4 \times 10^{-7} \text{m}^2\text{s}^{-1} \]

\[
R = \frac{g\alpha_T(T_0 - T_1)}{\kappa_T \nu} h^3 = 0.86 \times 10^{21} (T_0 - T_1)[^\circ\text{C}^{-1}],
\]

\[
\tilde{R} = \frac{g\alpha_S(S_0 - S_1)}{\kappa_T \nu} h^3 = 3.75 \times 10^{21} (S_0 - S_1)(\text{psu})^{-1}.
\]

To fit the length scale of the THC, we need to consider the Boussinesq Equation with added friction term in its nondim form with vertical length scaled to 1:

**Model:** \( \text{BE} + (\delta_0 u_1, \delta_0 u_2, \delta_1 u_3) \) with \( \delta_0 = 1.17 \times 10^8 \), \( \delta_1 = 2.33 \times 10^{23} \)
Results

1. Deduced Critical Parameters:

\[ \alpha_c^2 = \pi^2 \left[ \frac{\delta_0}{\delta_1} \right]^{1/2} = 2.24 \times 10^{-7} \quad \text{critical wave number} \]

\[ L_c = \frac{\pi}{\alpha_c} = \left[ \frac{\delta_1}{\delta_0} \right]^{1/4} = 0.67 \times 10^4 \quad \text{critical horizontal length scale} \]

\[ \sigma_c = (\pi^2 + \alpha_c^2)\delta_1 + \frac{\pi^4 \delta_0}{\alpha_c^2} = 2.33 \times 10^{24} \quad \text{critical R-Rayleigh number} \]

\[ \eta_c = (1 + L_e)\sigma_c \quad \text{critical C-Rayleigh number} \]

\[ \sigma = R - L_e^{-1} \tilde{R}, \quad \eta = R - \tilde{R} \quad \text{R- and C-Rayleigh numbers} \]
2. **Case $\tilde{R} < 2.35 \times 10^{20}$**: The system undergoes a dynamic transition at $\sigma_c$ to a local attractor consisting of multiple equilibria and their unstable manifolds:

- If $\tilde{R} < 2.33 \times 10^{18}$, the transition is continuous. In fact, the problem bifurcates to two stable steady state solutions $\psi_1^{\sigma}$ and $\psi_2^{\sigma}$ for $\sigma > \sigma_c$ with basin of attractions of $U_1$ and $U_2$.

In addition, the initial value $\tilde{\psi} \in U_i$, then there is a time $t_0$ such that as $t > t_0$, the flow structure of the solution $\psi(t, \psi_0)$ is topologically equivalent to:
\[ v^\pm = \left( \pm C \beta^{1/2}(\sigma) L_1 \sin \frac{\pi x_1}{L_1} \cos \pi x_3, 0, \pm C \beta^{1/2}(\sigma) \cos \frac{\pi x_1}{L_1} \sin \pi x_3 \right) \]

\[ T^\pm = T_0 + (T_1 - T_0)x_3 \mp \frac{C \beta^{1/2}(\sigma)}{\alpha_c^2 + \pi^2} \cos \frac{\pi x_1}{L_1} \sin \pi x_3, \]

\[ S^\pm = S_0 + (S_1 - S_0)x_3 \mp \frac{\text{sign}(S_0 - S_1) C \beta^{1/2}}{\text{Le} (\alpha_c^2 + \pi^2)} \cos \frac{\pi x_1}{L_1} \sin \pi x_3, \]

\[ \beta(\sigma) = k(\sigma - \sigma_c) + o(|\sigma - \sigma_c|) \]

\[ v_{\text{max}} = C L_1 k h^{-1} \beta^{1/2} = 0.64 \times 10^{-7} \beta^{1/2}(\sigma) \text{m/s}, \quad \frac{v_3}{v_1} = 1.56 \times 10^{-4} \]

- If \( 2.33 \times 10^{18} < \tilde{R} < 2.35 \times 10^{20} \), the transition is jump, and there are two saddle-node bifurcations from \((\psi_1^*, \sigma^*)\) and \((\psi_2^*, \sigma^*)\) with \(\sigma^* < \sigma_c\).
3. **Case \( \tilde{R} > 2.35 \times 10^{20} \):** The system undergoes a dynamic transition at \( \eta_c \) to an attractor consisting of spatiotemporal oscillations (Hopf bifurcation).

- For \( 2.35 \times 10^{20} < \tilde{R} < 2.35 \times 10^{24} \), the transition is continuous, leading to a stable periodic solution. The period is about \( 1.1 \times 10^6 \) s. NOT realistic.

- For \( \tilde{R} > 2/35 \times 10^{24} \), the transition is jump.
V. Conclusions

- A nondimensional parameter $K$ is introduced to distinguish the multiple steady state and oscillatory spatiotemporal patterns, which play an important role in understanding the mechanism of THC in different oceanic basins.

- For both the multiple equilibria and periodic solutions transitions, both Type-I (continuous) and Type-II (jump) transitions can occur, depending respectively on the signs of two computable nondimensional parameters $b_1$ and $b_2$.

- A convection scale law is introduced, providing a method to introduce proper friction terms in the model in order to derive the correct circulation length scale.

- The analysis of the idealized model with the proper friction terms shows that the THC appears to be associated with the continuous transitions to stable multiple equilibria.
VI. References


