Time dependent Ginzburg-Landau equations of superconductivity

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Received 9 January 1995; accepted 14 April 1995
Communicated by R. Temam

Abstract

We study in this article the existence, uniqueness and long time behavior of the solutions of a nonstationary Ginzburg-Landau superconductivity model. We first prove the existence and uniqueness of solutions with $H^1$ initial data, which are crucial for the study of the global attractor. We also obtain, for the first time, the existence of global weak solutions of the model with $L^2$ initial data.

It is then proved that the Ginzburg-Landau system admits a global attractor, which represents exactly all the long time dynamics of the system. The global attractor obtained consists of exactly the set of steady state solutions and its unstable manifold. Its Hausdorff and fractal dimensions are estimated in terms of the physically relevant Ginzburg-Landau parameter, diffusion parameter and applied magnetic field.

We construct explicitly absorbing sets for some abstract semigroups having a Lyapunov functional and consequently prove the existence of global attractors. This abstract result is applied to the Ginzburg-Landau system, for which the existence of global attractor does not seem to be the direct consequence of some a priori estimates of solutions.

Keywords: Ginzburg-Landau equations; Superconductivity; Magnetic fields; Solutions; Long time behavior; Global attractors; Hausdorff and fractal dimensions

1. Introduction

In this article, we establish some general existence, uniqueness and long time asymptotic analysis results for a nonstationary Ginzburg-Landau system linked with the description of a superconductivity model. We establish the existence of strong solutions by abolishing the assumption that the initial order parameter is bounded in $L^\infty$ thus generalizing the results obtained in [3], [6] and [22]. Thanks to this existence theorem, the Ginzburg-Landau system defines a semigroup on a Hilbert space instead of a Banach space. This is crucial for the study

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of the global attractor later. In the past, when one studied this system, one always assumed that the initial condition of the order parameter satisfied

$$|\psi_0|_{L^\infty} \leq 1.$$  

This assumption prevents one, for instance, from using the known theory of Lyapunov exponents (in which the Hilbert space structure is crucial) to estimate the dimension of the global attractor of the system.

It is also established that the Ginzburg-Landau system admits a global weak solution when the space dimension $n = 2$ and when the initial data for both the order parameter and magnetic potential are in $L^2$, thus completing the problem of existence in this case. When $n = 3$, the problem of existence for $L^2$ initial data remains open. We prove the existence theorems by the Galerkin method. The advantages of this method are three-fold: First it is simpler than the Leray-Schauder method. Second, this method enables us to obtain the existence of the global weak solutions ($n = 2$). Third, it provides a way of discretizing the Ginzburg-Landau equations.

We also prove that the Ginzburg-Landau system admits a global attractor, which represents exactly all the long time dynamics of the system. It is proved in [18] and [22] that each trajectory of the system converges to the set of steady state solutions. The global attractor obtained in this article consists of exactly the set of steady state solutions and its unstable manifold. The unstable manifold provides complete information of the long time dynamics; it contains, for instance, not only the steady state solutions, but also the heteroclinic orbits, joining the steady state solutions.

For most classical infinite dimensional dynamical systems, the global attractors are obtained by proving the existence of absorbing sets, which are the direct consequence of some a priori energy estimates. For the Ginzburg-Landau system, however, we are unable to obtain these a priori energy estimates. To overcome this difficulty, we construct explicitly the absorbing sets of the system, utilizing the Ginzburg-Landau energy functional. This technique can also be applied to many other systems which admit a Lyapunov functional. We also obtain in this article some upper bounds of the Hausdorff and fractal dimensions of the global attractor in terms of the physically relevant Ginzburg-Landau parameter, diffusion parameter and applied magnetic field. Some explicit bounds depending on the physically meaningful constants of the set of steady states, as well as the global attractor, are also given.

The paper consists of the following parts. In Section 2, we discuss the mathematical formulation, scaling and gauge invariance. We fix the parameter system under which we are going to estimate the dimension of the attractors and the bounds of the steady state solutions. In Section 3, the focus is on the variational formulation, function spaces and the actual proofs of the theorems. The main method is to combine the Galerkin method, a priori estimates, Lyapunov functional and various inequalities to obtain the existence and uniqueness of solutions. In Section 4, we construct explicitly an absorbing set for some abstract semigroups admitting a Lyapunov functional and consequently, prove the existence of a global attractor. In Section 5, we establish the existence of global attractors for the Ginzburg-Landau system. At the same time, we give a more substantiated proof of the fact that the long time dynamics of the Ginzburg-Landau system requires that the time dependent terms and the terms involving the electric potential, i.e. $A_t + \text{grad} \phi$ and $\psi_t + i\psi \phi$, go to zero together. Finally, the Hausdorff and fractal dimensions of the attractors are estimated in Section 6.
2. The Ginzburg-Landau superconductivity model

2.1. The Ginzburg-Landau equations

In this article, we consider the existence, uniqueness and long time asymptotic behavior of the solutions of an evolutionary Ginzburg-Landau model in a bounded domain $\Omega$ in $\mathbb{R}^n$ with $n = 2$ or 3.

To start with, let $\Omega$ be a bounded, smooth and simply connected region in $\mathbb{R}^n$ with coordinate system $(x_1, \cdots, x_n)$ with $n = 2$ or 3. The following three quantities are involved in the mathematical formulation: a complex valued function $\psi: \Omega \rightarrow \mathbb{C}$, for the order parameter, a vector valued function $A: \Omega \rightarrow \mathbb{R}^n$, for the magnetic potential and a scalar valued function $\phi: \Omega \rightarrow \mathbb{R}$, for the electric potential.

The time-dependent Ginzburg-Landau equations can now be written as follows:

\[
\frac{\hbar^2}{2m_s D} \left( \frac{\partial}{\partial t} + \frac{i e_s}{\hbar} \phi \right) \psi - |a| \psi + b |\psi|^2 \psi + \frac{1}{2m_s} \left( - \frac{\hbar}{i} \text{grad} + \frac{e_s}{c} A \right)^2 \psi = 0, \tag{2.1}
\]

\[
\text{curl}^2 A - \text{curl} \, H = -\frac{4\pi \sigma}{c} \left( \frac{1}{c} A_t + \text{grad} \, \phi \right) + \frac{4\pi}{c} \left( \frac{e_s \hbar}{2m_s} (\psi^* \text{grad} \, \psi - \psi \text{grad} \, \psi^*) - \frac{e_s^2}{m_s c} |\psi|^2 A \right). \tag{2.2}
\]

Here $\hbar$ is the Planck constant; $e_s$ and $m_s$ are the charge and mass of a Cooper pair; $\sigma$ is the conductivity of the normal phase; $D$ is the diffusion coefficient; $c$ is the speed of light; $H$ is the applied magnetic field, $\psi^*$ is the complex conjugate of $\psi$. The parameters $a = a(T)$ and $b = b(T)$ are coefficients satisfying the following conditions (see, among others, [1] and [10]):

\[
a = a(T) > 0 \quad \text{for} \quad T > T_c; \quad a = a(T) < 0 \quad \text{for} \quad T < T_c, \tag{2.3}
\]

\[
b = b(T) > 0. \tag{2.4}
\]

Here $T_c$ is the critical temperature where incipient superconductivity property can be observed. In the BCS theory, for instance, they are given by (see [10]):

\[
\begin{cases}
    a(T) = N(0) \frac{T - T_c}{T_c}, \\
    b(T) = 0.098 \frac{N(0)}{(k_b T_c)^2}.
\end{cases} \tag{2.5}
\]

In the above equations, the unknown functions are the order parameter $\psi$, the magnetic potential $A$ and the electric potential $\phi$. We only have equations for solving $\psi$ and $A$. In principle, we need one more equation for solving $\phi$. This could be resolved by fixing a choice of gauge. It is easy to verify that if $(\psi, \phi, A)$ is a triple of solutions of (2.1)-(2.2), then for any given smooth function $\theta$, $(\psi e^{i\theta}, \phi - (h/e_s) \theta_t, A + (hc/e_s) \text{grad} \, \theta)$ is also a triple solution of (2.1)-(2.2). The following transformation

\[
(\psi, \phi, A) \rightarrow (\psi e^{i\theta}, \phi - \frac{h}{e_s} \theta_t, A - \frac{hc}{e_s} \text{grad} \, \theta) \tag{2.5}
\]

is called gauge transformation. Following [4] and [7], by fixing a choice of $\theta$, we add the following equation and boundary condition to (2.1)-(2.2):

\[
\begin{cases}
    \text{div} \, A = 0, & \text{in} \, \Omega, \\
    A \cdot n = 0, & \text{on} \, \partial \Omega. \tag{2.6}
\end{cases}
\]

We are now in a position to propose the complete set of initial and boundary conditions, they read...
\begin{equation}
A \cdot n = 0, \quad \text{curl} \, A \times n = H \times n, \quad \frac{\partial \psi}{\partial n} = 0 \quad \text{on } \partial \Omega,
\end{equation}

\begin{equation}
(\psi, A) \big|_{r=0} = (\psi_0, A_0) \quad \text{in } \Omega,
\end{equation}

where \( n \) is the outward normal unit vector of \( \partial \Omega \) at the corresponding point, \( H, \psi_0 \) and \( A_0 \) are given functions.

Remark 2.1. The boundary conditions in (2.7) are a particular case (under fixed gauge) of the following “gauge invariant form”:

\begin{equation}
\text{curl} \, A \times n = H \times n, \quad \left( -\frac{h}{c} \text{grad} \psi + \frac{e^*_s}{c} A \psi \right) \cdot n = 0, \quad \text{on } \partial \Omega.
\end{equation}

When \( n = 2 \), the boundary condition curl \( A \times n = H \times n \) in (2.7) and (2.9) should be replaced by

\begin{equation}
\text{curl} \, A = H.
\end{equation}

2.2. Mathematical scaling

From both the mathematical and physical points of view, we introduce here the nondimensionalized form of the Ginzburg-Landau equations. To this end, we introduce some physical parameters:

\begin{equation}
\begin{aligned}
|\psi_0|^2 &= |a|/b, \\
H_c &= \left(4\pi|a|^2/b\right)^{1/2}, \\
\lambda &= \lambda(T) = \left(m_s c^2 b/4\pi e^2 |a|\right)^{1/2}, \\
\xi &= \xi(T) = h/(2m_s |a|)^{1/2}, \\
\kappa &= \lambda/\xi, \\
\eta &= 4\pi\sigma D/c^2, \\
\tau &= \lambda^2/D.
\end{aligned}
\end{equation}

Physically, \( |\psi_0|^2 \) stands for the equilibrium density, \( H_c \) for the thermodynamic critical field, \( \lambda = \lambda(T) \) for the penetration depth, \( \xi(T) \) for the coherence length and \( \tau \) for the relaxation time. The ratio of the two characteristic lengths \( \kappa = \lambda/\xi \) is called the Ginzburg-Landau parameter of the substance. When \( 0 < \kappa < 1/\sqrt{2} \), the material is of the first type, when \( \kappa > 1/\sqrt{2} \), the material is of the second type.

We now introduce the nondimensional variables (those with prime):

\begin{equation}
\begin{aligned}
x &= \lambda x', \\
t &= \tau t', \\
\psi &= \psi_0 \psi', \\
A &= \sqrt{2} H_c \lambda A', \\
\phi &= D \sqrt{2} H_c \phi', \\
H &= \sqrt{2} H_c H'.
\end{aligned}
\end{equation}

Then we have the following nondimensional time dependent Ginzburg-Landau equations (we henceforth drop the primes):
\[
\begin{align*}
\frac{1}{\kappa^2} (\psi_t + i\kappa \phi \psi) + (|\psi|^2 - 1) \psi + \left( \frac{i}{\kappa} \text{grad} + A \right)^2 \psi &= 0, \\
\eta (A_t + \text{grad} \phi) + \frac{i}{2\kappa} (\psi^* \text{grad} \psi - \psi \text{grad} \psi^*) + |\psi|^2 A + \text{curl}^2 A - \text{curl} H &= 0,
\end{align*}
\]

In the above equations, \(\psi_t\) and \(\Delta \psi = \text{grad}^2 \psi\) have the same coefficient \(1/\kappa^2\). To make the computations and estimates of Hausdorff and fractal dimensions of the attractors in this article simpler, we rescale the unknown functions \(\psi, A, \phi\) and the given function \(H\) as follows. We set

\[
\begin{align*}
A &= \frac{1}{\kappa} A', \\
\phi &= \frac{1}{\kappa} \phi', \\
H &= \frac{1}{\kappa} H', \\
\psi &= \psi'.
\end{align*}
\]

Then the resulting equations for the new unknown functions \(\psi', A'\) and \(\phi'\) read (we drop the primes again):

\[
\begin{align*}
\psi_t + i\phi \psi + \kappa^2 (|\psi|^2 - 1) \psi + (i \text{grad} + A)^2 \psi &= 0, \\
\eta (A_t + \text{grad} \phi) + \frac{1}{2} i (\psi^* \text{grad} \psi - \psi \text{grad} \psi^*) + |\psi|^2 A + \text{curl}^2 A - \text{curl} H &= 0, \\
\text{div} A &= 0.
\end{align*}
\]

The corresponding nondimensional initial and boundary conditions then read

\[
\begin{align*}
(i \text{grad} \psi + A \phi) \cdot n &= 0, \quad \text{curl} A \times n = H \times n, \quad A \cdot n = 0 \quad \text{on } \partial \Omega \times (0, T), \\
\psi(x, 0) &= \psi_0(x), \quad A(x, 0) = A_0(x) \quad \text{on } \Omega.
\end{align*}
\]

Here \(\psi_0\) and \(A_0\) are given functions. Here we remark that we used the same notation \(\Omega\) for the nondimensionalized domain.

2.3. The Ginzburg-Landau functional and the stationary problem

The nondimensionalized Ginzburg-Landau energy is

\[
E(\psi, A) = \frac{1}{2} \int_\Omega \left| (i \text{grad} + A) \psi \right|^2 + \frac{1}{2} \kappa^2 (|\psi|^2 - 1)^2 + |\text{curl} A - H|^2 \, dx.
\]

The steady state Ginzburg-Landau equations of superconductivity under our choice of gauge are

\[
\begin{align*}
(i \text{grad} + A)^2 \psi + \kappa^2 (|\psi|^2 - 1) \psi &= 0, \quad \text{in } \Omega, \\
\text{curl}^2 A - \text{curl} H + \frac{1}{2} i (\psi^* \text{grad} \psi - \psi \text{grad} \psi^*) + |\psi|^2 A &= 0, \quad \text{in } \Omega, \\
\text{div} A &= 0, \quad \text{in } \Omega.
\end{align*}
\]

(2.17-1) and (2.17-2) are the variational equations of the above Ginzburg-Landau energy, then fixing the gauge choice, we obtain the divergence free constraint \(\text{div} A = 0\) for the magnetic potential \(A\).

The boundary conditions of steady state Ginzburg-Landau equations (2.17) are given by (2.15-1).

The existence and properties of solutions of the above steady state Ginzburg-Landau equations are given in Lemma 5.2 hereafter.
3. Existence and properties of solutions of the Ginzburg-Landau system

3.1. Mathematical setting

Hereafter, we use $W^{s,p} (\Omega)$ for the standard Sobolev spaces for real functions defined on $\Omega$ and as usual, $W^{s,2} (\Omega)$ is denoted by $H^s (\Omega)$. Sobolev spaces of complex valued functions are denoted by $W^{s,p} (\Omega)$ and $H^s (\Omega)$ with calligraphic letters. In particular, we have

$$
\mathcal{H}^1 (\Omega) = \{ \psi = \psi_1 + i \psi_2 | \psi_1, \psi_2 \in H^1 (\Omega) \}.
$$

(3.1)

Moreover, we use $W^{s,p}$ and $H^s$ with bold faced letters to denote Sobolev spaces of the vector valued functions.

It is easy to see that in the Ginzburg-Landau system the unknown function $\phi$ is the Lagrange multiplier of the magnetic field (cf. (2.14)). Therefore, we introduce the following notation $u$ for the real unknowns:

$$
u = (\psi, A).
$$

As in the study of the Navier-Stokes equations of an incompressible fluid, we set

$$
V_2 = \{ A \in C^\infty (\Omega) | A \cdot n|_{\partial \Omega} = 0, \text{div} A = 0 \}.
$$

(3.2)

Then we define

$$
\begin{aligned}
& H = H_1 \times H_2, \\
& V = V_1 \times V_2, \\
& H_1 = L^2 (\Omega), \\
& V_1 = \mathcal{H}^1 (\Omega), \\
& H_2 = \text{the closure of } V_2 \text{ for the } L^2\text{-norm}, \\
& V_2 = \text{the closure of } V_2 \text{ for the } H^1\text{-norm}.
\end{aligned}
$$

(3.3)

The $L^2$-norms and the inner products in $H_1$, $H_2$ and $H$ are denoted by $\| \cdot \|$ and $(\cdot, \cdot)$. The $H^1$-norm and the inner products of $V_1$ and $V_2$ are denoted by $\| \cdot \|$ and $(\cdot, \cdot)$ respectively. It is easy to see that

$$
\| A \| = \left( \int_H |\text{curl} A|^2 d\Omega \right)^{1/2}
$$

(3.4)

is an equivalent norm of $V_2$, the corresponding inner product is also denoted by $(\cdot, \cdot)$. We also use the notations $\| \cdot \|$ and $(\cdot, \cdot)$ for the induced norm and the inner product in $V$.

Moreover, it is classical that (see among others, [13], [19] and [24])

$$
\begin{aligned}
& H_2 = \{ A \in L^2 (\Omega) | \text{div} A = 0, A \cdot n|_{\partial \Omega} = 0 \}, \\
& V_2 = \{ A \in H^1 (\Omega) | \text{div} A = 0, A \cdot n|_{\partial \Omega} = 0 \}.
\end{aligned}
$$

(3.5)

(3.6)

Hereafter, we also use

$$
P = (P_1, P_2) : L^2 (\Omega) \times L^2 (\Omega) \rightarrow H
$$

(3.7)

to denote the orthogonal projection.

By identifying $H$ (resp. $H_j$, $j = 1, 2$) with its dual space $H'$ (resp. $H'_j$, $j = 1, 2$), we have
\( V \subset H = H' \subset V', \quad V_j \subset H_j = H_j' \subset V'_j, \)

for \( j = 1, 2. \)

We need to specify explicitly the function \( \phi \) in the weak formulation of the system. It is easy to see that the function \( \phi \) is determined uniquely up to constants as follows:

\[
\eta \text{grad} \phi + (I - P_2) \left( \frac{1}{2} i (\psi^\ast \text{grad} \psi - \psi \text{grad} \psi^\ast) + |\psi|^2 A \right) = 0, \quad \forall \psi = (\psi, A) \in V
\]

(3.8)

the operator \( I \) being the identity on \( L^2 \). Therefore, we define a nonlinear operator \( \phi = \Phi(u) \) to be the unique solution \( \phi \) of (3.8) satisfying

\[
\int_I \phi dx = 0.
\]

(3.9)

**Lemma 3.1.** The mapping \( \phi = \Phi(u) \) satisfies

\[
\begin{aligned}
|\phi|_{L^2(I, \Omega)} & \leq \frac{c}{\eta} \left\{ |\psi|_{L^1}^2 \| \psi \| + |\psi|_{L^2}^2 \| u \|^{1/2} \| u \|^{1/2} \right\}, & \text{if} \ u \in V, \\
|\phi|_{L^1} & \leq \frac{c}{\eta} \left( \| u \|^2 + \| \psi \|_{L^2}^2 \right), & \text{if} \ u \in V, \\
|\phi|_{H^1(I, \Omega)} & \leq \frac{c}{\eta} \left[ \| u \|^3 + \| u \|^3 \cdot \| u \|^{1/2} \| u \|^{1/2} \right], & \text{if} \ u \in V \cap H^2(\Omega).
\end{aligned}
\]

(3.10)

The constant \( c \) here is independent of \( \eta, \kappa \) and \( u \). Consequently, \( \Phi \) is a continuous mapping from \( V \) into \( L^2(\Omega) \) (resp. from \( V \cap H^2 \) into \( H^1 \)).

**Proof.** The first and third equations of (3.10) follow directly from the following computations:

\[
\begin{aligned}
\int_I \psi \text{grad} \psi^\ast \cdot A dx & \leq |\psi|_{L^1} \| \text{grad} \psi^\ast \|_{L^1} |\bar{A}|_{L^6} \leq c |\psi|_{L^1} \| \psi \| \| \bar{A} \|, \\
\int_I |\psi|^2 A \cdot A dx & \leq |\psi|_{L^1}^2 \| A \|_{L^1} |\bar{A}|_{L^6} \leq C |\psi|_{L^1}^2 \| u \|^{1/2} \| u \|^{1/2} \| \bar{A} \|, \\
\int_I \psi \text{grad} \psi^\ast \cdot A dx & \leq |\psi|_{L^1} \| \text{grad} \psi \|_{L^1} |\bar{A}|_{L^2} \leq C \| u \| \| \text{grad} \psi \|_{H^{1/2}} |\bar{A}| \leq C \| u \|^3 \| u \|^{1/2} |\bar{A}|, \\
\int_I |\psi|^2 A \cdot A dx & \leq |\psi|_{L^1}^2 \| A \|_{L^1} |\bar{A}|_{L^2} \leq C \| u \|^3 |\bar{A}|.
\end{aligned}
\]

On the other hand, we have

\[
\begin{aligned}
\int_I \psi \text{grad} \psi^\ast \cdot A dx & \leq c |\psi|_{L^1} \| \text{grad} \psi^\ast \|_{L^1} \leq c |\psi|^2 \| \bar{A} \|_{W^{1,2}}, \\
\int_I |\psi|^2 A \cdot A dx & \leq c |\psi|_{L^1}^2 \| A \|_{L^1} |\bar{A}|_{L^2} \leq c |\psi|_{L^1}^2 \| u \| \| \bar{A} \|_{W^{1,2}}.
\end{aligned}
\]
Hence
\[ \| (I - P_2)[\frac{1}{2}i(\psi^* \text{grad} \psi - \psi \text{grad} \psi^*) + |\psi|^2 A] \|_{W^{-1,3}} \leq c(\|u\|^2 + |\psi|^2 \|u\|). \]
This proves the second equation (3.10). Q.E.D.

We now define some operators related to different terms in the Ginzburg-Landau equations. First of all, for the dissipative terms in (2.14), we define two linear operators \( L_j : V_j \rightarrow V'_j \) \((j = 1, 2)\) by
\[ \langle L_1 \psi, \tilde{\psi} \rangle = \int_{\Omega} \text{grad} \psi \cdot \text{grad} \tilde{\psi}^* dx, \] (3.11)
\[ \langle L_2 A, \tilde{A} \rangle = \int_{\Omega} \text{curl} A \cdot \text{curl} \tilde{A} dx, \] (3.12)
for all \( u = (\psi, A) \) and \( \tilde{u} = (\tilde{\psi}, \tilde{A}) \in V \). Classically, \( L_j \) \((j = 1, 2)\) can be extended naturally as unbounded linear self-adjoint operators on \( H_j \) with domains
\[ D(L_j) = V_j \cap H^2, \quad j = 1, 2. \] (3.13)
\( L_2 \) is also positive definite with compact inverse \( L_2^{-1} : H_2 \rightarrow H \) while \( L_1 \) is semi-positive definite with \((L_1 + I)^{-1}\) compact and satisfies the following Garding inequality:
\[ \langle L_1 \psi, \psi \rangle + |\psi|^2 \geq \|\psi\|^2. \] (3.14)
For the remaining terms other than the time derivative terms, we define also two (nonlinear) operators \( R_1 \) and \( R_2 \) as follows:
\[ (R_1(u), \tilde{\psi}) = \int_{\Omega} [i\Phi(u)\psi\tilde{\psi}^* + \kappa^2(|\psi|^2 - 1)\psi\tilde{\psi}^* + i(\tilde{\psi}^* \text{grad} \psi - \psi^* \text{grad} \tilde{\psi}) \cdot A + |A|^2 \psi\tilde{\psi}^*] dx. \] (3.15)
\[ (R_2(u), \tilde{A}) = \frac{1}{\eta} \int_{\Omega} [\frac{1}{2}i(\psi^* \text{grad} \psi - \psi \text{grad} \psi^*) \cdot \tilde{A} + |\psi|^2 A \cdot \tilde{A} - H \cdot \text{curl} \tilde{A}] dx \] (3.16)
for some \( u, \tilde{u} \in V \). These two operators enjoy the following properties:

**Lemma 3.2.**
\[ |R_1(u)| \leq C(1 + \|u\|^4 + \|u\|^5/2 \|u\|^{1/2}), \] (3.17)
\[ |R_2(u)| \leq C(1 + \|u\|^3 + \|u\|^{3/2} \|u\|^{1/2}) \] (3.18)
where \( C \) depends on \( \kappa \) and \( \eta \) but not on \( u \) and \( \tilde{u} \).

**Proof.** We only have to check each term in the definition of \( R_j(u) \) given by (3.15)–(3.16):
\[
\left| \int \nabla \cdot (\phi \psi^* \nabla \psi^* dx) \right| \leq \left| \phi \right|_{L^3} \left| \psi \right|_{L^3} \left| \psi^* \right|_{L^2} \leq c \| \phi \|_{H^1} \| \psi \|^{1/2} \| \psi^* \|^{1/2} \| \psi \|_{L^2} \leq c\left( \| u \|^3 + \| u \|^{3/2} \| u \|_{H^2}^{1/2} \right) \| u \|_{L^2} \leq c\left( \| u \|^3 + \| u \|^{3/2} \| u \|_{H^2}^{1/2} \right)\| u \|_{H^2}^{1/2} \| \bar{u} \|,
\]

\[
\left| \int \nabla \cdot (\psi^* \nabla \psi^* dx) \right| \leq c\left( \| u \|^3 + \| u \|^{3/2} \| u \|_{H^2}^{1/2} \right) \| u \|^{1/2} \| \bar{u} \|.
\]

\[
\left| \int \psi^* \nabla \psi^* dx \right| \leq c\| \psi \|^3 \| \psi^* \|_{L^2} \leq c\| u \|^3 \| \bar{u} \|.
\]

\[
\left| \int \psi^* \nabla \psi^* dx \right| \leq c\| \psi \|^3 \| \psi^* \|_{L^2} \leq c\| u \|^3 \| \bar{u} \|.
\]

\[
\left| \int \psi^* \nabla \psi^* dx \right| \leq c\left( \| u \|^3 + \| u \|^{3/2} \| u \|_{H^2}^{1/2} \right) \| u \|_{L^2} \leq c\left( \| u \|^3 + \| u \|^{3/2} \| u \|_{H^2}^{1/2} \right)\| u \|_{H^2}^{1/2} \| \bar{u} \|.
\]

Other terms in (3.15) and (3.16) can be estimated in the same fashion.

We are now in a position to state the weak formulation of the Ginzburg-Landau system as follows:

\textit{Problems 3.1 (Weak formulation): For } u_0 = (\psi_0, A_0) \in H \text{ given, find a solution } u = (\psi, A) \text{ of the Ginzburg-Landau system in the following sense:}

\begin{align*}
&u \in L^\infty(0,T; H) \cap L^2(0,T; V), \text{ for all } T > 0, \\
&\frac{d}{dt} (\psi, \bar{\psi}) + (L_1 \psi, \bar{\psi}) + (R_1(u), \bar{\psi}) = 0, \quad \forall \bar{\psi} \in V_1, \\
&\frac{d}{dt} (A, \bar{A}) + \frac{1}{\eta} (L_2 A, \bar{A}) + (R_2(u), \bar{A}) = 0, \quad \forall \bar{A} \in V_2, \\
&u|_{t=0} = u_0.
\end{align*}

Alternatively, (3.20) and (3.21) can be written as

\begin{align*}
&\frac{d}{dt} (u, \bar{u}) + (Lu, \bar{u}) + (R(u), \bar{u}) = 0, \quad \forall \bar{u} \in V,
\end{align*}

where

\[
\begin{cases}
Lu = (L_1 \psi, \frac{1}{\eta} L_2 A), \\
R(u) = (R_1(u), R_2(u)).
\end{cases}
\]

\textit{Remark 3.3.} The interpretation of the above weak formulation can be made in the following fashion. First, for any smooth solution } u = (\psi, A) \text{ and } \phi \text{ of the Ginzburg-Landau system (2.14)-(2.15), } u \text{ satisfies (3.20)-(3.22). On the other hand, if } u \text{ given by (3.19) is a solution of (3.20)-(3.22), as in the study for the Navier-Stokes equations of an incompressible fluid (see among others [19] and [24]), there is a function } \phi \text{ such that } u \text{ and } \phi \text{ satisfies (2.14) at least in the distributional sense.}
3.2. Existence and uniqueness of solutions

We present in this section two main existence theorems. The first theorem, Theorem 3.4, shows the existence of global (in time) weak solutions of the Ginzburg-Landau system with arbitrary $L^2$-initial data when the space dimension is two. In the second one, Theorem 3.5, we obtain the existence and uniqueness of global (in time) strong solutions of the Ginzburg-Landau equations for arbitrary $H^1$ initial data when the space dimension is two or three.

Theorem 3.4. When the space dimension $n = 2$, there is at least one global weak solution $u$ of Problem 3.1, the weak formulation of the Ginzburg-Landau system.

Theorem 3.5. For any $u_0 \in V$, there is a unique solution for Problem 3.1, i.e., the initial-boundary value problem (3.22)–(3.23), such that, for any $T > 0$,

$$ u \in L^2(0,T; V \cap H^2) \cap C([0,T]; V), \quad \phi \in L^2(0,T; H^1(\Omega)) \cap C([0,T]; L^2(\Omega)), $$

(3.25)

where $\phi = \Phi(u)$ defined by (3.8) and (3.9) is the Lagrange multiplier. Moreover, if $u_0 \in D(L)$, the unique solution $(u, \phi)$ satisfies

1) If $u_0$ satisfies some compatibility conditions, then $u \in C([0, \infty); D(L))$.
2) For any $T > 0$,

$$ \begin{align*}
&\left\{ u \in L^2(0,T; V \cap H^3(\Omega)) \cap L^\infty(0,T; D(L)), \\
&\phi \in L^2(0,T; H^2(\Omega)) \cap L^\infty(0,T; H^1(\Omega)) \right. \\
&\text{where } |\psi_0|_{L^\infty} \leq 1, \quad (3.27)
\end{align*}$$

Remark 3.6. In the case where $u_0 \in V$ satisfying

the existence and uniqueness of solutions of the system were obtained in [3], [6] and [9]. Here, however, the assumption (3.27) is removed. Thanks to this we will be able to define a semigroup on the Hilbert space $V$ (see Section 4 hereafter), fully describing the dynamics of the Ginzburg-Landau system. It is easy to understand that this will be crucial for the existence and dimension estimates of the global attractors later in this article.

Remark 3.7. As far as we know, Theorem 3.4 is the first existence theorem of weak solutions so far.

Remark 3.8. It is easy to see that the evolutionary Ginzburg-Landau system and the 3-D Navier-Stokes equations of an incompressible fluid possess the same type of nonlinear terms, namely, $\psi \text{ grad } \psi^*$ here and $\nu \cdot \text{ grad } \nu$ for the Navier-Stokes equations. The existence of global strong solutions of the 3-D Navier-Stokes equation is still open. But for the Ginzburg-Landau system, as we will see from the proof of Theorem 3.5, it is the Lyapunov functional which provides us enough a priori estimates to show the existence of the global strong solutions.

Remark 3.9. (Open question): When the space dimension $n = 3$, the existence of a global and/or local (in time) weak solutions with $L^2$ initial data (i.e. $u_0 \in H$) is still open. Theorem 3.4, however, provides only a complete solution to this problem in the case where $n = 2$. 
3.3. Proof of Theorem 3.5

We prove this theorem by the Galerkin method. As we shall see later, the existence of the Ginzburg-Landau energy functional (cf. (2.16)) is essential in proving the global existence of solutions, but it is not available for the truncated system of equations in the Galerkin procedure. Therefore, as we mentioned at the beginning of this article, we first use the Galerkin method to obtain the existence of local (in time) solutions of the equations, then we apply the Ginzburg-Landau functional to the original Ginzburg-Landau equations to prove that the local solution can be extended naturally to a unique global solution.

We divide the proof into 4 steps.

Step 1. Approximate solutions. We first observe that the linear operator $L$ possesses an orthonormal family of eigenfunctions $w_0, w_1, \cdots, w_m, \cdots$ with corresponding eigenvalues $\lambda_0, \lambda_1, \cdots, \lambda_m, \cdots$ such that

$$\begin{cases}
0 = \lambda_0 < \lambda_1 < \cdots < \lambda_j \to \infty, \\
w_j \in D(L).
\end{cases} \tag{3.28}$$

Now, we want to find an approximate solution of Problem 3.1 as follows:

$$u_m = \sum_{j=0}^{m} g_j(t) w_j \in W_m, \ g_j(t) \in \mathbb{R}, \ j = 1, 2, \cdots, m,$$

$$W_m = \text{span}\{w_0, w_1, \cdots, w_m\}, \tag{3.29}$$

$$\frac{d}{dt}(u_m, w_j) + (Lu_m, w_j) + (R(u_m), w_j) = 0, \ j = 1, 2, \cdots, m,$$ \tag{3.30}

$$u_m(0) = P_m u_0, \tag{3.31}$$

the linear operator $P_m$ being the orthogonal projection into $W_m$.

It is easy then to see that (3.31) and (3.32) are a system of ordinary differential equations of $g_j(t)$, $j = 0, 1, \cdots, m$ and there exists a unique local solution.

Step 2. Energy estimates. We now need to prove some a priori estimates to pass to the limit $m \to \infty$ to obtain a solution of the original problem. To this end, since $w_j$ is an eigenfunction of $L$ corresponding to the eigenvalue $\lambda_j$, we multiply both sides of (3.31) by $\lambda_j g_j(t)$ and add them up for $j = 0, 1, \cdots, m$. It follows then easily that

$$\frac{1}{2} \frac{d}{dt} \left( |u_m|^2 + (Lu_m, u_m) + (R(u_m), u_m) \right) = 0. \tag{3.33}$$

We infer from (3.31) that

$$\frac{1}{2} \frac{d}{dt} |u_m|^2 + (Lu_m, u_m) + (R(u_m), u_m) = 0. \tag{3.34}$$

Therefore,

$$\frac{1}{2} \frac{d}{dt} \left[ |u_m|^2 + (Lu_m, u_m) \right] + (Lu_m, u_m) + |Lu_m|^2 \leq |R(u_m)||u_m| + |R(u_m)||Lu_m| \leq |u_m||R(u_m)| + 2|R(u_m)|^2 + \frac{1}{4} |Lu_m|^2.$$ 

This yields
\[
\frac{d}{dt} \left[ |u_m|^2 + 2(L_{u_m}, u_m) + (L_{u_m}, u_m) + \frac{3}{2}|L_{u_m}|^2 \right] \leq 4|R(u_m)|^2 + 2|u_m||R(u_m)|
\]
\[
\leq (\text{by (3.17) and (3.18)}) \leq c\left(1 + \|u_m\|^8 + \|u_m\|^5\|u_m\|_{H^2}\right)
\]
\[
\leq (\text{by } |u_m|_{H^2} \leq c(|u_m| + |L_{u_m}|)) \leq c\left(1 + \|u_m\|^8 + \|u_m\|^5|L_{u_m}|\right)
\]
\[
\leq \frac{1}{2}|L_{u_m}|^2 + c\left(1 + \|u_m\|^{10}\right) \leq \left(\text{by } \|u_m\|^2 \leq c(|u_m|^2 + (L_{u_m}, u_m))\right)
\]
\[
\leq \frac{1}{2}|L_{u_m}|^2 + c\left[1 + (|u_m|^2 + (L_{u_m}, u_m))^5\right].
\]

Namely,
\[
\frac{d}{dt} \left[ |u_m|^2 + (L_{u_m}, u_m) + (L_{u_m}, u_m) + |L_{u_m}|^2 \right] \leq c\left\{1 + \left[|u_m|^2 + (L_{u_m}, u_m)^5\right]\right\}. \tag{3.35}
\]

Set
\[
y = 1 + |u_m|^2 + (L_{u_m}, u_m), \tag{3.36}
\]
then
\[
\frac{dy}{dt} \leq cy^5.
\]

Integrating this differential inequality, we find
\[
0 < y(t) \leq (y(0)^{-4} - ct)^{-1/4}
\]
for \(0 \leq t \leq T_0 = T(\|u_0\|) = c/\|u_0\|^8\). In other words, we have
\[
\|u_m\|^2 \leq C, \quad \text{for } 0 \leq t \leq T_0 = T(\|u_0\|). \tag{3.37}
\]
The combination of (3.35) and (3.37) implies finally that
\[
u_m \in \text{ a bounded set of } L^2(0, T_0; D(L)) \cap L^\infty(0, T_0; V). \tag{3.38}
\]
Moreover, (3.31) amounts to saying that
\[
(u_m)_t + Lu_m + P_mR(u_m) = 0.
\]

Hence
\[
|(u_m)_t| \leq |Lu_m| + |P_mR(u_m)| \leq c\|u_m\|_{H^2} + c\left(1 + \|u_m\|^4 + \|u_m\|^5/2\|u_m\|_{H^2}^{1/2}\right).
\]

It follows then that, also by (3.31),
\[
|(u_m)_t|^2_{L^2(0, T_0; H)} \leq c \left(1 + \int_0^{T_0} \|u_m(t)\|^4 dt + \int_0^{T_0} \|u_m(t)\|_{H^2}^2 dt\right) \leq C,
\]
\(C\) being independent of \(m\) and \(u_m\). That is,
\[
(u_m) \in \text{ a bounded set of } L^2(0, T_0; H). \tag{3.39}
\]

**Step 3. Passage to the limit \(m \to \infty\) and existence of solutions.** By (3.38) and (3.39), there is a subsequence \(u_{m'}\) of \(u_m\) such that as \(m' \to \infty\),
\[\begin{align*}
&\{u_{m'} \to u\ \text{weakly in } L^2(0, T_0; D(L)), \\
&u_{m'} \to u\ \text{weak-star in } L^\infty(0, T_0; V), \\
&(u_{m'})_t \rightharpoonup u_t\ \text{weakly in } L^2(0, T_0; H). \\
\end{align*}\]

Thanks to the compactness of the embedding of \(D(L)\) in \(V\), the inclusion
\[\{u \in L^2(0, T_0; D(L)) | u' \in L^2(0, T_0; H)\} \subset L^2(0, T_0; V)\]
is compact. Therefore without loss of generality, we may assume
\[u_{m'} \to u\ \text{strongly in } L^2(0, T_0; V).\]

Then, it is classical to pass to the limit \(m' \to \infty\) in (3.31)-(3.32) to obtain that \(u\) is a solution of the Ginzburg-Landau system described in Lemma 3.5. The proof of uniqueness of solutions can be conducted in a similar way as in [3], [6] and [22], we omit the details.

**Step 4. Global solutions.** We only have to prove now the local strong solution obtained can be extended uniquely to a global (in time) strong solution of the Ginzburg-Landau system. More precisely, let \(T_{\text{max}}\) be the maximum time where the local solution \(u\) is obtained, namely, \(u\) satisfies (3.22)-(3.23) and
\[\begin{align*}
&\{u \in L^2_{\text{loc}}(0, T_{\text{max}}; V \cap H^2) \cap C(0, T_{\text{max}}), V), \\
u_t \in L^2_{\text{loc}}(0, T_{\text{max}}; H), \\
\end{align*}\]

we now prove that \(T_{\text{max}} = +\infty\). To achieve this, it is enough to establish some \textit{a priori} estimates for the above local strong solutions.

First of all, we consider the Ginzburg-Landau functional \(E: V \to \mathbb{R}\) as follows:
\[E(u)(t) = \int \left\{ \frac{1}{2} \left| (i \text{grad } + A) \psi \right|^2 + \frac{1}{4} \kappa^2 (|\psi|^2 - 1)^2 + \frac{1}{2} |\text{curl } A - H|^2 \right\} dx.\]

Introduce the following gauge transformation:
\[\begin{align*}
\zeta &= \psi e^{i\chi}, \\
B &= A + \text{grad } \chi, \\
\chi &= \int_0^t \phi(x, s) ds.
\end{align*}\]

Noticing that
\[\begin{align*}
\zeta_t &= (\psi_t + i \phi \psi) e^{i\chi}, \\
B_t &= A_t + \text{grad } \phi, \\
(i \text{grad } + B) \zeta &= ((i \text{grad } + A) \psi) e^{i\chi},
\end{align*}\]

we multiply the complex conjugate of Eq. (2.14-1) by \(\psi_t + i \phi \psi\) and take the real part, then multiply (2.14-2) by \(A_t + \text{grad } \phi\). Adding up the results and using gauge invariance, we obtain, noticing that \(H\) is independent of \(t\),
\[0 \geq -|\psi_t + i \phi \psi|^2 - \eta |A_t + \text{grad } \phi|^2 = \int_t \left[ \text{Re} \left( (\psi_t + i \phi \psi) \left( -i \text{grad } + A \right)^2 \bar{\psi} + \kappa^2 (|\psi|^2 - 1) \psi^* \right) \right] dt + \left( A_t + \text{grad } \phi \right) \cdot \left( (\text{curl}^2 A - \text{curl } H) + \frac{i}{2} (\psi^* \text{grad } \psi - \psi \text{grad } \psi^*) + |\psi|^2 A \right) \]
\[
\begin{align*}
\frac{d}{dt} [\text{Re}(i \nabla (\xi + B) \zeta^* (-i \nabla (\xi + B) \zeta^* )] + \text{Re}(B \cdot ( \zeta + B^*) ] dx & \\
= \frac{dE}{dt}(\xi(t), B(t)) = \frac{dE}{dt}(u(t)). \tag{3.45}
\end{align*}
\]

In the integration process, it is easy to see that the boundary terms concerning \( A \) and \( \psi \) vanish due to the boundary conditions they satisfy.

It is also easy to verify that \( E(u(t)) \rightarrow 0 \) implies the \( H^1 \) norms of \( \psi \) and \( A \) tend to \( 0 \).

From the existence of the Lyapunov functional, it is easy to verify that \( T_{\text{max}} \) can be extended to be equal to \( +\infty \). Then the remaining parts of the proof are easy; we omit the details. Q.E.D.

### 3.4. Proof of Theorem 3.4

We also proceed by the Galerkin method. As in the previous subsection, we need first to solve (3.31)-(3.32). However, since now we only have \( u_0 \in H \), we cannot expect to obtain the energy estimates by using the Ginzburg-Landau functional. We divide the proof into 4 steps.

**Step 1. Approximate solutions.** We first observe that the eigenfunctions \( w_j \) \((j = 0, 1, \cdots)\) of the linear operator \( L \) can be written as

\[
w_j = (w_j^{(1)}, w_j^{(2)}), \quad j = 0, 1, \cdots
\]

with superindex \((1)\) for the order parameter and \((2)\) for the magnetic potential. Therefore, the approximate equations (3.31) are equivalent to the following equations:

\[
\begin{align*}
\frac{d}{dt} (\psi_m, w_j^{(1)}) + (L_1 \psi_m, w_j^{(1)}) + (R_1(u_m), w_j^{(1)}) &= 0, \\
\frac{d}{dt} (A_m, w_j^{(2)}) + \frac{1}{\eta} (L_2 A_m, w_j^{(2)}) + (R_2(u_m), w_j^{(2)}) &= 0,
\end{align*}
\]

for \( j = 0, 1, \cdots, m \).

**Step 2. Energy estimates.** We infer then from (3.46) that

\[
\frac{1}{2} \frac{d}{dt} |\psi_m|^2 + \int \left[ i\Phi(u_m) |\psi_m|^2 + \kappa^2 (|\psi_m|^2 - 1) |\psi_m|^2 + |(i \nabla + A_m) \psi_m|^2 \right] dx = 0.
\]

Taking the real part of the above equation implies that

\[
\frac{d}{dt} |\psi_m|^2 + 2\kappa^2 |\psi_m|_{L^4}^4 + 2 |(i \nabla + A_m) \psi_m|_{L^2}^2 \leq 2\kappa^2 |\psi_m|^2 \leq \kappa^2 |\psi_m|_{L^4}^4 + c.
\]

Hence

\[
\frac{d}{dt} |\psi_m|^2 + \kappa^2 |\psi_m|_{L^4}^4 + 2 |(i \nabla + A_m) \psi_m|_{L^2}^2 \leq c. \tag{3.48}
\]

Integrating (3.48), we obtain that for any \( T > 0 \),

\[
|\psi_m(t)|^2 \leq c, \quad \text{for} \ t \in [0, T], \tag{3.49}
\]
\[ \int_0^T |\psi_m|^2 dt \leq c, \quad (3.50) \]

\[ \int_0^T |(\text{grad} + A_m)\psi_m|^2 dt \leq c, \quad (3.51) \]

where \( c \) is a constant independent of \( m \) and \( u_m \).

On the other hand, we deduce, from (3.47) that

\[
\frac{n}{2} \frac{d}{dt} |A_m|^2 + |\text{curl} A_m|^2 = (H, \text{curl} A_m) - \text{Re} \int_\Omega (i \text{grad} \psi_m + A_m \psi_m) \psi_m^* \cdot A_m dx
\]

\[
\leq c |\text{curl} A_m| + |(i \text{grad} \psi_m + A_m \psi_m)| |\psi_m|_{L^2} |A_m|_{L^2}
\]

\[
\leq c |\text{curl} A_m| + |(i \text{grad} \psi_m + A_m \psi_m)|^2 + c |\psi_m|_{L^2}^2 \|A_m\|_{H^{1/2}}^2
\]

\[
\leq c |\text{curl} A_m| + |(i \text{grad} \psi_m + A_m \psi_m)|^2 + c |\psi_m|_{L^2}^2 |A_m| |\text{curl} A_m|
\]

\[
\leq \frac{1}{2} |\text{curl} A_m|^2 + c(1 + |(i \text{grad} + A_m)\psi_m|^2) + c|\psi_m|_{L^2}^2 |A_m|^2.
\]

Therefore

\[ \frac{1}{2} \frac{d}{dt} |A_m|^2 + |\text{curl} A_m|^2 \leq c(1 + |(i \text{grad} + A_m)\psi_m|^2) + c|\psi_m|_{L^2}^2 |A_m|^2. \quad (3.52) \]

By the Gronwall inequality, we have for any \( T > 0 \),

\[ |A_m(t)|^2 \leq |P_{m_0}|^2 \exp\left(c \int_0^t |\psi_m(t)|_{L^2}^4 dt\right) \]

\[ + c \int_0^t (1 + |(i \text{grad} + A_m(s))\psi_m(s)|^2) \exp(-c \int_t^s |\psi_m(\tau)|_{L^2}^4 d\tau) d\tau \]

\[ \leq (\text{by (3.49)} - (3.51)) \leq c(|u_0|^2 + 1), \quad t \in [0, T] \]

where \( c \) is a constant depending on \( T \) but not on \( m \). Hence, it is easy to see that for any \( T > 0 \),

\[ |A_m(t)| \leq c, \quad \forall t \in [0, T], \quad (3.53) \]

\[ \int_0^T |\text{curl} A_m|^2 dt \leq c. \quad (3.54) \]

Since we have

\[ \int_0^T |\text{grad} \psi_m|^2 dt \leq \int_0^T |(i \text{grad} \psi_m + A_m \psi_m)|^2 dt \]

\[ + c \int_0^T \int_\Omega \left| (\psi_m^* \text{grad} \psi_m - \psi_m \text{grad} \psi_m^*) A_m \right| + |A_m|^2 |\psi_m|^2 dx dt, \quad (3.55) \]
by the Hölder inequality, we obtain
\[ \int |\nabla_m \psi_m \cdot A_m| \, dx \leq |\nabla \psi_m| |\psi_m| |A_m| |A_m| \leq c |\nabla \psi_m| \frac{3}{2} |\psi_m| \frac{1}{2} |A_m| \frac{1}{2} |A_m| \frac{1}{2} \leq \epsilon |\nabla \psi_m|^2 + c(\epsilon) |\psi_m|^2 |A_m|^2 |A_m|^2. \]

Similar computations apply also to other terms in the right hand side of (3.55) and therefore,
\[ \int_0^T |\nabla \psi_m|^2 \, dt \leq c + c \int_0^T |\psi_m|^2 |A_m|^2 \, dt \leq c \]
(3.56)
with \(c\) independent of \(m\).

**Step 3. Estimate of \((u_m)_t\).** As in the proof of Lemma 3.1, it is easy to obtain that for any \(\delta > 0\),
\[ |\phi_m| L_2^\infty \leq c |u_m| L_2^\infty (|\psi_m| |A_m| |A_m|). \]
(3.57)
Then for any \(\delta > 0\),
\[ \int i \phi_m \psi_m \tilde{\psi} \, dx \leq c |\phi_m| |\psi_m| L_2^\infty \|	ilde{\psi}\|, \]
which implies that for any \(\delta > 0\),
\[ \|i \psi_m \phi_m\| L_\infty^\prime \leq c |\phi_m| |\psi_m| L_2^{\infty} \leq c |u_m| L_2^{\infty} \left( |\psi_m| + |\psi_m|^2 \right) \]
\[ \leq c \|u_m\|^{\frac{2\delta}{2+3\delta} |u_m|^{\frac{2}{2+3\delta}} \left( |\psi_m| + |\psi_m|^2 \right). \]
(3.59)
Thanks to (3.49)–(3.50), (3.53)–(3.54) and (3.56), we infer from (3.58) that
\[ P_1 (i \psi_m \phi_m) \in a \text{ bounded set of } L^{2-\delta_1/(2+3\delta)} (0, T; V_\delta^\prime) \text{ independent of } m. \]
(3.60)
By definition,
\[ (u_m)_t + PU_m + PR(u_m) = 0. \]
(3.61)
All terms in \(PR(u_m)\) can be handled in the same fashion as in (3.59) for \(P_1 (i \psi_m \phi_m)\), then we obtain that for any \(T > 0\), there is a \(\delta_1 > 0\) such that
\[ (u_m)_t \in a \text{ bounded set of } L^{2-\delta_1/(2+3\delta)} (0, T; V_\delta^\prime) \text{ independent of } u_m \text{ and } m. \]
(3.62)

**Step 4. Passage to the limit \(m \to \infty\).** In summary, for any \(T > 0\),
\[ \begin{align*}
(u_m) & \in a \text{ bounded set of } L^2(0, T; V) \cap L^\infty(0, T; H), \\
\psi_m & \in a \text{ bounded set of } L^4(0, T; L^4(\Omega)), \\
(u_m)_t & \in a \text{ bounded set of } L^{2-\delta_1}(0, T; V_\delta^\prime)
\end{align*} \]
(3.63)
independently of \(u_m\) and \(m\). Here \(0 < \delta_1 < 1\). We infer also from (3.62) that
\[ u_m \in \text{ a precompact subset of } L^2(0, T; H) \text{ independent of } u_m \text{ and } m. \]
(3.64)
Then it is classical to pass to the limit to obtain the existence of a global weak solution of the Ginzburg-Landau system. We omit the details. Q.E.D.
4. An abstract existence theorem of global attractors of semigroups

In this section, we prove an existence theorem of global attractors for some abstract semigroup, whose absorbing sets are difficult to obtain directly from the \textit{a priori} estimates of the solutions.

We consider, in this section, a general reflexive Banach space \( X \). Let \( \{S(t)\} \) be a one parameter semigroup of nonlinear operators \( S(t) : X \rightarrow X \), enjoying the following semigroup properties:

\[
\begin{align*}
S(0) &= I = \text{identity: } X \rightarrow X, \\
S(t + s) &= S(t)S(s) : X \rightarrow X, \quad \forall s, t \geq 0.
\end{align*}
\] (4.1)

First, we recall some definitions about the absorbing set and the global attractor (cf. [25]):

**Definition 4.1** (absorbing set). A subset \( B_0 \) of \( X \) is an absorbing set in \( X \) for the semigroup \( \{S(t)\}_{t \geq 0} \), if for every bounded set \( B \) of \( X \), there exists \( t = t(B) \) such that

\[
S(t)B \subset B_0, \quad \forall t \geq t(B).
\] (4.2)

**Definition 4.2** (global attractor).

1. A set \( A \subset X \) is a functional invariant set for the semigroup \( S(t) \) if

\[
S(t)A = A, \quad \forall t \geq 0.
\] (4.3)

2. The global attractor \( A \), when it exists, is the unique compact invariant set which attracts every bounded set in \( X \), i.e., for any bounded set \( B \subset X \),

\[
\lim_{t \to \infty} \text{dist}(S(t)B, A) = 0
\] (4.4)

where for two sets \( B_1 \) and \( B_2 \) of \( X \),

\[
\text{dist}(B_1, B_2) = \sup_{x \in B_1} \inf_{y \in B_2} |x - y|.
\]

A general existence result (see among many others, [25]) states that the existence of the global attractor is guaranteed by the following conditions:

1. There exists a bounded absorbing set \( B_0 \).
2. The operators \( S(t) \) are uniformly compact. i.e., \( \forall B \) bounded, there exists \( t_1 = t_1(B) \geq 0 \) such that

\[
\bigcup_{t \geq t_1} S(t)B
\]

is precompact in \( H \).

Normally, the existence of a bounded absorbing set is the consequence of the following \textit{a priori} estimates of solutions:

\[
\|S(t)u_0\| \leq c, \quad \forall u_0 \in B \quad \text{and} \quad t \geq t_1(B)
\] (4.5)

with \( c \) independent of \( t_1(B) \) and \( u_0 \); As we indicated before, we are unable to obtain the estimate (4.5) via direct estimates.

The main result in this subsection is the following theorem which provides an explicit way of obtaining an absorbing set and hence the global attractor for the Ginzburg-Landau system (or any other system with a Lyapunov functional). Before we give the theorem, we need the following definition:
Definition 4.3. A Lyapunov functional of the semigroup \( S(\cdot) \) on a set \( B \subset X \) is a continuous function \( E: X \to \mathbb{R} \) such that

1) for any \( u_0 \in B \), the function \( t \to E(S(t)u_0) \) is decreasing;

2) if \( E(S(t+\tau)u_0) = E(S(t)u_0) \) for some \( \tau > 0 \), then \( S(t)u_0 \) is a fixed point of the semigroup.

Theorem 4.4. Let \( S(t): X \to X \) be a continuous semigroup satisfying the following properties:

1) the semigroup \( S(t) \) admits a continuous Lyapunov functional \( E(u) \) which satisfies \( E(u) \to \infty \) if and only if \( |u|_X \to \infty \);

2) the set \( S \) of stationary solutions is bounded in \( X \);

3) for any bounded subset \( B \) of \( X \), there exists a \( t_B > 0 \) such that \( S(t_B)B \) is a precompact subset of \( X \).

Then,

1) for each \( \alpha > 0 \), the set
\[
B_\alpha = \{ u \in V \mid E(u) < \sup_{v \in S} E(v) + \alpha \} \tag{4.6}
\]
is an absorbing set in \( V \) of the semigroup \( S(t) \);

2) the semigroup \( S(t) \) admits a global attractor which consists of exactly the unstable manifold of the set \( S \) of all steady state solutions, i.e.
\[
A = \mathcal{M}_+(S). \tag{4.7}
\]

Proof. First of all, it is easy to prove that for any \( u_0 \in X \), we have
\[
\lim_{t \to \infty} \text{dist}(S(t)u_0, S) = 0. \tag{4.8}
\]

In other words, \( S \) attracts each orbit of the semigroup \( S(t) \).

We now show that \( B_\alpha \) is an absorbing set of the semigroup in \( X \). First, \( B_\alpha \) is invariant because the Lyapunov functional is nonincreasing in time \( t \).

Let \( D \) be any given bounded set in \( V \) and assume that \( t_D = 1 \), so \( S(1)D \) is precompact in \( V \). By (4.8) and the continuity of the semigroup \( S(t) \), we know that for any \( y \in \overline{S(1)D} \), there exists a neighborhood \( N_y \) of \( y \) and a \( t_y > 0 \) such that
\[
S(t_y)N_y \subset B_\alpha.
\]

Since all these sets \( N_y \) forms a cover of \( \overline{S(1)D} \), there exists finite many of them, say \( N_1, \ldots, N_l \), which also forms a cover for \( \overline{S(1)D} \). Let
\[
T_D = \max\{t_1, \ldots, t_l\},
\]
by the invariance of \( B_\alpha \), we get
\[
S(t)D \subset B_\alpha \quad \text{for all } t \geq T_D.
\]

The existence of a global attractor then follows immediately. The structure of the global attractor given by (4.7) can be obtained using Theorem 4.1 on p. 401 of [25]. The proof is complete. Q.E.D.
5. Existence of the global attractor of the Ginzburg-Landau system

In this section, we study the long time behavior of the evolutionary Ginzburg-Landau system of superconductivity. We first prove the existence of a global attractor of the system, which represents exactly all the long time dynamics of the system. Then in the next section, we establish an upper bound for the Hausdorff and fractal dimensions of the global attractor in terms of the physically relevant parameters, i.e., the Ginzburg-Landau parameter $\kappa$, the diffusion related parameter $\eta$ and the applied field $H$.

As we indicated before, we are not able to obtain directly the a priori estimates of (3.22)-(3.23), to ensure the existence of the bounded absorbing set. We apply the abstract theorem developed in the previous section to the evolutionary Ginzburg-Landau system. First of all, by Theorem 3.5, we denote by $S(t)$, the semigroup generated by the Ginzburg-Landau system, which is defined as follows: for any $u_0 \in V$,

$$S(t)u_0 = u(t) = (\psi(t), A(t)),$$

(5.1)

where $u(t)$ is the unique solution of Problem 3.1. The main result in this section is

Theorem 5.1. The dynamical system associated with the Ginzburg-Landau system (2.14), supplemented by the boundary condition (2.15), possesses a global attractor $\mathcal{A} \subset V$, which is compact, connected and maximal in $V$. Moreover, $\mathcal{A} \subset V \cap H^2$ consists of exactly the unstable manifold of the set of all steady-state solutions, i.e.

$$\mathcal{A} = \mathcal{M}_+(S)$$

(5.2)

where

$$S \subset V \cap H^2$$

(5.3)

is the set of all solutions of the steady-state problem.

Proof. Now we only have to verify the assumptions of Theorem 4.4 which are trivial following our studies, we omit the details.

However, we will study the assumption 2) in Theorem 4.4 a little further. The reason is that we want to establish an explicit bound in terms of the physical parameter for the set of steady state solutions which will be useful later and also, the definition of the set of steady state solutions is interesting and worth an independent investigation.

Formally, there are two steady state Ginzburg-Landau systems, i.e. the steady state Ginzburg-Landau equations (2.17) with boundary conditions (2.15-1), and the following steady state Ginzburg-Landau equations obtained directly from the time dependent Ginzburg-Landau equations (2.14):

$$\begin{cases}
    i\psi \phi + \kappa^2(|\psi|^2 - 1)\psi + (i \text{grad} + A)^2 \psi = 0, \\
    \eta \text{grad} \phi + \frac{1}{2} i (\psi^* \text{grad} \psi - \psi \text{grad} \psi^*) + |\psi|^2 A + \text{curl}^2 A - \text{curl} H = 0, \\
    \text{div} A = 0
\end{cases}$$

(5.4)

with boundary conditions (2.15-1). The following lemma shows that these two steady state problems are exactly the same. That is, $\psi_t + i\phi_t$ and $A_t + \text{grad} \phi$ disappear together as $t \to \infty$ is an accurate description of the long time dynamics of the Ginzburg-Landau system with respect to terms involving time derivatives.
Lemma 5.2.
1. If $u = (\psi, A) \in V$ and $\phi \in H^1(\Omega)$ is a solution of (5.4) and (1.15-1), then $i\psi \phi = 0$ and $\text{grad} \phi = 0$.
2. There is at least one solution $u = (\psi, A) \in V$ to the steady state problem (2.17) and (2.15-1).
3. The steady state solutions $u = (\psi, A)$ satisfy the following estimates:

$$
\begin{cases}
|\psi|_{L^2}^2 \leq c, \\
\|u\|^2 \leq c(\kappa^2 + |H|^2), \\
\forall u \in S,
\end{cases}
$$

(5.5)

where $S$ is the set of all stationary solutions in $V$ and $c$ is independent of $\kappa$, $H$ and $u$.

Proof. 1. By (3.45), we have

$$
\frac{dE(u(t))}{dt} + |\psi_t + i\phi\psi|^2 + \eta |A_t + \text{grad}\phi|^2 = 0.
$$

Since $u$ is a stationary solution, we have

$$
|\psi_t|^2 + \eta |\text{grad}\psi|^2 = 0.
$$

2. The existence of steady state solutions can be obtained by either using the Galerkin method or looking for the minimizers of the Ginzburg-Landau functional; we omit the details.

3. First of all, we infer from (2.17-1) that

$$
\kappa^2 |\psi|_{L^2}^4 + |(i\text{grad} + A)\psi|^2 = \kappa^2 |\psi|^2 \leq \frac{1}{2}\kappa^2 |\psi|^2_{L^4} + c\kappa^2
$$

which implies that

$$
\begin{cases}
|(i\text{grad} + A)\psi|^2 \leq c\kappa^2, \\
|\psi|^2_{L^4} \leq c,
\end{cases}
$$

(5.6)

c being independent of $u = (\psi, A)$ and $\kappa^2$.

Then noticing that

$$
\frac{1}{2}(\psi^*\text{grad}\psi - \psi \text{grad}\psi^*) + |\psi|^2 A = \text{Re} (i\text{grad}\psi + A\psi)\psi^*,
$$

we deduce from (2.17-2) that

$$
|\text{curl} A|^2 \leq \left| \text{Re} \int (i\text{grad}\psi + A\psi)\psi^* \cdot Adx \right| + |\int H \cdot \text{curl} Adx|
\leq c|(i\text{grad} + A)\psi||\psi|_{L^4}||A|_{L^4} + |H||\text{curl} A|
\leq (\text{by (5.6)}) \leq c|A|^2_{L^4} + c|H|^2 + \frac{1}{4}|\text{curl} A|^2 \leq c(\kappa^2 + |H|^2) + \frac{1}{4}|\text{curl} A|^2,
$$

which implies that

$$
|\text{curl} A|^2 \leq c(\kappa^2 + |H|^2).
$$

(5.7)

Therefore,

$$
|i\text{grad} \psi|^2 + 2|A\psi|^2 \leq c + c|A|^2_{L^4}|\psi|^2_{L^4} + c + c|A|^2_{L^4}
\leq c + c|\text{curl} A|^2 \leq c(\kappa^2 + |H|^2).
$$

The proof is complete.

Q.E.D.
6. Physical bounds on the dimension of the global attractor

6.1. The main theorem

In this section, we shall estimate the dimension (fractal and Hausdorff) of the global attractor in terms of the physically relevant parameters $K$ (the Ginzburg-Landau parameter), $\eta$ (the diffusion constant) and the applied field $H$. The definition of Hausdorff dimension of a set is classical, we do not repeat it here. For any compact set $C$ in a Banach space $X$, let $N_{e}(C)$ be the minimum number of balls of radius $e$ necessary to cover $C$, then the fractal dimension of $C$ is defined by

$$\dim_{F}(C) = \limsup_{e \to 0} \frac{\log N_{e}(C)}{\log e}.$$  \hspace{1cm} (6.1)

Our main result in this section is the following:

**Theorem 6.1.** The global attractor $\mathcal{A}$ of the Ginzburg-Landau system (2.14)-(2.15) given by Theorem 5.1 has finite dimension. We have the following estimates of the Hausdorff and Fractal dimension in terms of the Ginzburg-Landau parameter $K$ and the diffusion related constant $\eta$:

$$\dim \mathcal{A} \leq \left\{ \begin{array}{ll} c \left( K^6 + \frac{1}{\eta^3} \right) \left( K^6 + |H|^6 \right) \left( 1 + \frac{|H|^3}{K^3} \right), & n = 3, \\ c \left( K^4 + \frac{1}{\eta^2} \right) \left( K^4 + |H|^4 \right) \left( 1 + \frac{|H|^2}{K^2} \right), & n = 2. \end{array} \right. \hspace{1cm} (6.2)$$

6.2. Proof of Theorem 6.1 when $n = 3$

We use the theory developed by Constantin, Foias and Temam (see [5]). We divide the proof into 4 steps.

**Step 1. Linearized problem.** For any $u_0 = (\phi_0, A_0) \in \mathcal{A}$ and $u(t) = S(t)u_0 = (\phi(t), A(t))$, we consider the linearized equations of the Ginzburg-Landau system around $u$:

$$\frac{\partial \Psi}{\partial t} + F_1'(u)(\Psi, B) = 0,$$  \hspace{1cm} (6.3)

$$\frac{\partial B}{\partial t} + \text{grad} \Pi + F_2'(u)(\Psi, B) = 0,$$  \hspace{1cm} (6.4)

$$\text{div} B = 0$$  \hspace{1cm} (6.5)

together with the linearized boundary conditions

$$\frac{\partial \Psi}{\partial n} = 0, \quad B \cdot n = 0, \quad \text{curl} B \times n = 0, \quad \text{on } \partial \Omega.$$  \hspace{1cm} (6.6)

In (6.3) and (6.4), $F_1'$, $F_2'$ and $\Pi$ are given by

$$F_1'(u)(\Psi, B) = -\Delta \Psi + 2i(A \text{grad} \Psi + B \text{grad} \psi) + |A|^2 \Psi + 2AB \psi + i(\Phi(u)\Psi + i\psi \Pi) + \kappa^2 [ (|\psi|^2 - 1) \Psi + \Psi|\psi|^2 + \Psi^* \psi^2 ],$$  \hspace{1cm} (6.7)
\[ F_2^j(u)(\Psi, B) = \frac{1}{\eta} \left\{ \frac{1}{2} i (\Psi^* \text{grad} \Psi - \Psi \text{grad} \Psi^* + \psi^* \text{grad} \psi - \psi \text{grad} \psi^*) \right. \\
\left. + |\psi|^2 B + (\psi \Psi^* + \Psi \psi^*) A + \text{curl}^2 B \right\}, \]  
\( (6.8) \)

\[ \text{grad} II + (I - P_2) F_2^j(u)(\Psi, B) = 0, \]  
\( (6.9) \)

\[ \int_{\Omega} II \, dx = 0. \]  
\( (6.10) \)

As in Lemma 3.1 (cf. (3.9)-(3.10)), \( \Pi \) is uniquely defined by (6.9) and (6.10).

It is easy to see that there is a unique strong solution of (6.3)-(6.6) with given initial data in \( V \) and given \( u(t) = S(t) u_0 \) with \( u_0 \in \mathcal{A} \).

**Step 2. CFT theory.** We proceed now by introducing the Constantin-Foias-Temam theory from [5]. We set

\[ T_m(\tau) = - \sum_{j=1}^{m} (F_2^j(u(\tau)) (\Psi^j(\tau) B^j(\tau), -\Delta \Psi^j(\tau) + \Psi^j(\tau))) \]

\[ - \sum_{j=1}^{m} (P_2 F_2^j(u(\tau)) (\Psi^j(\tau) B^j(\tau), \text{curl} B^j(\tau)). \]  
\( (6.11) \)

Then we consider solutions \( U_1(t), \ldots, U_m(t) \) of the linearized system (6.3)-(6.6) with initial values \( \xi_1, \ldots, \xi_m \in V \) respectively. Then we choose \( \{(\Psi^j(t), B^j(t))\}_{j=1}^{m} \) to be an orthonormal basis of

\[ Q_m V = \text{span}\{U_1(t), \ldots, U_m(t)\}, \]  
\( (6.12) \)

\( Q_m \), being an orthonormal projector of \( V \) into \( Q_m V \).

Then it is easy to see that (cf. among others, [25]),

\[ |U_1(t) \wedge \cdots \wedge U_m(t)|_{\wedge^* V} = |\xi_1 \wedge \cdots \wedge \xi_m|_{\wedge^* V} \exp \int_0^t \text{Re} T_m(\tau) \, d\tau. \]  
\( (6.13) \)

Moreover, we introduce

\[ q_m(t) = \sup_{u_0 \in \mathcal{A}} \sup_{\xi_j \in \mathcal{V} \| \xi_j \| \leq 1} \left[ \frac{1}{t} \int_0^t \text{Re} T_m(\tau) \, d\tau \right], \]  
\( (6.14) \)

\[ q_m = \lim_{t \to \infty} q_m(t). \]  
\( (6.15) \)

Then we quote without proof the main result of the Constantin-Foias-Temam theory from [5] which will be used in the remaining part of this section.

**Theorem 6.2.** If

\[ q_j \leq -\alpha j^d + \beta, \quad \forall j \geq 1, \]  
\( (6.16) \)

then the Hausdorff and fractal dimension
\( d_H(\mathcal{A}) \leq m, \quad d_F(\mathcal{A}) \leq 2m, \) \hfill (6.17)

\( m - 1 < \left( \frac{2\beta}{\alpha} \right)^{1/\theta} \leq m. \) \hfill (6.18)

**Step 3. Estimates of \( \text{Re} T_m \).** To obtain (6.16) type estimates, we now estimate \( \text{Re} T_m \). First of all, we have

\[
\text{Re} \sum_{j=1}^{m} \left[ (I - \Delta) \Psi_j, -\Delta \Psi_j + \Psi_j \right] + \frac{1}{\eta} \left( \text{curl}^2 \mathcal{B}_j, \text{curl}^2 \mathcal{B}_j \right) \\
= -\sum_{j=1}^{m} \left[ ((I - \Delta) \Psi_j)^2 - \|\Psi_j\|^2 + \frac{1}{\eta} |\text{curl}^2 \mathcal{B}_j|^2 \right] \\
\leq -\sum_{j=1}^{m} \left[ ((I - \Delta) \Psi_j)^2 + \frac{1}{\eta} |\text{curl}^2 \mathcal{B}_j|^2 \right] + m. \tag{6.19}
\]

\[
\text{Re} \sum_{j=1}^{m} (2iA \text{ grad } \Psi_j, -\Delta \Psi_j + \Psi_j) \leq c \sum_{j=1}^{m} |\text{grad } \Psi_j| L^1 \| (I - \Delta) \Psi_j \| A_L^0 \\
\leq c \| A \| \sum_{j=1}^{m} \| \Psi_j \|^{1/2} \| (I - \Delta) \Psi_j \|^{3/2} \leq c \| A \| \sum_{j=1}^{m} \| (I - \Delta) \Psi_j \|^{3/2} \\
\leq cm \| A \|^4 + \varepsilon \sum_{j=1}^{m} \| (I - \Delta) \Psi_j \|^2. \tag{6.20}
\]

\[
\text{Re} \sum_{j=1}^{m} (2iB_j \text{ grad } \psi, (I - \Delta) \Psi_j) \leq c \sum_{j=1}^{m} \int_{\Omega} |\text{grad } \psi||B_j|| (I - \Delta) \Psi_j| dx \\
\leq c \sum_{j=1}^{m} |\text{grad } \psi||B_j||_{L^\infty} \| (I - \Delta) \Psi_j \| \leq (\text{By Agmon's inequality, see [26]}) \\
\leq c \sum_{j=1}^{m} |\text{grad } \psi||B_j||^{1/2}||B_j||_{H^2}^{1/2} \| (I - \Delta) \Psi_j \| \leq c \| \psi \| \sum_{j=1}^{m} |\text{curl}^2 \mathcal{B}_j||^{1/2} \| (I - \Delta) \Psi_j \| \\
\leq cm \| \psi \|^4 + \varepsilon \sum_{j=1}^{m} \| (I - \Delta) \Psi_j \|^2 + |\text{curl}^2 \mathcal{B}_j|^2. \tag{6.21}
\]

\[
\text{Re} \sum_{j=1}^{m} (|A|^2 \Psi_j, (I - \Delta) \Psi_j) \leq c |A|^2 L^2 \sum_{j=1}^{m} \| (I - \Delta) \Psi_j \|_{L^4} \| \Psi_j \|_{L^6} \leq cm \| A \|^4 + \varepsilon \sum_{j=1}^{m} \| (I - \Delta) \Psi_j \|^2. \tag{6.22}
\]

\[
\text{Re} \sum_{j=1}^{m} (2A \cdot \mathcal{B}_j \psi, (I - \Delta) \Psi_j) \leq cm \| u \|^4 + \varepsilon \sum_{j=1}^{m} \| (I - \Delta) \Psi_j \|^2. \tag{6.23}
\]
\[ -\text{Re} \sum_{j=1}^{m} \kappa^2 (|\psi_j|^2 - 1) \psi_j + \psi_j |\psi_j|^2 + \psi_j^* \psi_j^2, (I - \Delta) \psi_j \]
\[ \leq c \kappa^2 \sum_{j=1}^{m} (1 + \|u\|^2) |(I - \Delta) \psi_j| \leq c \kappa^2 m (1 + \|u\|^4) + \varepsilon \sum_{j=1}^{m} |(I - \Delta) \psi_j|^2. \quad (6.24) \]

\[ -\text{Re} \sum_{j=1}^{m} \frac{1}{\eta} (P \frac{1}{2} \tilde{i} (\psi_j^* \text{grad} \psi - \psi_j \text{grad} \psi^*), \text{curl}^2 B_j) \]
\[ \leq \frac{c}{\eta} \int \sum_{j=1}^{m} |\text{curl}^2 B_j||\psi_j||\text{grad} \psi| dx \leq (\text{as in } (6.20)) \]
\[ \leq \frac{c}{\eta} |\psi_j| \sum_{j=1}^{m} |(I - \Delta) \psi_j| |\text{curl}^2 B_j| \leq \frac{cm}{\eta^2} \|\psi\|^4 + \varepsilon \sum_{j=1}^{m} \left[ |(I - \Delta) \psi_j|^2 + \frac{1}{\eta} |\text{curl}^2 B_j|^2 \right]. \quad (6.25) \]

\[ -\text{Re} \sum_{j=1}^{m} \frac{1}{\eta} (P \frac{1}{2} \tilde{i} (\psi_j^* \text{grad} \psi^* - \psi_j \text{grad} \psi), \text{curl}^2 B) \]
\[ \leq \frac{c}{\eta} \sum_{j=1}^{m} \|\psi_j\|_{L^2} |\text{curl}^2 B_j||\text{grad} \psi_j|_{L^2} \leq \frac{c}{\eta} \|\psi\| \sum_{j=1}^{m} |\psi_j|_{H^2} |\text{curl}^2 B_j| \]
\[ \leq \frac{c}{\eta} \|\psi\|^2 \sum_{j=1}^{m} |\psi_j|_{H^2}^2 + \varepsilon \sum_{j=1}^{m} |\text{curl}^2 B_j|^2 \leq \varepsilon \sum_{j=1}^{m} \left[ |(I - \Delta) \psi_j|^2 + \frac{1}{\eta} |\text{curl}^2 B_j|^2 \right] + \frac{c}{\eta^2} m \|\psi\|^4. \quad (6.26) \]

By Lemma 6.3 below, we have
\[ -\text{Re} \sum_{j=1}^{m} (i \phi \eta^j, (I - \Delta) \psi_j) \leq c \sum_{j=1}^{m} |(I - \Delta) \psi_j||\phi\|_{L^2} |\psi_j|_{L^2} \]
\[ \leq \frac{c}{\eta} (\|u\| + \|\phi\|_{L^2}) \|\psi\| \sum_{j=1}^{m} |(I - \Delta) \psi_j| \leq \varepsilon \sum_{j=1}^{m} |(I - \Delta) \psi_j|^2 + \frac{c}{\eta^2} m (\|u\|^4 + \|u\|^2 |\psi_j|^4). \quad (6.27) \]

Moreover, using Lemma 3.1, we obtain
\[ -\text{Re} \sum_{j=1}^{m} (i \phi \eta^j, (I - \Delta) \psi_j) \leq c \sum_{j=1}^{m} |(I - \Delta) \psi_j||\phi(u)\|_{L^2} |\psi_j| \]
\[ \leq \frac{c}{\eta} \sum_{j=1}^{m} |(I - \Delta) \psi_j| (\|u\|^2 + \|u\| \cdot |\psi_j|^2) \leq \varepsilon \sum_{j=1}^{m} |(I - \Delta) \psi_j|^2 + \frac{cm}{\eta^2} (\|u\|^4 + \|u\|^2 |\psi_j|^4). \quad (6.28) \]

Introducing (6.19)–(6.28) in (6.14) leads to
\[ \text{Re} T_m(\tau) \leq (1 - 10 \varepsilon) \sum_{j=1}^{m} \left[ |(I - \Delta) \psi_j|^2 + \frac{1}{\eta} |\text{curl}^2 B_j|^2 \right] + cm \left( 1 + \kappa^4 + \frac{1}{\eta^2} \right) (\|u\|^4 + \|u\|^2 |\psi_j|^4). \]

Choosing \( \varepsilon = \frac{1}{20} \), the above inequality yields
\[ \text{Re} T_m(\tau) \leq -\frac{1}{2} X_m(\tau) + c m \left( \kappa^4 + \frac{1}{\eta^2} \right) (\|u\|^4 + \|u\|^2 \|\psi\|_{L^4}^4) \] 

(6.29)

where

\[ X_m(\tau) = \sum_{j=1}^{m} \left[ (I - \Delta) \psi_j \right]^2 + \frac{1}{\eta} \|B^j\|_{H^2}^2 \].

(6.30)

Then we have

\[ \frac{1}{t} \int_{0}^{t} \text{Re} T_m(\tau) d\tau \leq -\frac{1}{2} \frac{1}{t} \int_{0}^{t} X_m(\tau) d\tau + c \left( \kappa^4 + \frac{1}{\eta^2} \right) \frac{m}{t} \int_{0}^{t} (\|u\|^4 + \|u\|^2 \|\psi\|_{L^4}^4) d\tau \]

\[ \leq -\frac{1}{2} \frac{1}{t} \int_{0}^{t} X_m(\tau) d\tau + c \gamma m \left( \kappa^4 + \frac{1}{\eta^2} \right) \]

(6.31)

where

\[ \gamma = \sup_{u \in \mathcal{A}} (\|u\|^4 + \|u\|^2 \|\psi\|_{L^4}^4). \]

(6.32)

**Step 4. Completion of the proof.** By Lemma 6.4 below, we have now that

\[ q_m \leq -cm^{5/3} + c \gamma m \left( \kappa^4 + \eta^{-2} \right) \leq -cm^{5/3} + c \gamma \left( \kappa^4 + \eta^{-2} \right)^{5/2}. \]

(6.33)

Finally by Theorem 6.2, we have

\[ \dim \mathcal{A} \leq c \left( \gamma \left( \kappa^4 + \eta^{-2} \right)^{5/2} \right)^{3/5} \leq c \gamma \left( \kappa^4 + \eta^{-2} \right)^{3/2} \] (by Theorem 6.5 below)

\[ \leq c \left( \kappa^6 + \frac{1}{\eta^3} \right)^{1/2} \left( \kappa^6 + |H|^6 \right) \left( 1 + \frac{|H|^3}{\kappa^3} \right). \]

The proof is complete. Q.E.D.

The following estimate for \( \Pi^j \) was used in the above proof.

**Lemma 6.3.** The function \( \Pi^j \) defined by

\[ \text{grad} \, \Pi^j + (I - P_2) F_2^j(u)(\psi^j, B_j) = 0, \]

(6.34)

\[ \int_{\Omega} \Pi^j dx = 0 \]

(6.35)

satisfies the following estimates:

\[ |\Pi^j|_{L^1} \leq c \|(I - P_2) F_2^j(u)(\psi^j, B^j)\|_{W^{-1,1}(\Omega)} \leq c \|\psi\| + \|\psi\|_{L^4}^2. \]

(6.36)

**Proof.** We only have to prove the second inequality in (6.36). To this end, we consider each term in (6.8).

\[ \left| \int_{\Omega} \psi^j \text{grad} \psi \tilde{A} dx \right| \leq |\psi^j|_{L^6} \|\text{grad} \psi\|_{L^3} \leq c \|\psi\| \|\tilde{A}\|_{W^{1,3}(\Omega)}, \]

(6.36)
\[
\left| \int_{\Omega} \psi^* \text{grad} \Psi^j \tilde{A} \, dx \right| \leq c \| \psi \|_{L^q} \| \text{grad} \Psi^j \|_{L^q} \| \tilde{A} \|_{L^3} \leq c \| \psi \|_{W^{1,q/2}(\Omega)},
\]

\[
\left| \int_{\Omega} |\psi|^2 B^j \cdot \tilde{A} \, dx \right| \leq c \| \psi^2 \|_{L^q} \| B^j \|_{L^q} \| \tilde{A} \|_{L^3} \leq c \| \psi \|^2_{L^q} \| \tilde{A} \|_{W^{1,q/2}}.
\]

The other terms can be estimated in the same fashion, noticing that
\[
\text{curl}^2 B^j - P_2 \text{curl}^2 B^j = 0.
\]

The proof is complete. Q.E.D.

Lemma 6.4.

\[
\sum_{j=1}^{m} \left[ |(I - \Delta) \Psi^j|^2 + \| B^j \|^2_{H^2} \right] \geq c m^{5/3}
\]

(6.37)

with \( c \) independent of \( m \).

Proof. By Lemma 2.1, p. 302 of [25], we have
\[
\sum_{j=1}^{m} \left[ |(I - \Delta) \Psi^j|^2 + \| B^j \|^2_{H^2} \right] \geq \lambda_1^{1/2} + \cdots + \lambda_m^{1/2} \geq c \sum_{j=1}^{m} j^{2/3} \geq c m^{5/3}
\]

where \( \lambda_1, \ldots, \lambda_m \) are eigenvalues of the operator \( u = (\Psi, A) \rightarrow (I - \Delta) \Psi, -\Delta A) \). Q.E.D.

The following theorem provides an upper bound \( \gamma \) for the \( H^1 \)-norm of the global attractor defined by (6.32).

Theorem 6.5. We have
\[
\begin{aligned}
\sup_{u \in \mathcal{A}} |\psi|^2_{L^4} &\leq c \left( 1 + \frac{|H|^2}{\kappa^2} \right), \\
\sup_{u \in \mathcal{A}} \| u \| &\leq c (\kappa + |H|) \left( 1 + \sqrt{\frac{|H|}{\kappa}} \right)
\end{aligned}
\]

(6.38)

where \( \mathcal{A} \) is the global attractor of the Ginzburg-Landau system obtained in Theorem 5.1.

Proof. First of all, we claim that
\[
\sup_{u \in \mathcal{A}} E(u) = \sup_{u \in \mathcal{S}} E(u).
\]

(6.39)

The see this, for any \( u_0 \in \mathcal{A} \), by Theorem 5.1, there is a complete orbit \( \{ u(t) : t \in \mathbb{R} \} \) such that \( u(0) = u_0 \)
and
\[
d(u(t), S) \to 0 \quad \text{as} \ t \to \infty.
\]

(6.40)

On the other hand, \( E(u(t)) \) is decreasing. Therefore,
\[ \sup_{u \in S} E(u) \geq E(u_0). \]

Namely, (6.39) holds.

Then by definition (3.43) and the estimates (5.5), (6.39), it yields that
\[ \sup_{u \in A} E(u) \leq c(\kappa^2 + |H|^2). \]  
(6.41)

This shows that for any \( u \in A \),
\[ \begin{cases} 
|\left( i \text{grad} + A \right) \psi |^2 \leq c(\kappa^2 + |H|^2), \\
\kappa^2 |\psi|_{L^2}^4 \leq c(\kappa^2 + |H|^2), \\
|\text{curl} A|^2 \leq c(\kappa^2 + |H|^2).
\end{cases} \]  
(6.42)

Hence
\[ |\text{grad} \psi|^2 \leq c\left( |\left( i \text{grad} + A \right) \psi|^2 + |A\psi|^2_{L^2} \right) \leq c(\kappa^2 + |H|^2) + |A|^2_{L^2} |\psi|^2_{L^2} \]
\[ \leq c(\kappa^2 + |H|^2) \left( 1 + |\psi|^2_{L^2} \right) \leq c(\kappa^2 + |H|^2) \left( 1 + \frac{|H|}{\kappa} \right). \]

The proof is complete. Q.E.D.

6.3. Proof of Theorem 6.1 when \( n = 2 \)

First of all, all discussions in the previous subsection hold true in the case where \( n = 2 \). In order to get better dimension estimates, we modify some estimates presented in the previous subsection.

We only have to start with (6.33). Notice first that in the case where \( n = 2 \), (6.37) becomes
\[ n \sum_{j=1}^m |(I - \Delta) \Psi_j|^2 + \|B_j\|^2_{H^1} \geq c m^2. \]  
(6.43)

with \( c \) independent of \( m \). Therefore, we infer from (6.31) that
\[ q_m \leq -cm^2 + c\gamma m(\kappa^4 + \eta^{-2}) \leq -cm^2 + c[\gamma(\kappa^4 + \eta^{-2})]^2. \]  
(6.44)

By Theorem 6.2, we deduce that
\[ \dim A \leq c\gamma(\kappa^4 + \eta^{-2}) \leq c \left( \kappa^4 + \frac{1}{\eta^2} \right) \left( \kappa^4 + |H|^4 \right) \left( 1 + \frac{|H|^2}{\kappa^2} \right). \]

The proof is complete. Q.E.D.

Acknowledgement

The authors thank sincerely Roger Temam for his help. Tang was partially supported by a Nuffield foundation New Science Lecturer grant. Part of the work was accomplished when Tang visited the Institute for Scientific Computing and Applied Mathematics, Indiana University. He would also like to acknowledge the support given by P. Bauman and D. Phillips from Purdue University which makes the completion of this work possible. Wang was partially supported by the National Science Foundation under Grant NSF-DMS9400615, by the Department of Energy under Grant DOE-DE-FG02-92ER25120, and by the Research Fund of Indiana University.
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