BOUNDARY LAYER SEPARATION AND STRUCTURAL
BIFURCATION FOR 2-D INCOMPRESSIBLE FLUID FLOWS

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Abstract. The main objective of this article and the previous articles [2, 3, 7] is to provide a rigorous characterization of the boundary layer separation of 2-D incompressible viscous fluids. First we establish a simple equation linking the separation location and time with the Reynolds number, the external forcing the boundary curvature, and the initial velocity field. Second, we show that external forcing with reverse orientation to the initial velocity field leads to structural bifurcation at a degenerate singular point with integer index of the velocity field at the critical bifurcation time. Necessary and sufficient kinematic conditions are given to identify the case for boundary layer separation.

1. Introduction. This article along with [2, 3, 7] is part of a program to develop a rigorous theory on boundary layer separation of incompressible fluid flows. This is a long standing problem in fluid mechanics going back to the pioneering work of Prandtl (1904). Basically, in the boundary layer, the shear flow can detach/separate from the boundary, generating bubbles and leading to more complicated turbulent behavior. As pointed out by Jäger, and Lax, and Morawetz [4] and Chorin and Marsden [1], the crucial question of how the separation or breaking away of the boundary layer from the boundary takes place is still not resolved. The study in this article along with [2, 3, 7] provides a first rigorous account on this question.

In the previous articles [2, 3, 7], in collaboration with M. Ghil, we studied structural transitions for a family of divergence-free vector fields \( u(\cdot, t) \) with the Dirichlet boundary conditions or the no normal flow condition. We derived crucial kinematic conditions for the structural bifurcation and for boundary layer separation. The main results were obtained based on a complete classification of orbit structures near an isolated degenerate singular (or \( \partial \)-singular in the Dirichlet boundary conditions case) point.

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Technically speaking, the main ideas and results in this series of papers are as follows. The first step is to study the structural bifurcation for flows with free boundary conditions; see [2]. The key technical point is to fully classify the flow structure near the bifurcation location and time in terms of the index of the vector field at a degenerate singular point on the boundary and at the critical time, and the main technical tool is careful orbit analysis. The second step is to analyze the stability for flows with the Dirichlet boundary conditions; see [7]. The key ingredient is the introduction of the so-called $\partial$-singular point as in this case, all points on the boundary are singular points in the usual sense. Third, the flow structure near the bifurcation point and time are classified as well for flows with the Dirichlet boundary conditions based on the studies in the first two steps. This is done in [3]. Finally, the main result in the Dirichlet boundary condition case, which corresponds to the boundary layer separation, provides detailed information on flow transition near the critical time under four natural sufficient kinematic conditions; see Theorems 2.6 and 2.7. For fluid flows, these conditions are also sufficient. Hence, from the fluid mechanics point of view, Theorems 2.6 and 2.7 gives just complete kinematic picture of the structure bifurcation, hence the boundary separation.

This article is oriented toward structural bifurcation and boundary layer separation of the solutions of the Navier-Stokes equations. For this purpose, first we establish a simple equation, which we call separation equation, linking the separation location and time with the Reynolds number, the external forcing and the initial velocity field. It is hoped that this separation equation will lead to further applications in both theoretical and engineering studies of the boundary layer separation.

Second, we give an example to demonstrate how the external forcing with reverse orientation to the initial velocity field leads to structural bifurcation and boundary layer separation. Furthermore, we show that the structural bifurcation occurs at a degenerate singular point with integer index of the velocity field at the critical bifurcation time.

Finally, we establish some necessary and sufficient kinematic conditions are given to identify the typical case for the boundary layer separation. As we know, our theory are based on classification of the detailed orbit structure of the velocity field near the bifurcation time and location, in terms of the index of the instantaneous velocity field (at the critical bifurcation time) and at the bifurcation location. The typical case for the boundary layer separation is the case where the index is zero, which corresponds to the separation of one or multiple vortices from shear flows. The necessary and sufficient kinematic conditions derived eliminates also the possible transition of multiple vortices directly from the shear flow.

The paper is organized as follows. In Section 2, we recall the kinematic theory obtained in [2, 3, 7]. In Section, we establish the separation equation linking the separation time and location with the Reynolds number, the external forcing, and curvature of the boundary, and the initial data. In addition, dynamics sufficient conditions for structural bifurcation are also obtained. In Section 4, an example of boundary layer separation driven by the external forcing is given. Section 5 introduces conditions to distinguish structural bifurcation typically occurring in the boundary layer separation.
2. Recapitulation of Structural Bifurcation of Vector Fields.

2.1. Structural Stability. Let $M \subset \mathbb{R}^2$ be a closed and bounded domain with $C^{r+1}$ ($r \geq 2$) boundary $\partial M$, and $TM$ be the tangent bundle of $M$. Let $C^r(TM)$ be the space of all $r$-th differentiable vector fields on $M$. We set

$$C^r_u(TM) = \{ u \in C^r(TM) \mid u_n|_{\partial M} = 0, \},$$

$$D^r(TM) = \{ u \in C^r(TM) \mid u_n|_{\partial M} = 0, \text{ div } u = 0 \},$$

$$B^r(TM) = \{ u \in D^r(TM) \mid \frac{\partial u_n}{\partial n}|_{\partial M} = 0 \},$$

$$B_\nu^r(TM) = \{ u \in D^r(TM) \mid u|_{\partial M} = 0 \},$$

where $u_n = u \cdot n$ and $u_\tau = u \cdot \tau$, $n$ and $\tau$ are the unit normal and tangent vector on $\partial M$ respectively. The boundary condition $u_n = 0$ on $\partial M$ is the no-normal flow condition, which is the natural boundary condition for the Euler equations, and $u = 0$ on $\partial M$ is the Dirichlet boundary condition, which is usually used for the Navier-Stokes equations.

Let $X = D^r(TM)$ or $B_\nu^r(TM)$ in the following definitions.

**Definition 2.1.** Two vector fields $u, v \in X$ are called topologically equivalent if there exists a homeomorphism of $\varphi : M \rightarrow M$, which takes the orbits of $u$ to orbits of $v$ and preserves their orientation.

**Definition 2.2.** A vector field $v \in X$ is called structurally stable in $X$ if there exists a neighborhood $O \subset X$ of $v$ such that for any $u \in O$, $u$ and $v$ are topologically equivalent.

We recall next the structural stability theorems obtained in Ma and Wang [5, 7, 8]. For this purpose, consider a vector field $v \in C^r_u(TM)$. A point $p \in M$ is called a singular point of $v$ if $v(p) = 0$; a singular point $p$ of $v$ is called non-degenerate if the Jacobian matrix $Dv(p)$ is invertible; $v$ is called regular if all singular points of $v$ are non-degenerate.

**Theorem 2.1.** (Ma and Wang [5, 8]) A divergence-free vector field $u \in D^r(TM)$ is structurally stable in $D^r(TM)$ if and only if

1. $u$ is regular,
2. all interior saddle points of $u$ are self-connected, and
3. each boundary saddle point is connected only to a boundary saddle point on the same connected component of the boundary.

Moreover, all structurally stable vector fields in $X$ form an open and dense set of $D^r(TM)$.

For $u \in B_\nu^r(TM)$ ($r \geq 2$), a different singularity concept for points on the boundary was introduced in [7]. We proceed as follows.

1. A point $p \in \partial M$ is called a $\partial$–regular point of $u$ if $\partial u_\tau(p) / \partial n \neq 0$; otherwise, $p \in \partial M$ is called a $\partial$–singular point of $u$.
2. A $\partial$–singular point $p \in \partial M$ of $u$ is called nondegenerate if

$$\det \begin{pmatrix} \frac{\partial^2 u_\tau(p)}{\partial \tau \partial n} & \frac{\partial u_\tau(p)}{\partial n} \\ \frac{\partial^2 u_n(p)}{\partial \tau \partial n} & \frac{\partial^2 u_n(p)}{\partial n^2} \end{pmatrix} \neq 0.$$ 

A nondegenerate $\partial$–singular point of $u$ is also called a $\partial$–saddle point of $u$. 
3. A vector field \( u \in B^r_0(TM) \) \((r \geq 2)\) is called \( D \)-regular if \( u \) is regular in \( \mathring{M} \), the interior of \( M \), and all \( \partial \)-singular points of \( u \) on \( \partial M \) are nondegenerate. The \( \partial \)-regular and \( \partial \)-saddle points of \( u \in B^r_0(TM) \) can be characterized by the following properties.

4. No orbit of \( u \) in \( \mathring{M} \) connects to a \( \partial \)-regular point of \( u \).

5. Locally there is exactly one orbit of \( u \) in \( \mathring{M} \) connected to a \( \partial \)-saddle point.

The following theorem provides necessary and sufficient conditions for structural stability of divergence-free vector fields with the Dirchlet boundary conditions.

**Theorem 2.2.** (Ma and Wang [7]) Let \( u \in B^r_0(TM) \) \((r \geq 2)\). Then \( u \) is structurally stable in \( B^r_0(TM) \) if and only if

1. \( u \) is \( D \)-regular;
2. all interior saddle points of \( u \) are self-connected; and
3. each \( \partial \)-saddle point of \( u \) on \( \partial M \) is connected to a \( \partial \)-saddle point on the same connected component of \( \partial M \).

Moreover, the set of all structurally stable vector fields is open and dense in \( B^r_0(TM) \).

2.2. Singularity Classification. We recall the definition of indices of singular points of a vector field from Ma and Wang [6]. Let \( p \in \mathring{M} \) be an isolated singular point of \( u \in C^r(TM) \). Then

\[
\text{ind}(u, p) = \text{deg}(u, p),
\]

where \( \text{deg}(u, p) \) is the Brouwer degree of \( u \) at \( p \).

Let \( p \in \partial M \) be an isolated singular point of \( u \in C^r(TM) \), and \( \mathring{M} \subset \mathbb{R}^2 \) be an extension of \( M \), i.e. \( M \subset \mathring{M} \) such that \( p \in \mathring{M} \) is an interior point of \( \mathring{M} \). In a neighborhood of \( p \) in \( \mathring{M} \), \( u \) can be extended by reflection to \( \tilde{u} \in C^0(TM) \) such that \( p \) is an interior singular point of \( \tilde{u} \), thanks to \( u \cdot n|_{\partial M} = 0 \), the condition of no normal flow. Then we define the index of \( u \) at \( p \) in \( \partial M \) by

\[
\text{ind}(u, p) = \frac{1}{2} \text{ind}(\tilde{u}, p).
\]

Let \( p \in M \) be an isolated singular point of \( u \in C^r(TM) \). An orbit \( \gamma \) of \( u \) is said to be a stable orbit (resp. an unstable orbit) connected to \( p \), if the limit set \( \omega(x) = p \) (resp. \( \alpha(x) = p \)) for any \( x \in \gamma \).

We now recall the singularity classification theorem for incompressible vector fields, which is basic for the discussion of structural bifurcation.

**Theorem 2.3.** (Singularity Classification Theorem [2]) Let \( p \in M \) be an isolated singular point of \( u \in D^r(TM) \), \( r \geq 1 \). Then \( p \) is connected only to a finite number of orbits and the stable and unstable orbits connected to \( p \) alternate when tracing a closed curve around \( p \). Furthermore

1. if \( p \in \mathring{M} \), then \( p \) is connected by \( 2n \) \((n \geq 1)\) orbits, \( n \) of which are stable, and the other \( n \) unstable, while the index of \( u \) at \( p \) is:

\[
\text{ind}(u, p) = 1 - n;
\]

2. if \( p \in \partial M \), then \( p \) is connected by \( n + 2 \) \((n \geq 0)\) orbits, two of which are on the boundary \( \partial M \), and the index of \( u \) at \( p \) is

\[
\text{ind}(u, p) = -\frac{n}{2}.
\]
2.3. Structural Bifurcation. We recall in this subsection two structural theorems, one for one parameter family of divergence-free vector fields with the no-normal flow condition, and the other for one parameter family of divergence-free vector fields with the Dirichlet conditions.

Definition 2.3. Let $u \in C^1([0, T], X)$. We say that $u(x, t)$ has a bifurcation in its local structure in a neighborhood $U \subset M$ of $x_0$ at $t_0$ ($0 < t_0 < T$) if, for any $t^- < t_0$ and $t_0 < t^+$ with $t^-$ and $t^+$ sufficiently close to $t_0$, the vector fields $u(\cdot, t^-)$ and $u(\cdot, t^+)$ are not topologically equivalent locally in $U \subset M$, and we say that $u(\cdot, t)$ has a bifurcation at $t_0$ in its global structure if $U = M$.

We remark here that bifurcation in its local structure does not implies the bifurcation in its global structure. In fact, one can construct easily examples showing that flow structure changes in some local area $U \subset M$, but not on $M$.

Let $u \in C^1([0, T], D^r(TM))$ be Taylor expanded at $t_0$ ($0 < t_0 < T$) as

$$
\begin{align*}
    u(x, t) &= u^0(x) + (t - t_0)u^1(x) + o(|t - t_0|), \\
    u^0(x) &= u(x, t_0), \\
    u^1(x) &= \frac{\partial}{\partial t} u(x, t_0).
\end{align*}
$$

Then we proceed with the following assumption.

Assumption (H). Let $x_0 \in \Gamma$ be an isolated degenerate singular point of $u^0$, and $u^0 \in C^{k+1}$ near $x_0 \in \Gamma$ for some $k \geq 2$. Assume that

$$
\begin{align*}
    u^0(x_0) &= 0, \quad (2.1) \\
    \text{ind}(u^0, x_0) &\neq -\frac{1}{2}, \quad (2.2) \\
    \frac{\partial^k u^0(x_0)}{\partial x^k} &\neq 0, \quad (2.3) \\
    u^1(x_0) &\neq 0. \quad (2.4)
\end{align*}
$$

Then the following two theorems classify the structure of $u$ near $(x_0, t_0)$, providing sufficient conditions for structural bifurcation for divergence-free vector fields with the no normal flow condition.

Theorem 2.4. [2]. Let $u \in C^1([0, T], X)$ satisfy Assumption (H). Then in a neighborhood $\Gamma \subset \partial M$ of $p$, the singular points of $u(x, t_0 \pm \varepsilon)$ are nondegenerate for any $\varepsilon > 0$ sufficiently small. Moreover the following assertions hold true:

1. if the index $\text{ind}(u^0, p)$ is not an integer, then each of $u(x, t_0 \pm \varepsilon)$ has only one singular point on $\Gamma \subset \partial M$, and
2. if the index $\text{ind}(u^0, p)$ is an integer, then one of $u(x, t_0 \pm \varepsilon)$ has two singular points on $\Gamma$, and the other one has no singular points on $\Gamma$.

Theorem 2.5. [2] Let $u \in C^1([0, T], X)$ satisfy Assumption (H). We have

1. $u(x, t)$ has a bifurcation in its local structure at $(p, t_0)$, and
2. if $p \in \partial M$ is a unique degenerate singular point of $u^0$ on $\partial M$, then $u(x, t)$ has a bifurcation in its global structure at $t = t_0$.

Now we consider $u \in C^1([0, T], B_0^r(TM))$ ($r \geq 2$). Technically, the key ingredient is to study the normal derivative of the original field. We first modify Assumption (H) above as follows:
Assumption (H₀). Let \( x₀ \in Γ \) be an isolated degenerate \( ∂ \)-singular point of \( u^0 \), and \( u^0 \in C^{k+1} \) near \( x₀ \in Γ \) for some \( k \geq 2 \). Assume that
\[
\frac{∂u^0}{∂n} = 0,
\]
\( n ≠ -1, \)
\[
\frac{∂^{k+1}u^0_r(p)}{∂τ∂n} ≠ 0,
\]
\[
\frac{∂u^0_τ(p)}{∂n} ≠ 0,
\]
where \( n \) is the interior orbits of \( u^0(x) \) connected to \( x₀ \in ∂M \).

Theorem 2.6. [3] Let \( u \in C^1([0, t], B^0_r(TM)) \) satisfy Assumption (H₀). Then in a neighborhood \( Γ \subset ∂M \) of \( x₀ \), the \( ∂ \)-singular points of \( u(⋅, t₀ ± ε) \) are nondegenerate for \( ε > 0 \) sufficiently small. Moreover, we have
1. if the number \( n \) is even, then one of \( u(⋅, t₀ ± ε) \) has two \( ∂ \)-singular points on \( Γ \), and the other one has no \( ∂ \)-singular point on \( Γ \), and
2. if \( n = odd \), then each of \( u(⋅, t₀ ± ε) \) has only one \( ∂ \)-singular point on \( Γ \).

Theorem 2.7. [3] Let \( u \in C^1([0, T], B^0_r(TM)) \) satisfy Assumption (H₀). Then
1. \( u \) has a bifurcation in its local structure at \( (x₀, t₀) \), and
2. if \( x₀ \in ∂M \) is a unique degenerate \( ∂ \)-singular point of \( u^0 \) on \( ∂M \), then \( u \) has a bifurcation in its global structure at \( t₀ \).


3.1. Bifurcation time and location. In this section, we study the relationship between the bifurcation points of solutions of the following Navier–Stokes equations and the given data, e.g. the initial values, the Reynold number, etc.

Consider the Navier-Stokes equations on \( M \) given by
\[
\begin{aligned}
\frac{∂u}{∂t} + (u \cdot ∇)u - R^{-1}Δu + ∇p &= λf, \\
\text{div} u &= 0, \\
u|_{∂M} &= 0, \\
u(x, 0) &= ϕ(x),
\end{aligned}
\]
with \( ϕ|_{∂M} = 0 \).

The main theorem in this section is as follows.

Theorem 3.1. If \( (x₀, t₀) \) \( (x₀ \in ∂M, t₀ ≥ 0) \) is a bifurcation point of the solution \( u \) of (3.1), then
\[
\frac{∂ϕ_τ(x₀)}{∂n} = \int_0^{t₀} \left\{ R^{-1}[∇ × Δu - kΔu \cdot τ] + λ(∇ × f - kf_τ) \right\}(x₀, t)dt,
\]
where \( ∇ × Δu = -\frac{∂}{∂x}(Δu \cdot τ) + \frac{∂}{∂τ}(Δu \cdot n) \) is the vorticity of \( Δu \), and \( k(x₀) \) is the curvature of \( ∂M \) at \( x₀ \).

We remark here that (3.2) linking the separation location and time with the Reynolds number, the external forcing, boundary curvature, and the initial velocity field. It will useful in both theoretical and engineering future studies of the
boundary layer separation. For simplicity, we call (3.2) separation equation of the Navier-Stokes equation.

**Proof.** For simplicity, we prove only the case where \( f = 0 \). From the Navier–Stokes equations (3.1), we have

\[
\int_0^{t_0} \frac{d}{dt} \left[ \frac{\partial u_r(x_0,t)}{\partial n} \right] dt = \int_0^{t_0} \left[ R^{-1} \frac{\partial \Delta u \cdot \tau}{\partial n} - \frac{\partial p}{\partial n} - \frac{\partial (u \cdot \nabla) u \cdot \tau}{\partial n} \right] (x_0,t) dt.
\]

By assumption, \( (x_0,t_0) \) is a bifurcation point of \( u \), then \( x_0 \) is a \( \partial \)-singular point of \( u \). Hence, it follows from (3.3) that

\[
\frac{\partial \varphi \tau}{\partial n} (x_0) = - \int_0^{t_0} \left[ R^{-1} \frac{\partial \Delta u \cdot \tau}{\partial n} - \frac{\partial p}{\partial n} - \frac{\partial (u \cdot \nabla) u \cdot \tau}{\partial n} \right] dt.
\]

Observe that

\[
(u \cdot \nabla) u \cdot \tau = (u_r \frac{\partial u}{\partial \tau} + u_n \frac{\partial u}{\partial n}) \cdot \tau = u_r \frac{\partial u_r}{\partial \tau} - ku_r^2 + u_n \frac{\partial u_r}{\partial n} - u_n u \frac{\partial \tau}{\partial n},
\]

which implies, using the Dirichlet boundary condition and the incompressibility \( \partial u_n/\partial n = -\partial u_r/\partial \tau = 0 \) on \( \Gamma \), that

\[
\frac{\partial (u \cdot \nabla) u \cdot \tau}{\partial n} = 0 \quad \text{on } \partial M.
\]

For simplicity, let \( (x_1,x_2) \) be the coordinating system with \( x_0 \in \partial M \) at the origin, \( x_1 \)-axis pointing to the tangential direction, and \( x_2 \)-axis pointing to the normal direction. Then at the origin, \( \tau = (\tau_1, \tau_2) = (1,0) \) and \( n = (n_1, n_2) = (0,1) \). Hence

\[
\frac{\partial}{\partial n} \frac{\partial p}{\partial \tau} = n_1 \frac{\partial}{\partial x_1} \left( \tau_1 \frac{\partial p}{\partial x_1} + \tau_2 \frac{\partial p}{\partial x_2} \right) + n_2 \frac{\partial}{\partial x_2} \left( \tau_1 \frac{\partial p}{\partial x_1} + \tau_2 \frac{\partial p}{\partial x_2} \right)
\]

\[
= \frac{\partial^2 p}{\partial x_1 \partial x_2} + \frac{\partial \tau_1}{\partial x_2} \frac{\partial p}{\partial x_1} + \frac{\partial \tau_2}{\partial x_2} \frac{\partial p}{\partial x_2}
\]

\[
= \frac{\partial^2 p}{\partial x_1 \partial x_2}.
\]

Notice that

\[
\frac{\partial n}{\partial \tau} (x_0) = (k(x_0), 0),
\]

where \( k(x_0) \) is the curvature of \( \partial M \) at \( x_0 \). Hence we have

\[
\frac{\partial}{\partial \tau} \frac{\partial p}{\partial n} = \frac{\partial^2 p}{\partial x_1 \partial x_2} + \frac{n_1}{\partial x_1} \frac{\partial p}{\partial x_1} + \frac{n_2}{\partial x_2} \frac{\partial p}{\partial x_2}
\]

\[
= \frac{\partial^2 p}{\partial x_1 \partial x_2} + \frac{n_1}{\partial x_1} \frac{\partial p}{\partial \tau} + \frac{n_2}{\partial x_2} \frac{\partial p}{\partial \tau}
\]

\[
= \frac{\partial}{\partial n} \frac{\partial p}{\partial \tau} + k(x_0) \frac{\partial p}{\partial \tau}.
\]

Therefore (3.4) implies that

\[
\frac{\partial \varphi \tau}{\partial n} (x_0) = - \int_0^{t_0} \left[ R^{-1} \frac{\partial (\Delta u \cdot \tau)}{\partial n} - \frac{\partial p}{\partial \tau} - k(x_0) \frac{\partial p}{\partial \tau} \right] dt.
\]
Finally the theorem follows from the following observation

\[ \frac{\partial p}{\partial \tau} = R^{-1} \Delta u \cdot \tau, \quad \frac{\partial p}{\partial n} = R^{-1} \Delta u \cdot n, \quad \text{on } \partial M. \]

The proof is complete. \(\square\)

**Remark 3.1.** Equation (3.2), \(n\) is the outward normal vector, and \((\tau, n)\) forms a right hand frame. Thus, the curvature \(k(x_0) > 0\) (resp. \(k(x_0) < 0\)) implies that the boundary \(\partial M\) is convex (resp. concave) at \(x_0\).

**Remark 3.2.** Equation (3.2) provides a relationship between the bifurcation point \((x_0, t_0)\) \((x_0 \in \partial M, t_0 \geq 0)\) and the given data, i.e. the initial value \(\varphi\), the external forcing \(f\) and the Reynold number \(R\).

It is easy to see that the formula (3.2) is also a sufficient condition for \(x_0\) being a \(\partial\)–singular point of the solution \(u\) of (3.1). As seen in the proof of the Theorem, a singular point with integer index must be a bifurcation point. Thus, we immediately derive a sufficient condition of structural bifurcation of solutions for the Navier-Stokes equations with the Dirichlet boundary conditions (3.1) as follows.

**Theorem 3.2.** Assume that \(\frac{\partial \varphi}{\partial n} \neq 0\) on \(\Gamma\), where \(\Gamma \subset \partial M\) is a connected component of \(\partial M\). If \((x_0, t_0)\) with \(x_0 \in \Gamma, t_0 > 0\) satisfies the formula (3.2) and

\[
\left\{ \begin{array}{l}
\frac{\partial u_{\tau}(x,t)}{\partial n} \neq 0 \quad \forall \ x \in \Gamma \text{ and } 0 \leq t < t_0, \\
[R^{-1}(\nabla \times \Delta u + k\Delta u \cdot \tau) + \lambda(\nabla \times f + kf_{\tau})](x_0, t_0) \neq 0,
\end{array} \right. \tag{3.9}
\]

then \((x_0, t_0)\) must be a bifurcation point of the solution \(u\) of (3.1) in its global structure.

**Remark 3.3.** If we assume \(\Gamma \subset \partial M\) is an open (connected) portion of \(\partial M\), then \((x_0, t_0)\) must be a bifurcation point of the solution \(u\) of (3.1) in its local structure.

**Remark 3.4.** An important question is to establish an asymptotic relationship between the bifurcation time \(t_0\) and the Reynolds number \(R\) for the Navier-Stokes equations (3.1).

4. Structural Bifurcation Driven by Forcing. In this section, we study structural bifurcation of the solutions of the Navier–Stokes equations (3.1) with the (homogeneous) Dirichlet boundary conditions generated by the initial data and forcing.

For this purpose, let \(\varphi \in B^{2+\alpha}_0(TM), f \in D^\alpha(TM) (0 < \alpha < 1),\) and \(u \in C^1([0, \infty), B^{2+\alpha}_0(TM))\) be a solution of (3.1).

**Theorem 4.1.** Let \(\Gamma \subset \partial M\) be a connected component of \(\partial M\), and

\[ \frac{\partial f_{\tau}}{\partial n}, \frac{\partial \varphi_{\tau}}{\partial n} < 0, \quad \text{on } \Gamma. \]

Then, there is a constant \(\lambda_0 > 0\) depending on \(\|f\|_{C^1}, \|\varphi\|_{C^3}\) and the Reynolds number \(R\), such that for any \(\lambda > \lambda_0\), \(u\) has a bifurcation at \((x_0, t_0)\) in its global structure with \(t_0 > 0\) and \(x_0 \in \Gamma\). Consequently, \(x_0\) is a \(\partial\)–singular point of \(u^0\).

Furthermore, if \(x_0\) is an isolated \(\partial\)–singular point of \(u(\cdot, t_0)\), then the number of orbits connected to \(x_0\) in \(\hat{M}\) is even.
Proof. For simplicity, let the orientation of \( \varphi \) be the same as the tangent vector \( \tau \) on \( \Gamma \). The solution \( u \) of (3.1) can be Taylor expanded at \( t = 0 \) as follows 

\[
u(x, t) = \varphi + t\lambda f - t(\varphi \cdot \nabla)\varphi + tR^{-1}\Delta \varphi - t\nabla p_0 + g(x, t, \lambda), \tag{4.1}\]

Here \( g \) is given by \(^1\)

\[
g(x, t; \lambda) = \sum_{k=2}^{\infty} \frac{t^k}{k!} \frac{\partial^k u(x, 0, \lambda)}{\partial t^k}.
\]

By definition, we have

\[
\frac{\partial u(x, 0, \lambda)}{\partial t} = \lambda f - (\varphi \cdot \nabla)\varphi + R^{-1}\Delta \varphi - \nabla p_0,
\]

which implies that

\[
\frac{\partial u(x_0, 0, \lambda)}{\partial t} = c_1\lambda + c_0,
\]

for some constants \( c_0 \) and \( c_1 \). By induction, it is easy to see that \( \frac{\partial^k u(x_0, 0, \lambda)}{\partial t^k} \) is a polynomial of \( \lambda \) of degree \( k - 1 \) for \( k \geq 2 \). Hence, we obtain that

\[
|g(x_0, t, \lambda)| \leq t \sum_{k=2}^{\infty} \frac{(t\lambda)^k C_k}{k!},
\]

which implies that for any constant \( K \), we have

\[
g(x_0, t, \lambda) \to 0 \quad \text{as} \quad t\lambda = K, t \to 0, \lambda \to \infty. \tag{4.2}\]

By assumption, from (4.1) we derive that on \( \Gamma \),

\[
\frac{\partial u}{\partial n} \cdot \tau = \left| \frac{\partial \varphi}{\partial n} \right| - t\lambda \left| \frac{\partial f}{\partial n} \right| - t \frac{\partial}{\partial n} \left[ (\varphi \cdot \nabla)\varphi - R^{-1}\Delta \varphi + \nabla p_0 \right] \cdot \tau + \frac{\partial g}{\partial n} \cdot \tau. \tag{4.3}\]

Since \( u|_{\partial M} = 0 \) and \( \text{div} \ u = 0 \), we have

\[
\frac{\partial u}{\partial n} \cdot n = \frac{\partial u}{\partial n} \cdot \tau = 0.
\]

Hence a zero point of \( \partial u/\partial n \) on \( \Gamma \) is a \( \partial \)-singular point of \( u \). We infer from (4.2) and (4.3) that there exists a \( \lambda_0 > 0 \) depending on the modules \( \|f\|_{C^1}, \|\varphi\|_{C^3} \) and the Reynolds number \( R \), such that for any \( \lambda > \lambda_0 \) there is \( t_0 > 0 \) satisfying

\[
\begin{aligned}
&\frac{\partial u_r(x, t)}{\partial n} > 0, \quad 0 \leq t < t_0 \quad \text{and} \quad x \in \Gamma, \\
&\frac{\partial u_r(x_0, t_0)}{\partial n} = 0, \quad \text{for some} \quad x_0 \in \Gamma,
\end{aligned} \tag{4.4}\]

and

\[
\frac{\partial}{\partial t} \frac{\partial u_r(x_0, t_0)}{\partial n} = -\lambda \left| \frac{\partial f_r(x_0)}{\partial n} \right| - \frac{\partial}{\partial n} \left[ (\varphi \cdot \nabla)\varphi - R^{-1}\Delta \varphi + \nabla p_0 - \frac{t_0}{2} \cdot g \right] \cdot \tau < 0. \tag{4.5}\]

\(^1\)This Taylor expansion is ensured by the time-analyticity of \( u \) for positive time. Hence without loss of generality, we can assume \( u \) is also time-analytic at \( t = 0 \) otherwise, we need only to consider the expansion at \( t = \delta \) for an arbitrary small \( \delta > 0 \).
Then by Theorem 2.5 and (4.4) and (4.5), it is easy to show that $t_0$ is a structural bifurcation point in its global structure.

By the homotopy invariance of index sum of vector fields in an open set, as $x_0 \in \Gamma$ is an isolated singular point of $u(\cdot, t_0)$, we have

$$\text{ind}(u(\cdot, t_0), x_0) = \sum_i \text{ind}(u(\cdot, t_0 - \epsilon), z_i), \quad z_i \in U$$

for any $\epsilon > 0$ sufficiently small, where $U \subset M$ is a neighborhood of $x_0 \in \Gamma$ and $u(\cdot, t_0)$ has no singular point in $U \cap \hat{M}$. Since $u(\cdot, t_0 - \epsilon)$ has no boundary singular points in $\Gamma$, the index sum of $\sum_i \text{ind}(u(\cdot, t_0 - \epsilon), z_i)$ must be an integer.

The proof is complete.

5. Boundary Layer Separation.

5.1. Structural Bifurcation in Integer Index. By Theorem 2.3, the index $\text{ind}(u, x_0) = -n/2$ for $u \in D^\nu(TM)$ and $x_0 \in \partial M$ implies that there are $n$ orbits of $u$ in $M$ connected to $x_0$. For a vector field $u \in B^\nu_0(TM)$ and a $\partial$–singular point $x_0 \in \partial M$, we define the index

$$\text{ind}(u, x_0) = -\frac{n}{2}, \quad n \geq 0 \text{ an integer, } x_0 \in \partial M, \ u \in B^\nu_0(TM)$$

which amounts to saying that there are $n$ orbits of $u$ in $\hat{M}$ connected to $x_0 \in \partial M$.

Let $u \in C^1([0, T], B^\nu_0(TM))$ be a solution of (3.1). From the structural stability theorems and the structural bifurcation theorems in Section 2, we know that $(x_0, t_0)$ $(x_0 \in \partial M)$ is a bifurcation point of $u$ implies that $\text{ind}(u(\cdot, t_0), x_0) \neq -\frac{1}{2}$. In fluid dynamics, the structural bifurcation often occurs in the case with integer index as demonstrated in the theorems in the previous sections. Actually, in the real world, we can only observe the structural bifurcation of fluid flows whose index is either 0 or $-1$. The structural bifurcation from index zero is well known as the boundary layer separation. We proceed now with some schematic pictures characterizing the structural bifurcation when the index is either 0 or $-1$.

**Case 1. (Boundary layer separation.)** $\text{ind}(u(\cdot, t_0), x_0) = 0$.

*Figure 5.1. Boundary layer separation of shear flow*

In this case, the bifurcation occurs as shown in Figure 5.1. Figure 5.1(a) shows that the typical shear flow, and $u$ has no singular point near $x_0$ for $t < t_0$. The flow pattern for $u^0 = u(\cdot, t_0)$ given in Figure 5.1(b) has an isolated degenerate $\partial$–singular point $x_0 \in \partial M$. When $t > t_0$, the flow pattern given by Figure 5.1(c) has that $u(x, t)$ bifurcates from $x_0 \in \partial M$ one vortex or multi-vortices. Although solutions with either a single vortex or multiple vortices may occur, the case given by Figure 5.1(c) with one vortex is generic.
Case 2. \( \text{ind}(u(\cdot, t_0), x_0) = -1. \)

There are two types of flow transitions in this case, which are shown respectively in Figure 5.2 and Figure 5.3.

\[ \text{Figure 5.2. A type of structural bifurcation for the case of } \text{ind}(u(\cdot, t_0), x_0) = -1. \]

\[ \text{Figure 5.3. Another type of structural bifurcation for the case of } \text{ind}(u(\cdot, t_0), x_0) = -1. \]

The above two cases can be classified in the following theorem.

**Theorem 5.1.** Let \( X = D^+(TM) \) or \( B^+(TM) \) or \( B^0(TM) \), \( u \in C^1([0, T), X) \), \( x_0 \in \partial M \) be a singular (or \( \partial \)-singular) point of \( u \), and \( 0 < t_0 < T \). We have the following assertions:

1. As \( \text{ind}(u(\cdot, t_0), x_0) = 0 \), the type of structural bifurcation of \( u \) at \( (x_0, t_0) \) is unique, called the boundary layer separation, which is shown as in Figure 5.1. In other words, there are some vortices bifurcated from \( x_0 \in \partial M \), which are enclosed by orbit lines connecting to both bifurcated boundary saddle points \( x^- \) and \( x^+ \).

2. As \( \text{ind}(u(\cdot, t_0), x_0) = -1 \), there are only two types of structural bifurcation of \( u \) at \( (x_0, t_0) \), which are respectively shown as in Figure 5.2 and Figure 5.3.

**Remark 5.1.** The types of structural bifurcation mentioned in the above theorem are defined in the sense of the manner connecting the two bifurcated boundary saddle points, and not in the sense of topological equivalence.

The proof of the above theorem is a direct application of the stability lemma of orbit lines obtained in pp. 118–119 in Ma and Wang [8]. For convenience, we state it as follows. Let \( v \in C^r(TM) \) be a vector field. A curve \( \gamma \subset M \) is called an orbit line of \( v \), if \( \gamma \) is a union of curves

\[ \gamma = \bigcup_{i=1} \gamma_i \]
Lemma 5.1. (Stability of Orbit Lines) Let $\omega^n \in C'(TM)$ ($r \geq 1$) be a sequence of vector fields with $\lim_{n \to \infty} \omega^n = \omega \in C'(TM)$. Assume that $\gamma^n \subset M$ is an orbit line of $\omega^n$, and the starting points $p^n_1$ of $\gamma^n$ converge to $p_1$. Then the orbit line $\gamma$ of $\omega$ converges to an orbit line $\gamma$ of $\omega$ with starting point $p_1$.

Theorem 5.2. Let $u \in C^1([0,T], X)$, $x_0 \in \partial M$ be a degenerate singular (or $\partial$-singular) point of $u$ at $t = t_0$ ($0 < t_0 < T$). Let $u^0(x) = u(x, t_0)$ and $u^1(x) = \partial u(x, t_0)/\partial t$. We start with the following assumptions:

$$\begin{align*}
\frac{\partial^2 u^0_1(x_0)}{\partial \tau^2} &\neq 0, \quad \frac{\partial u^0_2(x_0)}{\partial n} \neq 0, \quad u^1_2(x_0) \neq 0, \quad \text{if} \ X = D'(TM), \quad (5.1) \\
\frac{\partial^2 u^0_2(x_0)}{\partial \tau^2} &\neq 0, \quad \frac{\partial^2 u^0_2(x_0)}{\partial n^2} \neq 0, \quad u^1_2(x_0) \neq 0, \quad \text{if} \ X = B'(TM), \quad (5.2) \\
\frac{\partial^3 u^0_2(x_0)}{\partial \tau^2 \partial n} &\neq 0, \quad \frac{\partial^2 u^0_2(x_0)}{\partial n^2} \neq 0, \quad \frac{\partial u^1_2(x_0)}{\partial n} \neq 0, \quad \text{if} \ X = B'_0(TM), \quad (5.3)
\end{align*}$$

Remark 5.2. By assumption, $x_0 \in \partial M$ is a degenerate singular (or $\partial$-singular) point of $u \in C^1([0,T], X)$ at $t = 0$. Thus, we have

$$\frac{\partial u^0_2(x_0)}{\partial \tau} = 0 \quad \left( \text{or} \quad \frac{\partial^2 u^0_2(x_0)}{\partial \tau \partial n} = 0 \right).$$

Hence, the above conditions are generic for a bifurcation point $(x_0, t_0)$ of $u \in C^1([0,T], X)$ with $(x_0) \in \partial M$ and $0 < t_0 < T$.

The following theorem determines the type of structural bifurcations.

Theorem 5.2. Let $u \in C^1([0,T], X)$, $n$ be an inward normal vector and $(\tau, n)$ form a right hand frame. Let the corresponding condition $(5.1)$, $(5.2)$ or $(5.3)$ hold. Then we have the assertions:

1. When the following corresponding condition is satisfied

$$\begin{align*}
\text{sign} \frac{\partial^2 u^0_2(x_0)}{\partial \tau^2} &u^0_2(x_0) = \text{sign} \frac{\partial u^0_2}{\partial n} u^0_1(x_0), \quad \text{if} \ X = D'(TM), \\
\text{sign} \frac{\partial^2 u^0_2(x_0)}{\partial \tau^2} &u^0_2(x_0) = \text{sign} \frac{\partial^2 u^0_2}{\partial n^2} u^0_1(x_0), \quad \text{if} \ X = B'(TM), \\
\text{sign} \frac{\partial^3 u^0_2(x_0)}{\partial \tau^2 \partial n} &u^0_2(x_0) = \text{sign} \frac{\partial^2 u^0_2}{\partial n^2} u^0_1(x_0), \quad \text{if} \ X = B'_0(TM),
\end{align*}$$

then the structural bifurcation of $u$ at $(x_0, t_0)$ is of the type of $\text{ind}(u^0, x_0) = 0$, i.e. the boundary layer separation, and the vortex separated from $x_0 \in \partial M$ is unique as shown in Figure 5.1(c).

2. When the corresponding condition in (5.4) is not valid, then the structural bifurcation of $u$ is of the type of $\text{ind}(u^0, x_0) = -1$, whose portrait of flows is as shown in Figure 5.2 or Figure 5.3.
Proof. We prove only the cases where \( u \in C^1([0, T], D'(TM)) \) or \( u \in C^1([0, T], B'(TM)) \), and the third case can be proved in the same fashion.

Without loss of generality, we assume that \( x_0 \in \partial M \) has a flat neighborhood \( \Gamma \subset \partial M \). We take a coordinate system \((x_1, x_2)\) with \( x_0 \) at the origin, and \( \Gamma = \{(x_1, 0) \mid |x_1| < \delta\} \), \( x_2\)-axis orienting the inward normal direction. By assumption, \( \partial u_1/\partial x_1 = \partial u_1/\partial x_2 \) and \( \partial u_2/\partial x_1 = \partial u_2/\partial x_2 \), etc.

The case of \( u \in C^1([0, T], D'(TM)) \). From (5.1) the vector field \( u^0 \) and \( u^1 \) have the Taylor expansion as follows:

\[
\begin{align*}
\begin{cases}
  u_1^0(x) = C_1 x_2 + C_2 x_1^2 + C_3 x_1 x_2 + o(|x|^2), & C_1, C_2 \neq 0, \\
  u_2^0(x) = -2C_2 x_1 x_2 + x_2 \cdot o(|x|),
\end{cases}
\end{align*}
\]

(5.5)

and

\[
\begin{align*}
\begin{cases}
  u_1^1(x) = \beta + O(|x|), & \beta \neq 0, \\
  u_2^1(x) = x_2 \cdot O(|x|).
\end{cases}
\end{align*}
\]

(5.6)

Without loss of generality, we assume that \( C_2 > 0 \) and \( \beta < 0 \). Obviously \( u(x, t_0 - \varepsilon) = u^0(x) - \varepsilon u^1(x) + o(|\varepsilon|) \) has no singular point on \( \Gamma = \{(x_1, 0) \mid |x_1| < \delta\} \) for \( \delta > 0 \) and \( \varepsilon > 0 \) sufficiently small, and \( u(x, t_0 + \varepsilon) = u^0(x) + \varepsilon u^1(x) + o(|\varepsilon|) \) has exactly two singular points on \( \Gamma \) near \( x_0 \). Hence we have that \( \text{ind}(u^0, x_0) = -n \) is integer.

When \( \text{ind}(u^0, x_0) = -n \), there are exactly \( 2n \) orbits of \( u^0 \) in \( \tilde{M} \) connecting to \( x_0 \).

Hence \( u_1^0(x) \) has at least \( 2n \) zero points for each \( x_2 > 0 \) sufficiently small because the sign of \( u_1^0(x) \) changes near \( x_0 \) \((x = 0)\) at least \( n \) times. Hence, as sign \( C_1 = \text{sign} C_2 \) i.e. \( C_1 > 0 \), from (5.5) we infer that \( u_1^0(x) \) has no zero point near \( x_1 = 0 \) for any \( x_2 > 0 \) sufficiently small. Thus we verify that \( \text{ind}(u^0, x_0) = 0 \).

From (5.5) and (5.6) we see that the interior singular points \((\tilde{x}_1, \tilde{x}_2)\) of \( u(x, t_0 + \varepsilon) \) with \( \tilde{x}_2 > 0 \) satisfying the equation

\[
\begin{align*}
\begin{cases}
  C_1 x_2 + C_2 x_1^2 + C_3 x_1 x_2 + \varepsilon \beta + o(|\varepsilon|, |x_2|, |x_1|^2) = 0, \\
  -2C_2 x_1 + \varepsilon \cdot O(|x|) + o(|\varepsilon|, |x|) = 0.
\end{cases}
\end{align*}
\]

(5.7)

From the implicit function theorem it follows that the solution \((\tilde{x}_1, \tilde{x}_2)\) of (5.7) with \( \tilde{x}_2 > 0 \) sufficiently small, if it exists, is unique for any \( \varepsilon > 0 \) sufficiently small.

And the existence of (5.7) can be derived by the invariance of index sum in a neighborhood of \( x_0 \) and Theorem 2.4. Thus, the first conclusion is proven.

When \( C_1 < 0 \), it is easy to see that \( u_1^0(x) = 0 \), i.e. the equation below has exactly two solutions near \( x_1 = 0 \) for any \( x_2 > 0 \) sufficiently small

\[
C_2 x_1^2 + C_4 x_2 + C_3 x_1 x_2 + o(|x|^2) = 0, \quad C_2 > 0, C_1 < 0.
\]

Moreover the domain

\[
D_\varepsilon = \{x \mid u_1^0(x) < 0, |x| < \varepsilon\} \neq \emptyset, \quad \forall \varepsilon > 0 \text{ sufficiently small}.
\]

It implies that there are exactly two orbits of \( u^0 \) in \( \tilde{M} \) connecting to \( x_0 \). Hence \( \text{ind}(u^0, x_0) = -1 \). The conclusion for \( u \in C^1([0, T], D'(TM)) \) is proven.

The case of \( u \in C^1([0, T], B'(TM)) \). By (5.3) we obtain that

\[
\begin{align*}
\begin{cases}
  u_1^0(x) = \alpha_1 x_1^2 + \alpha_2 x_2^2 + o(|x|^2), \\
  u_2^0(x) = -2\alpha_1 x_1 x_2 + x_2 \cdot o(|x|)
\end{cases}
\end{align*}
\]

(5.8)
and
\[
\begin{align*}
  u_1^1(x) &= \beta_0 + \beta_1 x_1 + O(|x|) \\
  u_2^1(x) &= -\beta_1 x_2 + x_2 \cdot O(|x|).
\end{align*}
\]
(5.9)

Assume that $\alpha_1 < 0$, $\beta_0 < 0$. In the same fashion as above, we can verify that
\[
\text{ind}(u^0, x^0) = \begin{cases} 
  0, & \text{as } \alpha_2 > 0 \\
  -1, & \text{as } \alpha_2 < 0.
\end{cases}
\]

We have only to prove that the equations below have only one solution near $x = 0$ for any $\varepsilon > 0$ sufficiently small
\[
\begin{align*}
  \alpha_1 x_1^2 + \alpha_2 x_2^2 + \varepsilon \beta_0 + o(|\varepsilon|, |x|^3) &= 0 \\
  -2\alpha_1 x_1 - \varepsilon \beta_1 + \varepsilon \cdot O(|\varepsilon|) + O(|x|^2) &= 0 \\
  x_2 > 0, \quad \alpha_1 > 0, \quad \alpha_2 > 0, \quad \beta < 0.
\end{align*}
\]
(5.10)

The equation (5.10) is equivalent to the equation
\[
\begin{align*}
  \alpha_2 x_2^2 + \varepsilon \beta_0 + \alpha_1 g^2(x_2) + o(|\varepsilon|, |x_2|^3) &= 0 \\
  g(x_2) &= -\frac{\beta_1}{2\alpha_1} + \varepsilon \cdot O(|x_2^2|) + O(|x_2|^2) \\
  x_2 > 0, \quad \alpha_2 > 0, \quad \beta < 0.
\end{align*}
\]
(5.11)

The existence and uniqueness of (5.11) is obvious. Thus, this theorem is proven.

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