Interior structural bifurcation and separation of 2D incompressible flows

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We study transitions in the topological structure of a family of divergence-free vector fields $u(\cdot,t)$ near an interior point. It is shown that structural bifurcation occurs at $t_0$ if $u(\cdot,t_0)$ has an isolated degenerate singular point $x_0 \in M$ with zero index and nonzero Jacobian at $x_0$, and with nonzero acceleration in the direction normal to the (unique) eigenspace of the Jacobian. This result is carried out by analyzing the orbit structure of $u$ near such an isolated degenerate interior singular point of $u(\cdot,t_0)$. Applications to typical interior separation phenomena in two-dimensional fluid flows are addressed as well. © 2004 American Institute of Physics. [DOI: 10.1063/1.1689005]

I. INTRODUCTION

The main objective of this paper is to derive a rigorous kinematic theory for the typical interior separation phenomena in two-dimensional fluid flows. This is part of a research program on the use of topological ideas to study the spatial–temporal structure of 2D incompressible fluid flows in physical space, along with its stability and bifurcations. This program consists of research in two areas: (a) the study of the topological structure of divergence-free vector fields, and its evolution in time or with respect to an arbitrary parameter, and (b) the study of the structure and evolution of velocity fields for 2D incompressible fluid flows governed by a class of equations that comprises the Navier–Stokes equations, the Euler equations, and the quasigeostrophic equations of rotating flows. The objectives of this research program are consistent with the program by Newton and his collaborators, using in particular the vorticity in their analysis; see Ref. 6 and the reference therein.

In this paper, we address structural bifurcation for a family of divergence-free vector fields $u(\cdot,t)$ near an interior point. Structural bifurcation near boundary singular points was carried out by the authors in collaboration with Michael Ghil, leading to a rigorous characterization for boundary layer separation for 2D incompressible fluid flows; see Refs. 2 and 3 and the survey article 5 for details.

More precisely, we shall show that structural bifurcation occurs at $t_0$ if $u(\cdot,t_0)$ has an isolated degenerate singular point $x_0 \in M$ with zero index and nonzero Jacobian at $x_0$, and with nonzero acceleration in the direction normal to the (unique) eigenspace of the Jacobian.

Technically speaking, the main results are carried out by analyzing the orbit structure of $u$ near such an isolated degenerate interior singular point of $u(\cdot,t_0)$. We now summarize the analysis. For this purpose, we consider the Taylor expansion of the divergence-free vector fields $u(\cdot,t)$ at $t_0$ ($0 < t_0 < T$)

$$u(x,t) = u^0(x) + (t-t_0)u^1(x) + o(|t-t_0|),$$

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First we observe by the structural stability theorem, Theorem 2.4, that structural bifurcation can only occur near a degenerate singular point $x_0$, i.e., the determinant of the Jacobian of $u$ at $x_0$ is zero: $\det Du(x_0) = 0$. Generically, we only have to examine then the case where Jacobian itself is not zero: $Du(x_0) \neq 0$; see Theorem 5.5.

Second, further analysis shows that a degenerate singular point $x_0 \in \bar{M}$ of $u \in D'(TM)$ ($r \geq 1$) with nonzero Jacobian $Du(x_0) \neq 0$ can only be one of the three cases:

1. a degenerate center,
2. a degenerate saddle such that the four orbits connected to $x_0$ are tangent to each other at $x_0$, and
3. a point with $\text{ind}(u, x_0) = 0$ such that the angle between the two orbits connected to $x_0$ is zero.

Again, the genericity result given in Theorem 5.5 shows that we only have to consider the structural bifurcation in the third case, i.e., near an interior degenerate singular point such that $\text{ind}(u, x_0) = 0$ and the angle between the two orbits connected to $x_0$ is zero.

Third, when $Du(x_0) \neq 0$ but $\det Du(x_0) = 0$, there is a unique eigendirection of $Du(x_0)$ corresponding to the zero eigenvalue. Let $e_1$ be the eigenvector, and $e_2$ is orthogonal to $e_1$. Then we are able to derive the main theorems of this paper, Theorems 4.4 and 4.5. In addition to detailed characterization of the structural transition near the bifurcation point, in particular, we are able to show that structural bifurcation occurs at $(t_0, x_0)$ if $x_0$ is an isolated degenerate singular point of $u^0(x)$, and

\[ \text{ind}(u^0, x_0) = 0, \]
\[ Du^0(x_0) \neq 0, \]
\[ u^1(x_0) \cdot e_2 \neq 0. \]

Fourth, the bifurcation obtained in these two theorems corresponds directly to typical interior separation phenomena in two-dimensional fluid flows, and is generic; see Theorems 5.2 and 5.5.

The paper is organized as follows. In Sec. II, we recall some preliminaries, including a structural stability theorem, and a singularity classification theory for 2D divergence-free vector fields. Section III classifies interior degenerate singular points, and identifies some useful and easy to verify kinematic conditions. Section IV states and proves the main structural bifurcation theorems. Genericity and connections to interior separation phenomena of fluid flows are given in Sec. V.

II. PRELIMINARIES

We first recall some basic facts and definitions on structural stability and bifurcation of divergence-free vector fields. Let $M \subset \mathbb{R}^2$ be a closed and bounded domain with $C^r$ ($r \geq 1$) boundary $\partial M$. Let $TM$ be the tangent bundle of $M$, and $C'(TM)$ be the space of all $C^r$ vector fields on $M$. Let

\[ D'(TM) = \{ v \in C'(TM) \mid v_n|_{\partial M} = 0, \ \text{div} v = 0 \}, \]

where $n$ is the unit outward normal vector on $\partial M$ and $v_n = v \cdot n$. 
The structural stability theorems of divergence-free vector fields with various boundary conditions, obtained by the authors in Ref. 4, are useful in the study of structural bifurcation. Here, we only introduce the structural stability theorem in $D'(TM)$ (for divergence-free vector fields with no-normal flows).

Definition 2.1: Two vector fields $u,v \in D'(TM)$ are called topologically equivalent in $D'(TM)$ if there exists a homeomorphism of $\varphi : M \to M$, which maps orbits of $u$ to orbits of $v$ and preserves their orientation.

Definition 2.2: A vector field $v \in D'(TM)$ is structurally stable if it is connected only to a finite number of orbits and the stable and unstable orbits connected to $p$ are connected.

Definition 2.3: Let $u \in C^1([0,T],D'(TM))$. We say that $u(x,t)$ has a bifurcation in its local structure in a neighborhood $U \subset M$ of $x_0$ at $t_0$ $(0 < t_0 < T)$ if, for any $t^- < t_0$ and $t_0 < t^+$ with $t^-$ and $t^+$ sufficiently close to $t_0$, the vector fields $u(\cdot, t^-)$ and $u(\cdot, t^+)$ are not topologically equivalent locally in $U \subset M$, and we say that $u(\cdot, t)$ has a bifurcation at $t_0$ in its global structure if $U = M$.

A point $p \in M$ is called a singular point of $u \in D'(TM)$ if $u(p) = 0$; a singular point $p$ of $u$ is called nondegenerate if the Jacobian matrix $Du(p)$ is invertible; $u$ is called regular if all singular points are nondegenerate. Then the following theorem provides necessary and sufficient conditions for structural stability of a divergence-free vector field in $D'(TM)$.

Theorem 2.4 (Ma and Wang): A divergence-free vector field $u \in D'(TM)$ ($r \geq 1$) is structurally stable in $D'(TM)$ if and only if

1. $u$ is regular;
2. all interior saddle points of $u$ are self-connected; and
3. each saddle point of $u$ on $\partial M$ is connected only to saddle points on the same connected component of $\partial M$.

Moreover, the set of all structurally stable vector fields is open and dense in $D'(TM)$.

From the above definitions and the structural stability theorem, the local structural bifurcation can only occur at degenerate singularity points. For bifurcation near boundary points, we refer the interested reader to Ref. 2. Here in this paper we address the bifurcation near an interior singular point.

For this purpose, let $p \in M$ be an isolated singular point of $v \in C^r_n(TM)$; then

\[ \text{ind}(v,p) = \text{deg}(v,p), \]

where $\text{deg}(v,p)$ is the Brouwer degree of $v$ at $p$.

Let $p \in \partial M$ be an isolated singular point of $v$, and $\bar{M} \subset \mathbb{R}^2$ be an extension of $M$, i.e., $M \subset \bar{M}$ such that $p \in \bar{M}$ is an interior point of $\bar{M}$. In a neighborhood of $p$ in $\bar{M}$, $v$ can be extended by reflection to $\bar{v}$ such that $p$ is an interior singular point of $\bar{v}$, thanks to the no-normal flow condition, i.e., $v \cdot n|_{\partial M} = 0$. Then we define the index of $v$ at $p \in \partial M$ by

\[ \text{ind}(v,p) = \frac{1}{2} \text{ind}(\bar{v},p). \]

Let $p \in M$ be an isolated singular point of $v \in C^r_n(TM)$. An orbit $\gamma$ of $v$ is said to be a stable orbit (respectively, an unstable orbit) connected to $p$, if the limit set $\omega(x) = p$ [respectively, $\alpha(x) = p$] for $x \in \gamma$.

Theorem 2.5 (Ref. 2): Let $p \in M$ be an isolated singular point of $v \in D'(TM)$, $r \geq 1$. Then $p$ is connected only to a finite number of orbits and the stable and unstable orbits connected to $p$ alternate when tracing a closed curve around $p$. Furthermore,

1. when $p \in \bar{M}$, $p$ has $2n$ ($n \geq 0$) orbits, $n$ of which are stable, and the other $n$ unstable, while the index of $p$ is

\[ \text{ind}(v,p) = 1 - n, \]
(2) when \( p \in \partial M \), \( p \) has \( n + 2 \) \((n \geq 2)\) orbits, two of which are on the boundary \( \partial M \), and the index of \( p \) is

\[
\text{ind}(u, p) = -\frac{n}{2}.
\]

### III. INTERIOR DEGENERATE SINGULARITIES

#### A. Characterization of degenerate singularities with nonzero Jacobian

The structural stability theorem (Theorem 2.4) suggests studying the structure of divergence-free vector fields near degenerate singular points. To this end, we now introduce some lemmas characterizing degenerate interior singularities with nonzero Jacobians, which are useful for studying interior structural bifurcation.

**Lemma 3.1:** Let \( u \in D'(TM) \) \((r \geq 1)\), and \( x_0 \in \tilde{M} \) be an isolated singular point of \( u \). If the index \( \text{ind}(u, x_0) \neq 0 \), \(-1\), then the Jacobian matrix

\[
Du(x_0) = 0.
\] (3.1)

**Proof:** By Theorem 2.5, the index of an interior singular point of a divergence-free vector field is determined by the \( 2n \) \((n \geq 0)\) orbits connected to \( x_0 \), i.e., \( \text{ind}(u, x_0) = 1 - n \). Hence, by assumption, \( n \geq 3 \).

Let \( \gamma \) be an orbit of \( u \) connected to \( x_0 \). Let \((x_1, x_2)\) be the orthogonal coordinate system with \( x_0 \) as its origin, and with its \( x_1\)-axis tangent to \( \gamma \) at \( x_0 \). Then \( u \) can be expressed locally by

\[
\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + o(|x|),
\]
(3.2)

\[
Du(x_0) = Du(0) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}.
\]

We shall prove that \( a = b = c = 0 \) in several steps as follows.

**Step 1:** We show that \( a = c = 0 \). By definition, the \( x_1\)-axis is tangent to \( \gamma \) at \( x_0 = 0 \), which yields

\[
\lim_{x \to 0} \frac{u_2(x)}{u_1(x)} = 0.
\] (3.3)

In addition, for \((x_1, x_2) \in \gamma, x_2 = o(|x_1|)\). Hence, we infer from (3.2) and (3.3) that

\[
\lim_{x \to 0} \frac{u_2(x)}{u_1(x)} = \lim_{x \to 0} \frac{cx_1 - ax_2 + o(|x|)}{ax_1 + bx_2 + o(|x|)} = \lim_{x \to 0} \frac{cx_1 + o(|x_1|)}{ax_1 + o(|x_1|)} = \frac{c}{a} = 0.
\]

Hence \( c = 0 \). By \( \text{ind}(u, x_0) \neq 1, -1 \), the singular point \( x_0 \) of \( u \) is degenerate. Therefore \( c = 0 \) implies that \( a = 0 \).

Hence when \( \text{ind}(u, x_0) \neq 1, -1 \), \( u \) can be expressed near \( x_0 = 0 \) as

\[
u_1(x) = bx_2 + o(|x|),
\]
(3.4)

\[
u_2(x) = o(|x|).
\]

**Step 2:** Consider the case where there is another orbit \( \gamma_1 \) of \( u \) connected to \( x_0 \), and the angle between \( \gamma_1 \) and \( \gamma \) is \( \theta \) different from \( 0 \) and \( \pi \). Then by (3.4) we deduce that
then (3.1) holds true.

In each $F_i$ of Theorem 2.5, it is obvious that for any $x \sim u$, the two orbits $g^1_0$ and $g^2_0$ connected to $x$, such that $u(x) = 0$, $x \sim u$ be a sufficiently small neighborhood of $x$, $F_i$ (1 ≤ $i$ ≤ 2n) be the domains in $O$ enclosed by the orbits connected to $x_0$, and the $\theta_i$ be the angle of the boundary of $F_i$ at $x_0$. It is easy to see that in each $F_i$ with $\theta_i = 0$, there exists at least a curvilinear segment $\ell_i$, with $x_0$ being its end point, such that $u_1(x) = 0$, $x \in \ell_i$ (see Fig. 1). Hence, there are at least 2$(n - 1)$ curvilinear segments in $O$ with $x_0$ as their common end point where $u_1 = 0$. On the other hand, by the implicit function theorem, if $b \neq 0$ in (3.4), then there is a unique curve $L < O$ with $x_0 \in L$ such that $u_1(x) = 0$, $x \in L$, i.e., there are only two line segments $L = \ell_1 \cup \ell_2$ in $O$ along which $u_1 = 0$; hence if $n \geq 3$ it follows that $b = 0$ and (3.1) holds true.

This completes the proof of this lemma.

Lemma 3.2: Let $u \in D'(TM)$ (r ≥ 1), and $x_0 \in \tilde{M}$ be an isolated singular point of $u$. If the index $\text{ind}(u, x_0) = 0$, and the angle $\theta$ between the two orbits connected to $x_0$ is different from 0, then (3.1) holds true.

Proof: By Step 2 in the proof of Lemma 3.1, it suffices to prove (3.1) when $\theta = \pi$. In this case the two orbits $\gamma_1$ and $\gamma_2$ connected to $x_0$ form a curve $\Gamma$ with the $x_1$-axis tangent to $\Gamma$ at $x_0$. By Theorem 2.5, it is obvious that for any $x_2 > 0$ sufficiently small, we have

$$\text{sign} u_1(0, x_2) = \text{sign} u_1(0, -x_2),$$

which, together with (3.4), yields that $b = 0$. This proof is complete.

From Lemma 3.1, we see that a degenerate singular point $x_0 \in \tilde{M}$ of $u \in D'(TM)$ (r ≥ 1) with nonzero Jacobian $Du(x_0) \neq 0$ can only be one of the three cases:

1. a degenerate center;
2. a degenerate saddle such that the four orbits connected to $x_0$ are tangent to each other at $x_0$; and
3. a point with $\text{ind}(u, x_0) = 0$ such that the angle between the two orbits connected to $x_0$ is zero.

We now take a further examination of these cases.

Let $x_0 \in \tilde{M}$ be an isolated degenerate singular point of $u \in D'(TM)$ (r ≥ 1) with nonzero Jacobian, $Du(x_0) \neq 0$. Since $Du(x_0)$ is a degenerate matrix, $Du(x)$ has an eigenvector $e_1$ satisfying

$$Du(x_0)e_1 = 0, \quad |e_1| = 1. \quad (3.5)$$

Let $e_2$ be a unit vector, which is orthogonal to $e_1$, and satisfies that

$$Du(x_0)e_2 = \alpha e_1, \quad (3.6)$$

FIG. 1. Sketch illustrating the proof of Lemma 3.1.
for some constant $\alpha \neq 0$.

For simplicity, we always take the orthogonal coordinate system $(x_1, x_2)$ with the origin at $x_0$, $x_1$-axis and $x_2$-axis pointing, respectively, in the $e_1$ and $e_2$ directions. In this case, the matrix $Du(x_0)$ and the vectors $e_1, e_2$ can be written as follows:

$$Du(x_0) = Du(0) = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}, \quad \alpha \neq 0,$$

$$e_1 = (1,0), \quad e_2 = (0,1).$$

(3.7)

Geometrically, $e_1$ and $e_2$ can be illustrated as in Fig. 2.

B. Index and kinematic conditions

We now make connections between the index of $u$ at $x_0$ and different orders of $u$ in its Taylor expansion near $x_0$. Let $x_0 \in \mathcal{M}$ be an isolated degenerate singular point of $u$, and

$$Du(x_0) \neq 0,$$

$$\frac{\partial^m(u(x_0) \cdot e_2)}{\partial e_1^m} = 0, \quad 1 \leq m < n, \quad \neq 0, \quad m = n.$$  

(3.8)  

(3.9)

Under the conditions (3.8) and (3.9), the vector field $u(x)$ has the Taylor expansion, by (3.7), as follows:

$$u(x) = \begin{cases} \alpha x_2 + f(x_1) + x_2 g_1(x), \\ \beta x_1^n - x_2 f'(x_1) + x_2^2 g_2(x) + o(|x_1|^n), \end{cases}$$

(3.10)

where $\alpha, \beta \neq 0$, $f(x_1) = o(|x_1|)$, and $g_i(0) = 0$ $(i = 1, 2)$. Let $k = \deg f$ be defined by

$$\lim_{z \to 0} \frac{f(z)}{z^k} = \lambda \neq 0, \quad k = \infty.$$

Lemma 3.3: Let $x_0 \in \mathcal{M}$ be an isolated degenerate singular point of $u$ satisfying (3.8) and (3.9).

(1) If $2k > n + 1$, then

$$\text{ind}(u, x_0) = \begin{cases} 0, & \text{as } n \text{ even}, \\ -1, & \text{as } n \text{ odd and } \alpha \cdot \beta > 0 \text{ in (3.10)}, \\ 1, & \text{as } n \text{ odd and } \alpha \cdot \beta < 0. \end{cases}$$

(2) If either $2k < n + 1$, or $2k = n + 1$ and $\alpha \beta \neq -k\lambda^2$, then $\text{ind}(u, x_0) = -1.$
Proof: Proof of Assertion (1). Let
\[ u(t,x) = \left[ ax_2 + tf(x_1) + x_2g_1(x) \right], \]
\[ \beta x_1^n + t[-xf'(x_1) + x_2^2g_2(x) + o(|x_1|^n)], \]
where \( 0 \leq t \leq 1 \). Since \( 2k-1 > n \), it is easy to see that there exists a neighborhood \( U \subseteq \mathbb{R}^n \) of \( x_0 \) (\( = 0 \)), such that \( u(t,x) \) has only a singular point \( x = 0 \) in \( U \) for all \( t \in [0,1] \). By the homotopy invariance of the index, we derive that
\[ \text{ind}(u_0, x_0) = \text{ind}(u_1, x_0) = \text{ind}(u, x_0). \] (3.11)

In a neighborhood of \( x = 0 \), orbits of \( u_0 = (ax_2, \beta x_1^n) \) are given by the following equations:
\[ \frac{\alpha}{2} x_2^2 - \frac{\beta}{n+1} x_1^{n+1} = C, \quad 0 \leq |C| < \delta. \] (3.12)

Obviously, we can see from (3.12) that if \( n = \text{odd} \), the flow of \( u_0 \) in a neighborhood of \( x = 0 \) is as shown in Fig. 2(c). If \( n = \text{even} \), when \( \alpha \cdot \beta > 0 \) (respectively, \( \alpha \cdot \beta < 0 \)) the flows of \( u_0 \) look as shown in Fig. 2(b) [respectively, as shown in Fig. 2(a)]. Thus, we derive from (3.11) this claim.

Proof of assertion (2): We take \( \varepsilon > 0 \) sufficiently small, and consider singular points of the following vector field near \( x = 0 \):
\[ u_\varepsilon = \left[ ax_2 + f(x_1) + x_2g_1(x) \right], \]
\[ \beta x_1^n - x_2^2f'(x_1) + x_2^2g_2(x) + o(|x_1|^n) - \varepsilon. \]

By assumption, \( f \) can be expressed near \( x = 0 \) by
\[ f(x_1) = \lambda x_1^k + o(|x_1|^k), \quad \lambda \neq 0, \quad 1 < k \leq \frac{n+1}{2}. \]
Thus, singular points of \( u_\varepsilon \) in a small neighborhood of \( x = 0 \) satisfy the equation below
\[ x_2 = -\frac{\lambda}{\alpha} x_1^k + o(|x_1|^k), \]
\[ \beta x_1^n + 1 = \frac{k}{\alpha} \lambda x_1^{2k-1} = \varepsilon + o(|x_1|^{2k-1}). \] (3.13)

Obviously, when \( 2k-1 < n \), or \( 2k-1 = n \) and \( \alpha \beta \neq -k\lambda^2 \), (3.13) has a unique solution
\[ x_\varepsilon \sim \left( Ce^{\lambda x_1^k}, -\frac{\lambda}{\alpha} Ce^{\lambda x_1^k} \right), \]
where
\[ C = \begin{cases} \frac{\lambda x_1^k}{\alpha}, & \text{as } 2k-1 < n, \\ \alpha(\alpha \beta + \lambda^2)^{-1}, & \text{as } 2k-1 = n \end{cases}, \quad \text{and } \alpha \beta \neq k\lambda^2. \]

It is easy to check that
\[ \text{sign det } Du_\varepsilon(x_\varepsilon) = -1, \] (3.14)
for any \( \varepsilon > 0 \) sufficiently small. By the invariance of index sums in a small domain with a perturbation we refer from (3.14) that \( \text{ind}(u, x_0) = -1 \). The proof of this lemma is complete.
IV. STRUCTURAL BIFURCATION NEAR INTERIOR SINGULAR POINTS WITH INDEX ZERO

A. Main theorems

Let $u \in C^1([0,T],D'(TM)) (r \geq 1)$ be a one-parameter family of divergence-free vector fields. We consider the Taylor expansion of $u(x,t)$ at $t_0$ ($0 < t_0 < T$),

$$u(x,t) = u^0(x) + (t-t_0)u^1(x) + o(|t-t_0|),$$

where $u^0(x) = u(x,t_0)$,

$$u^1(x) = \frac{\partial}{\partial t} u(x,t_0).$$

We start with the following assumptions for the structural bifurcation.

Assumption $(H_1)$: Let $x_0 \in \tilde{M}$ be an isolated degenerate singular point of $u^0(x)$. Suppose that

$$\text{ind}(u^0,x_0) = 0,$$

$$Du^0(x_0) \neq 0,$$

$$u^1(x_0) \cdot e_2 \neq 0,$$

where $e_2$ is the unit vector defined as in (3.6).

Assumption $(H_2)$: Under the conditions of Assumption $(H_1)$, we also assume that $u^0 \in C^n$ near $x_0 \in \tilde{M}$ for some $n \geq 2$, and

$$\frac{\partial^k (u^0(x_0) \cdot e_2)}{\partial t^k} = 0, \quad \text{for } 1 \leq k < n = \text{even},$$

$$\neq 0, \quad \text{for } k = n = \text{even}. \quad (4.5)$$

Remark 4.1: The conditions (4.2) and (4.3) imply by Lemma 3.2 that the flows of $u^0$ near $x_0$ is as shown in Fig. 2(c), i.e., both orbits of $u^0$ connected to $x_0$ are tangent to each other at $x_0$, and the eigenvector $e_1$ of $Du(x_0)$ is their common tangent vector. We shall see later that the conditions (4.2) and (4.3) are generic for the interior structural bifurcation.

Remark 4.2: In view of fluid mechanics applications, condition (4.4) is equivalent to nonzero acceleration of the flow in the orthogonal direction to the eigenvector $e_1$ of $Du(x_0)$. This is a natural condition for the structural bifurcation.

Remark 4.3: Condition (4.5) is a technical condition, which by Lemma 3.3 ensures the regularity of the bifurcated singular points of $u(x,t)$ from $(x_0,t_0)$. In addition, from (3.7) we can see that the integer number $n$ satisfying (4.5) must be $n \geq 2$.

The interior structural bifurcation of $u(x,t)$ near a singular point with index zero is described by the following theorems.

Theorem 4.4: Let $u \in C^1([0,T],D'(TM)) (r \geq 1)$ satisfy Assumption $(H_1)$. Then

1. the vector field $u$ has a bifurcation in its local structure at $(x_0,t_0)$. More precisely, $u(x,t)$ has no singular point in a small neighborhood of $x_0$ for any $t < t_0$ (or $t > t_0$) sufficiently close to $t_0$, and $u(x,t)$ bifurcates at least two singular points from $x_0$ as $t > t_0$ (or $t < t_0$), and

2. if $x_0 \in \tilde{M}$ is a unique singular point with index zero of $u^0$, then $u(x,t)$ has a bifurcation in its global structure at $t = t_0$.

Theorem 4.5: Let $u \in C^1([0,T],D'(TM)) (r \geq 1)$ satisfy Assumption $(H_2)$. Then, $u(x,t)$ bifurcates from $(x_0,t_0)$ exactly two nondegenerate singular points, one of which is a center and another is a saddle.
Remark 4.6: Both Theorems 4.4 and 4.5 study structural bifurcation of $u$ near an interior point $x_0$ with $\text{ind}(u^0, x_0) = 0$. When $\text{ind}(u^0, x_0)$ is different from zero, interior structural bifurcation may not occur as we shall see in Examples 5.6 and 5.7. In addition, Theorem 5.5 shows that the bifurcation given in Theorems 4.4 and 4.4 is generic. This is quite different from the structural bifurcation near a boundary singular point; see Refs. 2 and 3. We have shown in Refs. 2 and 3 that under suitable necessary conditions, $u(x,t)$ will always be a structural bifurcation near a boundary singular point with index different from $-1/2$. Reference 2 considers vector fields with free-slip boundary conditions, while Ref. 3 deals with vector fields with Dirichlet boundary conditions, which is related to boundary layer separation problems in viscous fluid flows.

B. Proof of Theorem 4.4

To investigate the structural bifurcation of $u(x,t)$ at $t_0$, by the Taylor expansion (4.1) and condition (4.4) it suffices to consider only the topological structure of the first-order approximation $u^0 \pm \varepsilon u^1$ of (4.1) for $\varepsilon > 0$ sufficiently small.

By Lemma 3.2, let $\gamma_1$ and $\gamma_2$ be the two orbits of $u^0(x)$ connected to $x_0 \in \bar{M}$. The eigenvector $e_1$ of $Du^0(x_0)$ is a common tangent vector of $\gamma_1$ and $\gamma_2$ at $x_0$ [see Fig. 3(a)].

Both orbits $\gamma_1$ and $\gamma_2$ divide a neighborhood of $x_0$ into open domains $I$ and $II$ as shown in Fig. 3(a). Since the angles between $e_1$ and the vectors of $u^0$ on $\gamma_1$ and $\gamma_2$ vary from $0$ to $\pi$, and by assumption that angle $\theta$ between $e_1$ with $u^1(x_0)$ satisfies $0<\theta<\pi$, there exist curves $\ell_1$ in domain I and curve $\ell_2$ in domain II connected to $x_0$, such that $u^1$ are parallel to $u^0$ on $\ell_1$ and $\ell_2$ [see Fig. 3(b)]. Obviously the singular points of $u^0 \pm \varepsilon u^1$ are only on the curves as $\ell_1$ and $\ell_2$. Without loss of generality, we assume that $u^0$ and $u^1$ have a reverse orientation on $\ell_1$ and $\ell_2$, i.e., $u^0$ and $-\varepsilon u^1$ have the same orientation on $\ell_1$ and $\ell_2$. By condition (4.4) it follows that $u^0 - \varepsilon u^1$ has no singular points in $\ell_1$ and $\ell_2$, and therefore has no singular points in a neighborhood of $x_0$. Because $x_0 \in \bar{M}$ is an isolated singular point of $u^0$, the values $|u^0(x)|$ are variant from 0 to a $\delta>0$ sufficiently small on $\ell_1$ and $\ell_2$, i.e.,

$$0<|u^0(x)|<\delta \quad \forall \ x \in \ell_1 \cup \ell_2,$$

$$\sup_{\ell_1 \cup \ell_2} |u^0| = \delta, \quad \inf_{\ell_1 \cup \ell_2} |u^0| = 0.$$
It follows from (4.6) that there is an \( \varepsilon_0 > 0 \) sufficiently small such that for any \( 0 < \varepsilon < \varepsilon_0 \) the vector field \( u^0 + \varepsilon u^1 \) has at least a singular point on each of \( \ell_1 \) and \( \ell_2 \), i.e., \( u^0 + \varepsilon u^1 \) has at least two singular points near \( x_0 \). Thus, \( u^0 + \varepsilon u^1 \) and \( u^0 - \varepsilon u^1 \) are not topologically equivalent in a neighborhood of \( x_0 \). The first assertion is proved.

Obviously, if \( x_0 \) is a unique singular point of \( u^0(x) \) with index zero, then \( u^0 + \varepsilon u^1 \) and \( u^0 - \varepsilon u^1 \) are not topologically equivalent on \( M \). Hence, \( u(x,t) \) has a globally structural bifurcation at \( t = t_0 \). This theorem is proved. \( \Box \)

### C. Proof of Theorem 4.5

By Assumption \( (H_2) \) and Lemma 3.3 the vector field \( u^0(x) \) has the Taylor expansion at \( x_0 \) \( (x=0) \) as follows:

\[
u^0(x) = \begin{cases} 
\alpha x_2 + f(x_1) + x_2 g_1(x), \\
\beta x_1^{2m} - x_2 f'(x_1) + x_2 g_2(x) + o(|x_1|^{2m}),
\end{cases}
\]

where \( \alpha \neq 0, \beta \neq 0, \) and

\[
f(x_1) = o(|x_1|^{m+1/2}),
\]

\[
f'(x_1) = o(|x_1|^{m-1/2}),
\]

\[
g_i(0) = 0 \quad (i = 1, 2).
\]

By (4.4), we have

\[
u^1(x) = \begin{bmatrix}
\lambda_1 + h_1(x), \\
\lambda_2 + h_2(x),
\end{bmatrix}
\]

where \( \lambda_2 \neq 0 \) and \( h_i(x) = O(|x|), \quad i = 1, 2. \)

By Theorem 4.4, one of \( u^0 \pm \varepsilon u^1 \) has no singular points, and another has at least two singular points near \( x_0 \) for all \( \varepsilon > 0 \) sufficiently small. We assume that \( u^0 - \varepsilon u^1 \) has singular points, i.e., \( \beta > 0 \) in (4.7) and \( \lambda_2 > 0 \) in (4.9). We need to prove that the equations below have exactly two solutions

\[
\alpha x_2 + x_2 g_1(x) = \lambda_1 e + \varepsilon h_1(x) - f(x_1),
\]

\[
\beta x_1^{2m} - x_2 f'(x_1) + x_2 g_2(x) = o(|x_1|^{2m}) = \lambda_2 e + \varepsilon h_2(x).
\]

By the implicit function theorem, we derive from (4.8) and (4.10) that

\[
x_2 = \alpha^{-1} \lambda_1 e - f(x_1) + G(\varepsilon, x_1),
\]

\[
G(\varepsilon, x_1) = o(\varepsilon |x_1|^{m+1/2}).
\]

Setting (4.12) in (4.11), we get the algebraic equation

\[
\beta x_1^{2m} = \lambda_2 e + \alpha^{-1} \lambda_1 e f'(x_1) + e \cdot O(|x|) + o(|\varepsilon| |x_1|^{2m}),
\]

where \( \beta \lambda_2 > 0 \). It is clear that for any \( \varepsilon > 0 \) sufficiently small, the equation (4.13) has exactly two solutions

\[
x_1 = \pm (\beta^{-1} \lambda_2)^{1/2} e^{1/2m} + o(e^{1/2m}).
\]

Thus, we deduce that the vector field \( u^0 - \varepsilon u^1 \) has exactly two singular points \( x(\varepsilon) = (x_1(\varepsilon), x_2(\varepsilon)) \) as follows:
\[ x_1^\pm(e) = \pm (\beta^{-1} \lambda_2)^{1/2m} e^{1/2m} + o(e^{1/2m}), \]
\[ x_2^\pm(e) = \alpha^{-1} \lambda_1 e + o(e|\pm|^{m+1/2}). \]

Finally, we shall show that \( x^\pm(e) \) are nondegenerate for all \( e > 0 \) sufficiently small. By \( \text{div} u = 0 \), we have

\[
\det D(u^0 - \varepsilon u^1)_{x=x^\pm(e)} = -\left( \frac{\partial}{\partial x_1} (u^0_1 - \varepsilon u^1_1) \right)^2 - \frac{\partial}{\partial x_2} (u^0_1 - \varepsilon u^1_1) \cdot \frac{\partial}{\partial x_1} (u^0_2 - \varepsilon u^1_2)
\]

\[
= \left[ \text{by (4.8) and (4.14)} \right]
\]

\[
= \pm 2m \alpha \beta (\beta^{-1} \lambda_2)^{(2m-1)/2m} e^2(2m-1)/2m + o(e^{(2m-1)/2m}),
\]

which yields that

\[
\det D(u^0 - \varepsilon u^1) \begin{cases} > 0, & \text{as } x = x^-, \\ < 0, & \text{as } x = x^+. \end{cases}
\]

Thus, we prove that \( u^0 - \varepsilon u^1 \) has exactly two singular points \( x^+ \) and \( x^- \) for any \( e > 0 \) sufficiently small, and \( x^- \) is a center, \( x^+ \) is a saddle, which are nondegenerate. This proof is complete. \( \square \)

V. APPLICATIONS TO INTERIOR SEPARATION OF FLUID FLOWS

A. Interior separation of fluid flows

We start with a typical example, illustrating how structural bifurcation occurs in the interior of fluid flows.

Let \( u \in C^1([0,T], D(TM)) \), and \( x_0 \in \hat{M} \) be an isolated singular point of \( u^0(x) = u(x, t_0) \), \( 0 < t_0 < T \).

**Example 5.1:** Consider the case where the index of \( u^0(x) \) at the singular point \( x_0 \in \hat{M} \) is zero, and the Jacobian matrix at \( x_0 \) is nonzero, i.e.,

\[
\text{ind}(u^0, x_0) = 0, \quad Du^0(x_0) \neq 0.
\]

The structural bifurcation occurs as shown in Fig. 4, which corresponds to interior separation phenomena in fluid mechanics.

When \( t = t_0 - \varepsilon \) with \( \varepsilon > 0 \) small, the flow of \( u(x, t_0 - \varepsilon) \) given by Fig. 4(a) exhibits no singular points in a neighborhood of \( x_0 \). At \( t = t_0 \), \( u^0 = u(x, t_0) \) is given by Fig. 4(b), which has an isolated singular point \( x_0 \in \hat{M} \) with index zero. When \( t = t_0 + \varepsilon \) for \( \varepsilon > 0 \) small, \( u(x, t_0 + \varepsilon) \) is given by either Fig. 4(c) or Fig. 4(c') or even more complicated circulation patterns in the back flow region. As we shall see in Theorem 5.5, the flow pattern given by Fig. 4(c) is generic. In other words, the flow transition from Fig. 4(a), to Fig. 4(b), and then to Fig. 4(c), or vice versa, is in general the pattern transition obtained both experimentally and numerically. For instance, in the axisymmetric plume shown Fig. 5, which is reproduced from Ref. 1, the three "bubbles" were caused by interior separation as described here.

We now address interior flow separation from a rigorous analysis point of view.

**Theorem 5.2:** Let \( u \in C^1([0,T], D(TM)) \) satisfy Assumption (H1). Then

(1) there must be some centers of \( u \) separated from \( x_0 \in \hat{M} \) as shown schematically in either Fig. 4(c) or Fig. 4(c');

(2) the centers (back flows) are enclosed by a closed orbit line \( \gamma(t) \) consisting of orbits of \( u(\cdot, t) \), and \( \gamma(t) \) converges/shrinks to \( x_0 \) as \( t \to t_0 \); and

(3) if Assumption (H2) is satisfied, then the center separated from \( x_0 \in \hat{M} \) is unique, as shown in Fig. 4(c).
The centers in Figs. 4(c) and 4(c') correspond, in a real fluid, to isolated vortices or, in the case of figure-eight ones, to pairs of co-rotating vortices. This theorem is a direct corollary of Theorems 4.4 and 4.5 and a stability lemma of extended orbits.

We now introduce this stability lemma.

FIG. 4. Schematic of structural bifurcation for the case index $(u^0, x_0)=0$, a typical case in fluid flows.

FIG. 5. The three "bubbles" in the axis symmetric plume, generated by interior bifurcations, reproduced from Ref. 1.
Definition 5.3: Let \( v \in C^r(TM) \) be a vector field. A curve \( \gamma \subset M \) is called an extended orbit of \( v \), if

(i) it is a union of curves

\[ \gamma = \bigcup_{i=1}^{\infty} \gamma_i; \]

(ii) either \( \gamma_i \) is an orbit of \( v \), or \( \gamma_i \) consists of both orbits and singular points of \( v \), and

(iii) if \( \gamma_i \) and \( \gamma_{i+1} \) are orbits of \( v \), then the \( \omega \)-limit set of \( \gamma_i \) is the \( \alpha \)-limit set of \( \gamma_{i+1} \),

\[ \omega(\gamma_i) = \alpha(\gamma_{i+1}); \]

namely, the end points of \( \gamma_i \) are singular points of \( v \), and the starting end point of \( \gamma_{i+1} \) is the finishing end point of \( \gamma_i \).

The point \( \varphi_1 = \alpha(\gamma_1) \) is called the starting point of the extended orbit \( \gamma \).

We have the following stability lemma for extended orbits. The result of this lemma has been proved by Ma and Wang\(^4\) in Step 2 of the proof of Lemma 4.5 in their paper. This lemma is quite useful in analyzing the families of vector fields, and thus in solving some problems in 2D incompressible fluid flows. Here we only state the result.

Lemma 5.4: (stability of extended orbits, Ref. 4) Let \( v^n \in C^r(TM) \) be a sequence of vector fields with \( \lim_{n \to \infty} v^n = v \in C^r(TM) \). Suppose that \( \gamma^n \subset M \) is an extended orbit of \( v^n \) and the starting points \( p^n_1 \) of \( \gamma^n \) converge to \( p_1 \). Then the extended orbits \( \gamma^n \) of \( v^n \) converge to an extended orbit \( \gamma \) of \( v \) with starting point \( p_1 \).

B. Genericity of structural bifurcation with index zero

In the following, we shall show that the type of structural bifurcation as shown in Fig. 4(c), i.e., one center interior separation, is generic in the interior structural bifurcation. This is remarkably different from the structural bifurcation near the boundary.\(^2,3,5\) It also explains why interior separation to multiple centers and the interior flow separation from the singularities with nonzero index are seldom observed in fluid motions.

Let \( x_0 \in M \) and \( 0 < t_0 < T \) be given. We define a topological space \( B \subset C^1([0,T],D^2(TM)) \) as follows:

\[ B = \{ u \in C^1([0,T],D^2(TM)) \mid u^0(x_0) = 0, \det Du^0(x_0) = 0, u^0 = u(\cdot,t_0) \} \]

with the topology of \( C^1([0,T],D^2(TM)) \). Obviously, the space \( B \) contains all vector fields in \( C^1([0,T],D^2(TM)) \), which have a bifurcation in their local structure at \((x_0,t_0)\). It is easy to see that the set

\[ B_0 = \bigg\{ u \in B \bigg| Du^0(x_0) \neq 0, \frac{\partial^2(u^0(x_0) \cdot e_2)}{\partial e_1^2} \neq 0, u^1(x_0) \cdot e_2 \neq 0 \bigg\} \]

is open and dense in \( B \), where \( e_1 \) and \( e_2 \) are as in (3.5) and (3.6), and \( u^1(x) = (\partial v/\partial t) u(x,t_0) \).

From Lemma 3.3, it immediately follows the following genericity theorem of structural bifurcation.

Theorem 5.5 (Genericity of structural bifurcation): For any \( u \in B_0 \), \( u \) has a bifurcation in its local structure at \((x_0,t_0)\). More precisely, \( u \) bifurcates from \((x_0,t_0)\) exactly two nondegenerate singular points, one of which is a center and another is a saddle, as shown in Figs. 4(a)-4(c).

Moreover, the set \( B_0 \) is open and dense in the topological space \( B \), which contains all vector fields in \( C^1([0,T],D^2(TM)) \) having a locally structural bifurcation at \((x_0,t_0)\).

Proof: If \( u \in B_0 \), then \( u^0 \in D^2(TM) \), and \( u^0 \) has the Taylor expansion (3.10) with \( n = 2 \), and

\[ \deg f = k \geq 2 \quad \text{[by } f \in C^2 \text{ and } f(z) = o(|z|)]. \]
Hence, by Lemma 3.3, we have
\[ \text{ind}(u^0, x_0) = 0. \]
Then this theorem follows from Theorems 4.4–4.5. The proof is complete.

\[ \square \]

C. Examples of no structural bifurcation

We know that for a locally structurally stable singular point of a vector field \( v \in D'(TM) \), its index obeys
\[
\text{ind}(v, x_0) = \begin{cases} 
-\frac{1}{2}, & x_0 \in \partial M, \\
-1 \text{ or } +1, & x \in \dot{M}.
\end{cases}
\]

For a vector field \( u(x,t) = u^0(x) + (t - t_0)u^1(x) + o(|t - t_0|) \) the structural bifurcation theorem given in Ref. 2 amounts to saying that if \( x_0 \) is a boundary singular point with \( \text{ind}(u^0, x_0) \neq -\frac{1}{2} \) and \( u^1(x_0) \neq 0 \) for \( x_0 \in \partial M \), then \( u(x,t) \) has a bifurcation in its local structure at \((x_0, t_0)\). However, for an interior singular point \( x_0 \in \dot{M} \) of \( u^0 \) with \( \text{ind}(u^0, x_0) \neq 1 \) or \(-1\), the vector field \( u(x,t) \) may have no structural bifurcation near \((x_0, t_0)\). In the following, we give two examples to show this.

**Example 5.6:** Figures 6(a)–6(c) illustrates a structural evolution of a vector field \( u(x,t) \) near \( x_0 \in \dot{M} \) as time \( t \) crosses \( t_0 \), where \( \text{ind}(u^0(x_0), x_0) = -n \) \((n > 1)\), \( u^0(x_0, t_0) \), and \( u^1(x_0) = (\partial / \partial t) u(x_0, t) \neq 0 \).

In Fig. 6, we see that the vector fields \( u(x,t_0 - \epsilon) \) given by (a) and \( u(x,t_0 - \epsilon) \) given by (c) are topologically equivalent for all \( \epsilon > 0 \) small. Hence, \( u(x,t) \) has no structural bifurcation at \((x_0, t_0)\).

**Example 5.7:** Let \( \text{ind}(u^0, x_0) = 0 \), and \( Du^0(x_0) = 0 \), and the structure of \( u^0 \) near \( x_0 \) is illustrated by Fig. 7(a).

---

**FIG. 7.** Sketch showing no bifurcation given in Example 5.7.
Let the $x_1$-axis of the coordinate system in Fig. 7(b) be tangent to the orbit line $\ell$ in (a) at $x_0$. The angles between $u^0(x)$ and the $x_1$-axis near $x_0$ vary in the shadow domain I in (b). Hence if the angle between $u^1(x_0)$ and the $x_1$-axis is in the domain II in (b), then $u^1(x)$ is transversal to $u^0(x)$ near $x_0$, which implies that the vector field $u(x,t) = u^0(x) + (t-t_0)u^1 + o(|t-t_0|^2)$ has no structural bifurcation near $(x_0,t_0)$.

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1 Dyke, M. V., *Album of Fluid Motion* (Parabolic, Stanford, CA, 1982).