Bifurcation and stability of superconductivity\textsuperscript{a)}

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In this article, we present a bifurcation and stability analysis on time-dependent Ginzburg–Landau model of superconductivity. It is proved in particular that there are two different phase transitions from the normal state to superconducting states or vice versa: one is continuous, and the other is jump. These two transitions are precisely determined by a simple nondimensional parameter, which links the superconducting behavior with the geometry of the material, the applied field and the physical parameters. The rigorous analysis is conducted using a bifurcation theory newly developed by the authors, and provides some interesting physical predictions. © 2005 American Institute of Physics.

I. INTRODUCTION

The main objective of this article is to study the nature of the phase transition from normal to superconducting states, which occurs when the temperature of a sample decreases. The rigorous analysis is conducted using a new bifurcation theory developed recently by the authors.

Superconductivity was first discovered in 1911 by H. Kamerlingh Onnes, who found that Mercury had zero electric resistance when the temperature decreases below some critical value $T_c$. Since then, one has found that large number of metals and alloys possess the superconducting property. In the superconducting state once a current is set up in a metal ring, it is expected that no change in this current occurs in times more that $10^{10}$ years (see Ref. 1). In 1933, the other important superconducting property, called the diamagnetism or the Meissner effect, was discovered by W. Meissner and R. Ochsenfield. They found that not only a magnetic field is excluded from entering a superconductor, but also that a field in an originally normal sample is expelled as it is cooled below $T_c$.

One central problem in the theory of superconductivity is the nature of the phase transition between a normal state, characterized by an order parameter that vanishes identically, and a superconducting state, characterized by the order parameter that is not identically zero. In this article, we address this problem by conducting rigorous bifurcation and stability analysis for the time dependent Ginzburg–Landau (TDGL) model of superconductivity.

The TDGL model of superconductivity involves an order parameter $\psi$, and the magnetic potential $A$. The problem is forced by an applied field $H_a$ [see (2.1)–(2.3)]. The associated Ginzburg–Landau free energy is given by (see Ref. 1)

\textsuperscript{a}Dedicated to Professor Louis Nirenberg on the occasion of his eightieth birthday with great affection and admiration.
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\[ f = f_{00} + a|\psi|^2 + \frac{b}{2} |\psi|^4 + \frac{1}{2m_s i} \left( \frac{h}{c} \nabla - \frac{e_s A}{c} \right) \psi^2 + \frac{h^2}{8\pi}, \]  

(1.1)

where \( h \) is the Planck constant, \( c \) the speed of light, \( e_s \) and \( m_s \) the charge and mass of a Cooper pair, and the parameters \( a = a(T) \) and \( b = b(T) \) are coefficients satisfying the following conditions (see, among others, Ref. 2):

\[
\begin{align*}
    a &= a(T) \\
    &\begin{cases} 
    > 0 & \text{for } T > T_c \\
    < 0 & \text{for } T < T_c,
    \end{cases}
\end{align*}
\]

Here \( T \) the temperature of the sample, and \( T_c \) the critical temperature where incipient superconductivity property can be observed.

With proper scaling, a nondimensional parameter \( \alpha \) plays a key role in the phase transition (or bifurcation), which is given in terms of dimensional quantities by

\[
\alpha = \alpha(T) = - \frac{2a\sqrt{bm_sD}}{e_s^3 h} = \frac{2\sqrt{bm_sDN_s T_c - T}}{e_s^3 h T_c},
\]

(1.2)

where \( D \) is the diffusion coefficient, and the last equality was derived using (2.4) based on the Bardeen-Cooper-Schrieffer (BCS) theory.

The main objectives of this article is to establish a nonlinear bifurcation and stability theory for the Ginzburg–Landau equations. It is clear that such a nonlinear bifurcation and stability theory should at least include

1. A bifurcation theorem when the parameter \( \alpha \) crosses some critical numbers for all physically sound boundary conditions and geometry of the domain,
2. Asymptotic stability of bifurcated solutions, and
3. The vortex structure and its stability and transitions in the physical space.

This bifurcation and stability analysis uses a new bifurcation theory for partial differential equations (PDEs) developed recently by the authors. This bifurcation theory is based on a new notion of bifurcation, called attractor bifurcation, and its corresponding theory introduced in Refs. 3 and 4. With the bifurcation theory, many bifurcation problems in science and engineering are becoming more accessible. In particular, applications are made for variety of PDEs from science and engineering, including, in particular, the Kuramoto-Sivashinsky equation, the Cahn-Hillard equation, the complex Ginzburg–Landau equation, Reaction-diffusion equations in biology and chemistry, the Rayleigh-Bénard convection problem, and the Taylor problem.

We now address different aspects of such a nonlinear bifurcation and stability theory for the TDGL model of superconductivity obtained in this article.

First, we proceed with the reduction of the infinite dimensional dynamical system governed by the TDGL equations to a finite dimensional system defined on the center manifold. One important ingredient of the analysis is the approximation of the center manifold function, which is part of the new bifurcation theory. With this reduction in our disposal, a general bifurcation theorem follows from the general strategy of attractor bifurcations, as required in part (1) mentioned previously. We prove in particular that there are two different phase transitions from the normal state to superconducting states or vice versa: one is continuous shown schematically in Fig. 1, and the other is jump shown in Figs. 2 and 3. These two transitions are precisely determined by a simple parameter \( R \) defined by (4.22) for transitions near a complex simple eigenvalue of the linearized problem [respectively, by two parameters \( R_1 \) and \( R_2 \) defined by (4.44) for transitions near eigenvalues with higher multiplicity]. The parameter \( R \) links the superconducting behavior with the geometry of the material, the applied field and the physical parameters.

Second, as an attractor, the bifurcated attractor has asymptotic stability in the sense that it attracts all solutions with initial data in the phase space outside of the stable manifold of the trivial
solution. Therefore bifurcation analysis for steady state problems provides in general only partial answers to the problem, and is not enough for solving the stability problem. Hence it appears that the right notion of asymptotic stability after the first bifurcation should be best described by the attractor near, but excluding, the trivial state. It is one of our main motivation for introducing attractor bifurcation.

Consider the TDGL model for the case where the first eigenvalue of the linearized problem is complex simple, i.e., has real multiplicity 2. When \( R < 0 \), we obtain a continuous transition from the normal state to superconducting states. In particular, we prove in Theorem 4.1 that (a) the bifurcated attractor is a circle \( S^1 \), consisting of only steady states, (b) the solutions in the bifurcated attractor are in superconducting states, and (c) the bifurcated attractor attracts bounded open sets \( U \setminus \Gamma \) in the phase space, where \( \Gamma \) is the stable manifold of the trivial solution. Consequently we prove in particular that under a fluctuation deviating both the normal and superconducting states, the sample will soon be restored to the superconducting states.

When \( R > 0 \), we obtain a jump transition from the normal state to superconducting states or vice versa. In particular, from Theorems 4.2 and 4.3, as shown in Figs. 2 and 3, there are two
critical temperatures $T_c^0$ and $T_c^1$ ($T_c^0 > T_c^1$) such that when $T_c^1 < T$ (or $\alpha < \alpha_i$), physically observable states consist of the normal state, and the superconducting states in $\Sigma_{\alpha_i}^2$, and when $T_c^1 > T$ (or $\alpha > \alpha_i$), physically observable states are in $\Sigma_{\alpha_i}^2$. In addition, the transitions at $T_c^1$ and $T_c^0$ are jump transitions.

Third, one important aspect of studies in superconductivity is the existence (appearance) and structure of vortices of the supercurrent. The associated mathematical question is to link the solutions of the PDEs (the TDGL model in this article) to the structure of the solutions in the “physical space.” In fluid mechanics context, this corresponds to linking the kinematics to dynamics, and an attempt by the authors is summarized in Ref. 5. For the TDGL model, the existence of vortices and the structure of the supercurrent is analyzed using this philosophy.

Fourth, it is noteworthy to mention that although we used the steady state equations to derive the existence of one of the steady state bifurcation branches, the dynamic properties (the basin attraction, stability, introduction of the parameters $R$, $R_1$ and $R_2$, etc) and the branch $\Sigma_{\alpha_i}^2$ are of a truly dynamical nature.

There have been extensive studies on bifurcation and stability analysis for superconductivity; see among others.6–10 In particular, we point out recent work by Berger and Rubinstein11 and Chapman,12 where similar questions on different phase transitions of the Ginzburg–Landau model were addressed. In Ref. 11, the study is based on evaluating the second variation of the Ginzburg–Landau functional, whereas in Ref. 12, an asymptotic method traced back to Ref. 13 is used.

This article is organized as follows. Section II introduces the TDGL model. In Sec. III, after a brief introduction of the attractor bifurcation theory and center manifold functions, we prove a special case of the attractor bifurcation called $S^1$ attractor bifurcation. Section IV states and proves the main theorems on stability and bifurcation of the TDGL equations. Conclusions and physical remarks are given in Sec. V.

II. TDGL MODEL

A. The equations

Let $\Omega \subset \mathbb{R}^n$ ($n=2$ or 3) be a bounded open set. We consider the attractor bifurcation of the TDGL equations of superconductivity defined on $\Omega$. The following three unknown functions are...
involved in the mathematical formulation: a complex valued function $\psi: \Omega \to \mathbb{C}$ for the order parameter, a vector valued function $A: \Omega \to \mathbb{R}^3$ for the magnetic potential and a scalar function $\phi: \Omega \to \mathbb{R}^1$ for the electric potential. The TDGL model reads

$$\frac{\hbar^2}{2m_s D} \left( \frac{\partial}{\partial t} + \frac{i e_s}{\hbar} \phi \right) \psi + a \psi + b |\psi|^2 \psi + \frac{1}{2m_s} \left( \hbar \nabla + \frac{e_s^2}{c} A \right)^2 \psi = 0,$$

(2.1)

$$J = -\sigma \left( \frac{1}{c} A_s + \nabla \phi \right) - \frac{e_s^2}{m_s c} |\psi|^2 A - \frac{c_e \hbar}{2m_s} (\psi^* \nabla \psi - \psi \nabla \psi^*),$$

(2.2)

$$\frac{4\pi}{c} J = \text{curl}^2 A - \text{curl} H_a,$$

(2.3)

where $\sigma$ the conductivity of the normal phase, $J$ the supercurrent, and $\psi^*$ the complex conjugate of $\psi$. In the BCS theory, the parameters $a=a(T)$ and $b=b(T)$ are given (see Ref. 2) by

$$a(T) = N(0) \frac{T - T_c}{T_c},$$

(2.4)

$$b(T) = 0.098 \frac{N(0)}{(k_B T_c)^2}.$$

Equations (2.1) and (2.2) are the TDGL equations generalized by P. L. Gor’kov and G. M. Éliashberg,1,14 and (2.3) is the classical Maxwell equation. The order parameter $\psi$ describes the local density $n_s$ of superconducting electrons: $|\psi|^2 = n_s$. In addition, $\psi$ is proportional to the energy gap parameter $\Delta$ near $T_c$, which appears in the BCS theory.

### B. Scaling

From both the mathematical and physical points of view, we introduce here two nondimensional forms of the TDGL equations: one of which is used often in the literature, and the other is more suitable for the bifurcation and stability analysis presented in this article.

For convenience, we start with the dimensions of various physical quantities. Let $m$ be the mass, $L$ the typical length scale, $t$ the time, and $E$ the energy. Then we have

$$E: L^2 m/t, \quad h: L^2 t, \quad c^2: EL,$$

$$\sigma: 1/t, \quad c: L/t, \quad a: E, \quad b: EL^3,$$

$$\psi: L^{3/2}, \quad A: (E/L)^{1/2}, \quad H: (E/L)^{3/2}.$$

Then we introduce some physical parameters:

$$|\psi_0|^2 = |a|/b,$$

$$H_s = (4 \pi |a|^2/b)^{1/2},$$

$$\lambda = \lambda(T) = (m_s c^2 b/4 \pi e_s^2 |a|)^{1/2},$$

$$\xi = \xi(T) = \hbar/(2m_s |a|)^{1/2},$$

$$\kappa = \lambda/\xi.$$
\[ \eta = 4 \pi \sigma D / c^2, \]
\[ \tau = \lambda^2 / D. \]

Physically, \(|\psi_0|^2\) stands for the equilibrium density, \(H_c\) for the thermodynamic critical field, \(\lambda = \lambda(T)\) for the penetration depth, \(\xi(T)\) for the coherence length, and \(\tau\) for the relaxation time. The ratio of the two characteristic lengths \(\kappa = \lambda / \xi\) is called the Ginzburg–Landau parameter of the substance. When \(0 < \kappa < 1 / \sqrt{2}\), the material is of the first type, and when \(\kappa > 1 / \sqrt{2}\), the material is of the second type.

We now introduce the nondimensional variables (those with a prime):

\[ x = \lambda x', \quad t = \pi t', \quad \psi = \psi_0 \psi', \]
\[ A = \sqrt{\frac{2 H \lambda}{\kappa}} A', \quad \phi = D \sqrt{\frac{2 H_c}{\kappa}} \phi', \quad H_a = \sqrt{\frac{2 H_c}{\kappa}} H_a'. \]

Then we have the following traditional nondimensional TDGL equations (we henceforth drop the primes)

\[ \psi_t + i \kappa \phi \psi + \kappa^2 (|\psi|^2 - 1) \psi + (i \nabla + A)^2 \psi = 0, \]
\[ \eta (A_t + \nabla \phi) + \frac{i}{2} (\psi \nabla \psi - \phi \nabla \phi') + |\psi|^2 A + \nabla \times A - \nabla H_a = 0, \]

for the case where \(a < 0\), or equivalently \(T < T_c\).

As mentioned before, we need to introduce another non-dimensional form for the stability and bifurcation study. To this end, we set

\[ l = \frac{\sqrt{b}}{e_s}, \quad \tau_0 = \frac{h l}{e_s}, \quad \phi_0 = \frac{e_s^2}{\sqrt{b}}, \]
\[ A_0 = \left( \frac{e_s h c^2}{D \sqrt{b}} \right)^{1/2}, \quad \alpha = -\frac{2 a \sqrt{b} m D}{e_s^3 h}, \quad \mu = \frac{h D}{\sqrt{b} e_s}, \]
\[ \beta = \frac{2 m D}{h}, \quad \xi = \frac{4 \pi \sigma \rho c^2}{e_s^3}, \quad \gamma = \frac{4 \pi e_s^2}{m c^2 l}, \]

and

\[ x = l x', \quad t = \tau_0 t', \quad \psi = l^{3/2} \psi', \]
\[ A = A_0 A', \quad \phi = \phi_0 \phi', \quad H_a = l^{-1} A_0 H_a'. \]

Then we have the second type of non-dimensional TDGL equations (we drop the primes also):

\[ \psi_t + i \phi \psi = - (i \mu \nabla + A)^2 \psi + \alpha \psi - \beta |\psi|^2 \psi, \]
\[ \zeta (A_t + \mu \nabla \phi) = - \nabla \times A + \nabla H_a - \gamma |\psi|^2 A - \frac{\gamma \mu l}{2} (\psi \nabla \psi - \phi' \nabla \phi'). \]

We shall see in later discussions that the parameter \(\alpha\) plays a key role in the phase transition (or bifurcation), which is given in terms of dimensional quantities by
\[ \alpha = \alpha(T) = \frac{2 \sqrt{b m c D N_0 T_c - T}}{e_i h} \]

### C. Boundary conditions

A physically sound boundary condition for the order parameter is given by

\[ C_1 \left( i h \nabla + \frac{e_s}{c} A \right) \psi \cdot n = - C_2 i h \psi \quad \text{on} \quad \partial \Omega, \tag{2.6} \]

which means that no current passes through the boundary, where \( n \) is the unit outward normal vector at \( \partial \Omega \), and \( C_1, C_2 \geq 0 \) are constants depending on the material to which the contact is made. Physically, they satisfy\(^1,2\)

\[ C_2 = 0, \quad C_1 \neq 0 \quad \text{for an insulator on} \quad \partial \Omega, \]
\[ C_1 = 0, \quad C_2 \neq 0 \quad \text{for a magnetic material}, \tag{2.7} \]
\[ 0 < C_2/C_1 < \infty \quad \text{for a normal metal}. \]

We note that Eqs. (2.1)–(2.3) with (2.6) is invariant under the following gauge transformation

\[ (\psi, A, \phi) \rightarrow \left( \psi e^{i \theta}, A - \frac{hc}{e_s} \nabla \theta, \phi - \frac{h}{e_s} \theta \right), \]

where \( \theta \) is an arbitrary function. If we take \( \theta \) such that

\[ \frac{hc}{e_s} \Delta \theta = \text{div} \ A \quad \text{in} \ \Omega, \]
\[ \frac{hc}{e_s} \frac{\partial \theta}{\partial n} = A \cdot n \quad \text{on} \ \partial \Omega, \]

then we obtain an additional equation and a boundary condition; see also Refs. 2 and 15:

\[ \text{div} \ A = 0, \tag{2.8} \]
\[ A_n = A \cdot n = 0 \quad \text{on} \quad \partial \Omega. \tag{2.9} \]

Another boundary condition often imposed for \( A \) is as follows:

\[ \text{curl} \ A \times n = H_n \times n \quad \text{on} \quad \partial \Omega. \tag{2.10} \]

### D. Nondimensional TDGL model

In summary, with the gauge taken such that (2.8) and (2.9) hold true, the nondimensional TDGL equations are

\[ \psi_t + i \phi \psi = - (i \mu \nabla + A)^2 \psi + \alpha \psi - \beta |\psi|^2 \psi, \]
\[ \zeta (A_t + \mu \nabla \phi) = - \text{curl}^2 A + \text{curl} H_n - \gamma |\psi|^2 A - \frac{\gamma \mu i}{2} (\psi^* \nabla \psi - \psi \nabla \psi^*), \tag{2.11} \]
\[ \text{div} \ A = 0. \]

The initial conditions are given by

\[ \psi(0) = \psi_0, \quad A(0) = A_0. \tag{2.12} \]

The boundary conditions are one of the following:

Neumann boundary condition: For the case where \( \Omega \) is enclosed by an insulator:

\[ \frac{\partial \psi}{\partial n} = 0, \quad A_n = 0, \quad \text{curl} \ A \times n = H_a \times n \quad \text{on} \ \partial \Omega. \tag{2.13} \]

Dirichlet boundary condition: For the case where \( \Omega \) is enclosed by a magnetic material:

\[ \psi = 0, \quad A_n = 0, \quad \text{curl} \ A \times n = H_a \times n \quad \text{on} \ \partial \Omega. \tag{2.14} \]

Robin boundary condition: For the case where \( \Omega \) is enclosed by a normal metal:

\[ \frac{\partial \psi}{\partial n} + C \psi = 0, \quad A_n = 0, \quad \text{curl} \ A \times n = H_a \times n \quad \text{on} \ \partial \Omega. \tag{2.15} \]

Remark 2.1: If the material is a loop, or a plate \( \Omega = \tilde{\Omega} \times (0, h) \) with the height \( h \) being small in comparison to the diameter of \( \tilde{\Omega} \), then it is reasonable to consider the boundary condition with periodicity either in \( x \) direction or in \( (x, y) \) directions.

### III. Dynamic Bifurcation Theory

#### A. Attractor bifurcation

We recall in this section a general theory on attractor bifurcation developed by the authors. \(^{3,4,16}\)

Let \( H \) and \( H_1 \) be two Hilbert spaces, and \( H_1 \hookrightarrow H \) be a dense and compact inclusion. We consider the following nonlinear evolution equations:

\[ \frac{du}{dt} = L_\lambda u + G(u, \lambda), \tag{3.1} \]

\[ u(0) = u_0, \]

where \( u : [0, \infty) \to H \) is the unknown function, \( \lambda \in \mathbb{R} \) is the system parameter, and \( L_\lambda : H_1 \to H \) are parameterized linear completely continuous fields depending continuously on \( \lambda \in \mathbb{R}^1 \), which satisfy

\[ L_\lambda = -A + B_\lambda \quad \text{a sectorial operator,} \]

\[ A : H_1 \to H \quad \text{a linear homeomorphism,} \tag{3.2} \]

\[ B_\lambda : H_1 \to H \quad \text{parameterized linear compact operators.} \]

It is easy to see \(^{17} \) that \( L_\lambda \) generates an analytic semigroup \( \{e^{-tL_\lambda}\}_{t \geq 0} \). Then we can define fractional power operators \( L_\lambda^\alpha \) for any \( 0 \leq \alpha \leq 1 \) with domain \( H_\alpha = D(L_\lambda^\alpha) \) such that \( H_{\alpha_1} \subset H_{\alpha_2} \) if \( \alpha_1 > \alpha_2 \), and \( H_0 = H \).

Further, we assume that the nonlinear terms \( G(\cdot, \lambda) : H_\alpha \to H \) for some \( 0 \leq \alpha < 1 \) are a family of parameterized \( C^r \) bounded operators (\( r \geq 1 \)) depending continuously on the parameter \( \lambda \in \mathbb{R}^1 \), such that

\[ G(u, \lambda) = o(\|u\|_{H_\alpha}), \quad \forall \ \lambda \in \mathbb{R}^1. \tag{3.3} \]
In this paper, we are interested in the sectorial operator \( L_\lambda = -A + B_\lambda \) such that there exist an eigenvalue sequence \( \{\rho_k\} \subset \mathbb{C}^1 \) and an eigenvector sequence \( \{e_k, h_k\} \subset H_1 \) of \( A \):

\[
A z_k = \rho_k z_k, \quad z_k = e_k + ih_k,
\]

\[
\text{Re} \ \rho_k \to \infty (k \to \infty),
\]

\[
|\text{Im} \ \rho_k/(\text{Re} \ \rho_k)| \leq C,
\]

for some \( C > 0 \), and such that \( \{e_k, h_k\} \) is a basis of \( H \).

Condition (3.4) implies that \( A \) is a sectorial operator. For the operator \( B_\lambda : H_1 \to H \), we also assume that there is a constant \( 0 < \theta < 1 \) such that

\[
B_\lambda : H_\theta \to H \text{ bounded}, \quad \forall \lambda \in \mathbb{R}^1.
\]  

Under conditions (3.4) and (3.5), the operator \( L_\lambda = -A + B_\lambda \) is a sectorial operator.

Let \( \{S_\lambda(t)\}_{t \geq 0} \) be an operator semi-group generated by Eq. (3.1), then the solution of (3.1) can be expressed as

\[
u(t) = S_\lambda(t)u_0, \quad t \geq 0.
\]

**Definition 3.1:** A set \( \Sigma \subset H \) is called an invariant set of (3.1) if \( S(t) \Sigma = \Sigma \) for any \( t \geq 0 \). An invariant set \( \Sigma \subset H \) of (3.1) is said to be an attractor if \( \Sigma \) is compact, and there exists a neighborhood \( U \subset H \) of \( \Sigma \) such that for any \( \varphi \in U \) we have

\[
\lim_{t \to \infty} \text{dist}_H(u(t, \varphi), \Sigma) = 0.
\]

The set \( U \) is called a basin of attraction of \( \Sigma \).

**Definition 3.2:** (1) We say that Eq. (3.1) bifurcates from \( (u, \lambda) = (0, \lambda_0) \) an invariant set \( \Omega_\lambda \), if there exists a sequence of invariant sets \( \{\Omega_\lambda_n\} \) of (3.1) such that \( 0 \in \Omega_\lambda_n \), and

\[
\lim_{n \to \infty} \lambda_n = \lambda_0,
\]

\[
\lim_{n \to \infty} \max_{x \in \Omega_\lambda_n} |x| = 0.
\]

(2) If the invariant sets \( \Omega_\lambda \) are attractors of (3.1), then the bifurcation is called attractor bifurcation.

A complex number \( \beta = \alpha_1 + i \alpha_2 \in \mathbb{C} \) is called an eigenvalue of \( L_\lambda \) if there are \( x, y \in H_1 \) such that

\[
L_\lambda x = \alpha_1 x - \alpha_2 y,
\]

\[
L_\lambda y = \alpha_2 x + \alpha_1 y.
\]

Now let the eigenvalues (counting the multiplicity) of \( L_\lambda \) be given by

\[
\beta_1(\lambda), \beta_2(\lambda), \ldots, \beta_k(\lambda) \in \mathbb{C},
\]

where \( \mathbb{C} \) is the complex space. Suppose that

\[
\text{Re} \beta_i(\lambda) \begin{cases}
< 0 & \text{if } \lambda < \lambda_0, \\
0 & \text{if } \lambda = \lambda_0, \\
> 0 & \text{if } \lambda > \lambda_0,
\end{cases} \quad \forall 1 \leq i \leq m,
\]

(3.6)
Let the eigenspace of $L_\lambda$ at $\lambda_0$ be

$$E_0 = \bigcup_{k \in \mathbb{N}} \bigcup_{j=m+1}^{\infty} \{ u, v \in H_1 | (L_{\lambda_0} - \beta_j(\lambda_0))^k w = 0, w = u + iv \}.$$ 

It is known that $\dim E_0 = m$.

**Theorem 3.3:** (attractor bifurcation) \(^{3,4}\) Assume that the conditions (3.3)–(3.7) hold true, and $u=0$ is a locally asymptotically stable equilibrium point of (3.1) at $\lambda = \lambda_0$. Then the following assertions hold true.

1. (3.1) bifurcates from $(u, \lambda) = (0, \lambda_0)$ to an attractor $A_\lambda$ for $\lambda > \lambda_0$, with $m-1 \leq \dim A_\lambda \leq m$, which is connected when $m > 1$;
2. The attractor $A_\lambda$ is a limit of a sequence of $m$-dimensional annulus $M_k$ with $M_{k+1} \subset M_k$; especially if $A_\lambda$ is a finite simplicial complex, then $A_\lambda$ has the homotopy type of $S^{m-1}$;
3. For any $u_\lambda \in A_\lambda$, $u_\lambda$ can be expressed as
   $$u_\lambda = v_\lambda + o(\|v_\lambda\|_{H_1}), \quad v_\lambda \in E_0;$$
4. There is an open set $U \subset H$ with $0 \in U$ such that the attractor $A_\lambda$ bifurcated from $(0, \lambda_0)$ attracts $U \setminus \Gamma$ in $H$, where $\Gamma$ is the stable manifold of $u=0$ with co-dimension $m$.

**B. Center manifold functions**

In this section, we introduce a method to derive the first order approximation of the central manifold functions, which was used in Ref. 4. For convenience, we first introduce the center manifold theorem in infinite dimensional spaces.

Let $H_1$ and $H$ be decomposed into

$$H_1 = E_1^\lambda \oplus E_2^\lambda,$$

$$H = \overline{E_1^\lambda} \oplus \overline{E_2^\lambda},$$

for $\lambda$ near $\lambda_0 \in \mathbb{R}^1$, where $E_1^\lambda$, $E_2^\lambda$ are invariant subspaces of $L_\lambda$, such that

$$\dim E_1^\lambda < \infty,$$

$$\overline{E_1^\lambda} = E_1^\lambda,$$

$$\overline{E_2^\lambda} = \text{closure of } E_2^\lambda \text{ in } H.$$ 

In addition, $L_\lambda$ can be decomposed into $L_\lambda = L_1^\lambda \oplus L_2^\lambda$ such that for any $\lambda$ near $\lambda_0$,

$$L_1^\lambda = L_{\lambda E_1^\lambda}: E_1^\lambda \rightarrow \overline{E_1^\lambda},$$

$$L_2^\lambda = L_{\lambda E_2^\lambda}: E_2^\lambda \rightarrow \overline{E_2^\lambda},$$

where the eigenvalues of $L_2^\lambda$ possess negative real parts, and the eigenvalues of $L_1^\lambda$ possess nonnegative real parts at $\lambda = \lambda_0$.

Thus, for $\lambda$ near $\lambda_0$, Eq. (3.1) can be written as
\[
\frac{dx}{dt} = L_1^\lambda x + G_1(x,y,\lambda),
\]
\[
\frac{dy}{dt} = L_2^\lambda y + G_2(x,y,\lambda),
\]  
where \( u = x + y \in H_1, \ x \in E_1^\lambda, \ y \in E_2^\lambda, \ G_i(x,y,\lambda) = P_i G(u,\lambda), \) and \( P_i : H \rightarrow \tilde{E}_i^\lambda \) are canonical projections. Further, let
\[
E_2^\lambda(\alpha) = \text{closure of } E_2^\lambda \text{ in } H_\alpha,
\]
with \( \alpha < 1 \) given in (3.3).

The following center manifold theorem is classical. \(^{17}\)

**Theorem 3.4:** Assume (3.2), (3.3), (3.8), and (3.9). Then there exists a neighborhood of \( \lambda_0 \) given by \( |\lambda - \lambda_0| < \delta \) for some \( \delta > 0 \), a neighborhood \( U_{\lambda} \subset E_1^\lambda \) of \( x = 0 \), and a \( C^1 \) function \( \Phi(\cdot, \lambda) : U_{\lambda} \rightarrow \tilde{E}_2^\lambda(\theta) \) depending continuously on \( \lambda \), such that

1. \( \Phi(0,\lambda) = 0, \ \Phi'(0,\lambda) = 0, \)
2. the set
\[
M_{\lambda} = \{(x,y) \in H | x \in U_{\lambda}, y = \Phi(x,\lambda) \in E_2^\lambda(\theta)\},
\]
called the center manifolds, are locally invariant for (3.1), i.e., for each \( u_0 \in M_{\lambda} \)
\[
u_{\lambda}(t,u_0) \in M_{\lambda}, \quad \forall \ 0 \leq t < t(u_0)
\]
for some \( t(u_0) > 0 \), where \( u_{\lambda}(t,u_0) \) is the solution of (3.1);
3. if \( (x_{\lambda}(t), y_{\lambda}(t)) \) is a solution of (3.10), then there is a \( \beta_{\lambda} > 0 \) and \( k_{\lambda} > 0 \) with \( k_{\lambda} \) depending on \( (x_{\lambda}(0), y_{\lambda}(0)) \) such that
\[
\|y_{\lambda}(t) - \Phi(x_{\lambda}(t),\lambda)\|_H \leq k_{\lambda} e^{-\beta_{\lambda} t}.
\]

Now we give a formula to calculate the center manifold function. Let the nonlinear operator \( G \) be given by
\[
G(u,\lambda) = G_k(u,\lambda) + o(|u|^k),
\]  
for \( k \gg 2 \), where \( G_k(u,\lambda) \) is a \( k \)-multilinear operator:
\[
G_k : H_1 \times \cdots \times H_1 \rightarrow H,
\]
\[
G_k(u,\lambda) = G_k(u,\ldots, u, \lambda).
\]

The following theorem gives an approximation of the center manifold function; see Ref. 4.

**Theorem 3.5:** Under the conditions of Theorem 3.4, the center manifold function \( \Phi(x,\lambda) \) can be expressed as
\[
\Phi(x,\lambda) = (L_2^\lambda)^{-1}P_2 G_k(x,\lambda) + O(|\text{Re } \beta(\lambda)|, \|x\|^k) + o(|x|^k),
\]  
where \( L_2^\lambda \) is given by (3.9), \( P_2 : H \rightarrow \tilde{E}_2^\lambda \) the canonical projection, \( x \in E_1^\lambda \), and \( \beta(\lambda) = (\beta_1(\lambda), \ldots, \beta_m(\lambda)) \) the eigenvalues of \( L_1^\lambda \).

**C. \( S^1 \) attractor bifurcation**

In this section, we prove that the bifurcated attractor \( \Omega_{\lambda} \) of (3.1) from an eigenvalue with multiplicity two is homeomorphic to a circle \( S^1 \).

Let \( \nu \) be a two-dimensional \( C^r (r \geq 1) \) vector field given by
\[ v_\lambda(x) = \lambda x - G(x, \lambda), \] 

for \( x \in \mathbb{R}^2 \). Here

\[ G(x, \lambda) = G_k(x, \lambda) + o(|x|^k), \]

where \( G_k \) is a \( k \)-multilinear field, which satisfies that

\[ C_1|x|^{k+1} \leq (G_k(x, \lambda), x) \leq C_2|x|^{k+1}, \]

for some constants \( C_1 > 0, k = 2m + 1, \) and \( m \geq 1 \).

**Theorem 3.6:** Under condition (3.14), vector field (3.13) bifurcates from \( x = 0 \) to an attractor \( \Omega_\lambda \), which is homeomorphic to \( S^1 \). Moreover, one and only one of the following is true.

1. \( \Omega_\lambda \) is a periodic orbit,
2. \( \Omega_\lambda \) consists of only singular points, or
3. \( \Omega_\lambda \) contains at most \( 2(k+1) = 4(m+1) \) singular points, and has \( 4N + n(N + n \geq 1) \) singular points, \( 2N \) of which are saddle points, \( 2N \) of which are stable node points (possibly degenerate), and \( n \) of which have index zero, as shown in Fig. 4 for \( N = 1 \) and \( n = 2 \).

**Proof:** We proceed in the following five steps.

1. Obviously (3.14) implies that \( x = 0 \) is asymptotically stable for (3.13) at \( \lambda = 0 \). Hence, by Theorem 3.3, the vector field \( v_\lambda \) bifurcates from \( (x, \lambda) = (0, 0) \) to an attractor \( \Omega_\lambda \) on \( \lambda > 0 \), which has the homology type of a circle \( S^1 \).
2. Let \( \Omega_\lambda \) have no singular points. Then, \( \Omega_\lambda \) must contain at least one periodic orbit. We need to show that \( \Omega_\lambda \) contains only one periodic orbit.

Take the polar coordinate system \( (x_1, x_2) = (r \cos \theta, r \sin \theta) \). Then the vector field \( v_\lambda \) becomes

\[ \frac{dr}{d\theta} = r \cos \theta v_1 + \sin \theta v_2, \quad \frac{d\theta}{d\theta} = \cos \theta v_1 - \sin \theta v_2. \]

We see that

FIG. 4. \( \Omega_\lambda \) has \( 4N + n \) singular points, where \( p_1, p_4 \) are saddles, \( p_3, p_6 \) are nodes, and \( p_2, p_5 \) are singular points with index zero.
\[
\begin{align*}
\cos \theta v_1 &= \lambda r \cos^2 \theta - \cos \theta g_1(r \cos \theta, r \sin \theta, \lambda), \\
\sin \theta v_2 &= \lambda r \sin^2 \theta - \sin \theta g_2(r \cos \theta, r \sin \theta, \lambda), \\
\cos \theta v_2 &= \lambda r \cos \theta \sin \theta - \cos \theta g_2(r \cos \theta, r \sin \theta, \lambda), \\
\sin \theta v_1 &= \lambda r \sin \theta \cos \theta - \sin \theta g_1(r \cos \theta, r \sin \theta, \lambda),
\end{align*}
\]

where \( G(x, \lambda) = (g_1(x, \lambda), g_2(x, \lambda)) \). Let
\[
g_k(x, \lambda) = g_k(x, \lambda) + o(|x|^k), \quad i = 1, 2.
\]

By (3.14) and (3.15) is rewritten as
\[
\frac{dr}{d\theta} = \frac{\lambda - r^{2m}(\cos \theta g_{k_1} + \sin \theta g_{k_2}) + o(r^{2m})}{r^{2m-1}(\sin \theta g_{k_1} - \cos \theta g_{k_2} + O(r))},
\]

(3.16)

Based on (3.14), we have
\[
C_1 \leq \cos \theta g_{k_1}(\cos \theta, \sin \theta, \lambda) + \sin \theta g_{k_2}(\cos \theta, \sin \theta, \lambda) \leq C_2.
\]

(3.17)

On the other hand, by assumption, \( \Omega_\lambda \) contains a periodic orbit for any \( \lambda > 0 \) sufficiently small. Hence
\[
0 < C \leq \sin \theta g_{k_1}(\cos \theta, \sin \theta, \lambda) - \cos \theta g_{k_2}(\cos \theta, \sin \theta, \lambda) + O(r),
\]

(3.18)

for any \( 0 \leq \theta \leq 2\pi \) and some constant \( C > 0 \). Condition (3.18) amounts to saying that the orbits of \( v_\lambda \) are circular around \( x = 0 \).

Let \( r(\theta, r_0) \) be the solution of (3.16) with initial value \( r(0, r_0) = r_0 \). Then we have the following Taylor expansion:
\[
r^{2m}(\theta, r_0) = r_0^{2m} + R(\theta) \cdot o(|r_0|^{2m}), \quad R(0) = 0.
\]

(3.19)

It follows from (3.16) and (3.19) that
\[
\frac{1}{2m} [r^{2m}(2\pi, r_0) - r^{2m}(0, r_0)] = \int_0^{2\pi} \frac{\lambda - r^{2m}(\alpha(\theta) + o(r^{2m}))}{\beta(\theta) + O(r)} d\theta = 2\pi(a\lambda - b r_0^{2m}) + o(r_0^{2m}),
\]

(3.20)

where
\[
a = \int_0^{2\pi} \frac{1}{\beta(\theta) + O(r)} d\theta,
\]
\[
b = \int_0^{2\pi} \frac{\alpha(\theta)}{\beta(\theta) + O(r)} d\theta,
\]
\[
\alpha(\theta) = \cos \theta g_{k_1} + \sin \theta g_{k_2},
\]
\[
\beta(\theta) = \sin \theta g_{k_1} - \cos \theta g_{k_2}.
\]

From (3.20) we see that the periodic solutions of \( v_\lambda \) near \( x = 0 \) correspond to positive solutions of
\[2\pi(a\lambda - br_0^{2m}) + o(r_0^{2m}) = 0.\]  
(3.21)

By (3.17) and (3.18), \(a > 0\) and \(b > 0\). Therefore, (3.21) has a unique positive solution near \(r=0\):

\[r_0 = \left(\frac{a\lambda}{b}\right)^{1/2m} + o(\lambda^{1/2m}),\]

for any \(\lambda > 0\) sufficiently small. Thus, \(\Omega_\lambda\) has a unique periodic orbit.

(3) We claim that if \(\Omega_\lambda\) contains either finite number of singular points or a cycle of singular points, and if it contains finite number of singular points, then there are at most \(2(k+1)\) of them near \(x=0\).

In fact, if

\[\frac{g_1(x,\lambda)}{g_2(x,\lambda)} = \frac{x_1}{x_2},\]

then \(\Omega_\lambda\) has a cycle of singular points. Otherwise, by (3.14), the number of singular points of \(v_\lambda\) is finite. The maximal number of singular points for \(v_\lambda\) is determined by the following equation

\[\lambda x - G_x(x,\lambda) = 0.\]  
(3.22)

Since \(G_x\) is a \(k\)-multilinear vector field, the singular points of (3.22) must be on the straight lines \(x_2 = zx_1\), where \(z\) satisfies

\[z = \frac{g_{k2}(x_1,x_2,\lambda)}{g_{k1}(x_1,x_2,\lambda)} = \frac{g_{k2}(1,z,\lambda)}{g_{k1}(1,z,\lambda)}.\]  
(3.23)

The number of solutions of (3.23) is at most \(k+1\). Since \(k=\text{odd}\), the number of solutions of (3.22) is at most \(2(k+1)\).

(4) Let \(\Omega_\lambda\) contain a circle \(S^1\) of singular points, then we shall see that \(\Omega_\lambda = S^1\).

Under the polar coordinate system, we have

\[v_r(\theta, r) = (v_r, x) = \lambda r^2 - r^{k+1} \alpha(\theta) + o(r^{k+1}),\]

where \(\alpha(\theta)\) is defined by (3.20). By (3.17),

\[0 < C_1 \leq \alpha(\theta) \leq C_2, \quad 0 \leq \theta \leq 2\pi.\]

It is clear that for each \(\theta\) \((0 \leq \theta \leq 2\pi)\), \(v_r\) has a unique zero point \(r_\lambda(\theta) > 0\) near \(r=0\). Hence the set \(\overline{\Omega_\lambda} = \{((\theta, r_\lambda(\theta))) | v_r(\theta, r_\lambda(\theta)) = 0, \quad 0 \leq \theta \leq 2\pi\}\)

is homeomorphic to a cycle \(S^1\), and all singular points of \(v\) near \(x=0\) are on \(\overline{\Omega_\lambda}\). It implies that \(\overline{\Omega_\lambda} \subset \Omega_\lambda\). Let

\[g_{kl} = \sum_{i+j=k} \alpha^i_l x_1^i x_2^j, \quad l = 1, 2.\]  
(3.24)

By Step 3, we know that \(x_2 g_{k1} = x_1 g_{k2}\). Then we infer from (3.14) that

\[0 < \alpha^1_{k0} = \alpha^2_{k-1}.\]  
(3.25)

For the singular point \((\tilde{x}_1, 0) \in \overline{\Omega_\lambda}\) of \(v_\lambda\), we have

\[\text{div} v_\lambda(\tilde{x}_1, 0) = 2\lambda - k \alpha^1_{k0} \tilde{x}_1^{k-1} - \alpha^2_{k-1} \tilde{x}_1^{k-1} + o(\tilde{x}_1^{k-1})\]

\[= (\text{by } \alpha^1_{k0} \tilde{x}_1^{k-1} = \lambda \text{ and (3.25)}) = -(k-1)\lambda + o(\lambda) < 0,

for any \(\lambda > 0\) sufficiently small.
In the same fashion, for any point \( \tilde{x} \in \tilde{\Omega}_\lambda \), we take an orthogonal system transformation such that \( \tilde{x} \) is on the \( \tilde{x}_1 \) axis, then we can prove that

\[
\text{div } v_\lambda(x) < 0, \quad \forall \ x \in \tilde{\Omega}_\lambda,
\]

which implies that \( \Omega_\lambda = \tilde{\Omega}_\lambda = S^1 \).

(5) \( \Omega_\lambda \) contains finite number of singular points. We show that \( \Omega_\lambda = S^1 \).

By the Brouwer degree theory, it follows from (3.14) that

\[
\deg(v_\lambda, \Omega, 0) = 1, \quad |\lambda| > 0 \text{ sufficiently small},
\]

in some neighborhood \( \Omega \subset \mathbb{R}^2 \) of \( x=0 \). It is known that

\[
\text{ind}(v_\lambda, 0) = 1, \quad |\lambda| \neq 0.
\]

Hence we have

\[
\sum_{z_i \in \Omega_\lambda} \text{ind}(v_\lambda, z_i) = 0. \quad (3.26)
\]

Let \( z \in \Omega_\lambda \) be a singular point of \( v_\lambda \). Without loss of generality, we take the orthogonal coordinate system such that \( z = (x_1, 0) \). Then by (3.14) and (3.24), the Jacobian matrix of \( v_\lambda \) at \( z \) is given by

\[
Dv_\lambda(z) = \begin{pmatrix}
-(k-1)\lambda + o(\lambda) & * \\
0 & (1 - \alpha_{k-1}^2/\alpha_0^2)\lambda
\end{pmatrix}, \quad (3.27)
\]

where \( \alpha_0^2 > 0 \). Obviously, \( Dv_\lambda(z) \) has an eigenvalue \( \beta = -(k-1)\lambda + o(\lambda) \neq 0 \). Hence for any singular point \( z \in \Omega_\lambda \) of \( v_\lambda \), the index of \( v_\lambda \) at \( z \) can only be either 1, -1 or 0. It is easy to see that if the index is 1, then \( z \) is a stable node point.

Let the index of \( v_\lambda \) at \( z \) be -1:

\[
\text{index}(v_\lambda, z) = -1. \quad (3.28)
\]

When \( \alpha_{k-1}^2 \neq \alpha_0^1 \), \( z \) is nondegenerate. Therefore, \( v_\lambda \) has a unique unstable manifold at \( z \). When \( \alpha_{k-1}^2 = \alpha_0^1 \),

\[
\text{div } u_\lambda(z) = -(k-1)\lambda + o(\lambda) < 0. \quad (3.29)
\]

If the unstable manifold of \( v_\lambda \) at \( z \) is not unique, then the local structure of \( v_\lambda \) at \( z \) is topologically equivalent to that as shown in Fig. 5.

On the other hand, (3.29) means that there is a neighborhood \( \mathcal{O} \subset \mathbb{R}^2 \) of \( x=0 \), such that

\[
\text{div } u_\lambda(x) < 0, \quad \forall \ x \in \mathcal{O},
\]

which implies that for any open set \( \tilde{\mathcal{O}} \subset \mathcal{O} \),

\[
|\tilde{\mathcal{O}}| > |\tilde{\mathcal{O}}_t|, \quad 0 < t < t_0, \quad (3.30)
\]

where \( t_0 > 0 \) depends on \( \tilde{\mathcal{O}} \), \( \tilde{\mathcal{O}} = S(t)\tilde{\mathcal{O}} \) and \( S(t) \) is the flow semigroup generated by \( v_\lambda \).

However, it is clear that for any open set \( \tilde{\mathcal{O}} \subset \mathcal{O} \) in domain \( P \) as shown in Fig. 5, the property (3.30) is not true. Therefore, the unstable manifold of \( v_\lambda \) at \( z \) must be unique.

We can prove in the same fashion that if the index of \( v_\lambda \) at \( z \) is 0, then the unstable manifold of \( v_\lambda \) at \( z \) is also unique.

By the Poincaré-Bendixson theorem, all unstable manifolds of the singular points of \( v_\lambda \) at \( z \) with index -1 and 0 are connected to the singular points with index 1 and 0, as shown in Fig. 4.
Thus by the uniqueness of unstable manifolds for each singular point with either index $-1$ or index $0$, the set of all singular points and unstable manifolds is a circle $S^1$, and $\Omega_\lambda = S^1$.

The proof is complete. \halmos

IV. ATTRACTOR BIFURCATION FOR TDGL EQUATIONS

A. Mathematical setting

It is known that for a given applied field $H_a$ with $\text{div} H_a = 0$, there exists a field $A_a$ such that

$$\text{curl} A_a = H_a \quad \text{in} \; \Omega,$$

$$\text{div} A_a = 0 \quad \text{in} \; \Omega,$$

$$A_a \cdot n = 0 \quad \text{on} \; \partial \Omega. \quad (4.1)$$

Let $A = A + A_a$. Then (2.11) are rewritten as

$$\psi_t + i \phi \psi = - (i \mu \nabla + A_a)^2 \psi + \alpha \psi - 2 A_a \cdot A \psi - 2 i \mu A \cdot \nabla \psi - |A|^2 \psi - \beta |\psi|^2 \psi,$$

$$\zeta(A, + \mu \nabla \phi) = - \text{curl}^2 A - \gamma A |\psi|^2 - \gamma A |\psi|^2 - \frac{\gamma \mu}{2} (\psi^* \nabla \psi - \psi \nabla \psi^*), \quad (4.2)$$

$$\text{div} A = 0,$$

with the following initial and boundary conditions

$$\psi(0) = \psi_0, \quad A(0) = A_0, \quad (4.3)$$
\[ A_n = 0, \quad \text{curl} \, A \times n = 0 \quad \text{on} \ \partial \Omega, \quad (4.4) \]

together with one of the following three boundary conditions for \( \psi \):

**Neumann boundary condition:**

\[ \frac{\partial \psi}{\partial n} = 0 \quad \text{on} \ \partial \Omega, \quad (4.5) \]

**Dirichlet boundary condition:**

\[ \psi = 0 \quad \text{on} \ \partial \Omega, \quad (4.6) \]

**Robin boundary condition:**

\[ \frac{\partial \psi}{\partial n} + C \psi = 0 \quad \text{on} \ \partial \Omega. \quad (4.7) \]

Hereafter we use \( H^k(\Omega, \mathbb{C}) \) for the Sobolev spaces of complex valued functions defined on \( \Omega \), and \( H^k(\Omega, \mathbb{R}^3) \) for the Sobolev spaces of vector valued functions. Let

\[ H^2(\Omega, \mathbb{C}) = \{ \psi \in H^2(\Omega, \mathbb{C}) | \psi \text{ satisfy one of (4.5) - (4.7)} \}, \]

\[ D^2(\Omega, \mathbb{R}^3) = \{ A \in H^2(\Omega, \mathbb{R}^3) | \text{div} \, A = 0, \ A \text{ satisfy (4.4)} \}, \]

\[ L^2(\Omega, \mathbb{R}^3) = \{ A \in L^2(\Omega, \mathbb{R}^3) | \text{div} \, A = 0, \ A_{n|\partial \Omega} = 0 \}. \]

We set

\[ H = L^2(\Omega, \mathbb{C}) \times L^2(\Omega, \mathbb{R}^3), \]

\[ H_1 = H^2(\Omega, \mathbb{C}) \times D^2(\Omega, \mathbb{R}^3). \]

Let

\[ P : L^2(\Omega, \mathbb{R}^3) \to L^2(\Omega, \mathbb{R}^3) \]

be the Leray projection. Then it is known that the function \( \phi \) in (4.2) is determined uniquely up to constants by

\[ \zeta \mu \nabla \phi = (I - P) \left[ \frac{\gamma \mu}{2} i(\psi \nabla \psi^* - \psi^* \nabla \psi) - \gamma |\psi|^2 (A + A_n) \right], \quad (4.8) \]

where \( I \) is the identity on \( L^2(\Omega, \mathbb{R}^3) \). Namely, for every \( u = (\psi, A) \in H_1 \), there is a unique solution of (4.8) up to constants. Therefore, we define a nonlinear operator \( \Phi : H_1 \to L^2(\Omega) \) by

\[ \Phi(u) = \phi = \text{the solution of (4.8) with} \int_\Omega \phi \, dx = 0. \quad (4.9) \]

**B. Eigenvalue problems**

In order to describe the dynamic bifurcation of the Ginzburg–Landau equations, it is necessary to consider the eigenvalue problems of the linearized equations.

Let \( \alpha_1 \) be the first eigenvalue of the following equation:

\[ \]
\begin{equation}
(i \mu \nabla + A_n)^2 \psi = \alpha \psi, \quad \forall x \in \Omega,
\end{equation}

with one of the boundary conditions \((4.5)-(4.7)\). It is clear that \((4.10)\) can be equivalently expressed as

\begin{align}
-\mu^2 \Delta \psi_1 + |A_n|^2 \psi_1 - 2 \mu A_n \cdot \nabla \psi_2 &= \alpha \psi_1, \\
-\mu^2 \Delta \psi_2 + |A_n|^2 \psi_2 + 2 \mu A_n \cdot \nabla \psi_1 &= \alpha \psi_2,
\end{align}

where \(\psi = \psi_1 + i \psi_2\).

It is not difficult to check that \((4.11)\) with one of the boundary conditions \((4.5)-(4.7)\) are symmetric. Therefore, there are an infinite real eigenvalue sequence of \((4.10)\):

\begin{equation}
\alpha_1 < \alpha_2 < \cdots,
\end{equation}

and an eigenvector sequence

\begin{equation}
\{e_n \in H^2_0(\Omega, \mathbb{C})| n = 1, 2, \ldots\},
\end{equation}

which is an orthogonal basis of \(L^2(\Omega, \mathbb{C})\).

The eigenvalues of \((4.10)\) always have even multiplicity, i.e., if \(\psi\) is an eigenvector of \((4.10)\), then \(e^{i\theta} \psi (\theta \in \mathbb{R})\) are also eigenvectors corresponding to the same eigenvalue. Let the first eigenvalue \(\alpha_1\) have multiplicity \(2m(m \geq 1)\) with eigenvectors

\begin{equation}
e_{2k-1} = \psi_{k1} + i \psi_{k2}, \quad e_{2k} = -\psi_{k2} + i \psi_{k1}, \quad 1 \leq k \leq m.
\end{equation}

We know that \(\alpha_1\) enjoys the following properties:

\begin{equation}
\alpha_1 = \alpha_1(A_n) \quad \text{depends continuously on } A_n,
\end{equation}

\begin{equation}
\alpha_1(A_n) > 0 \quad \text{for } A_n \neq 0,
\end{equation}

\begin{equation}
\alpha_1(0) = 0 \quad \text{for the boundary condition } (4.5),
\end{equation}

\begin{equation}
\alpha_1(0) > 0 \quad \text{for either } (4.6) \text{ or } (4.7).
\end{equation}

Now, we consider another eigenvalue problem, which is also crucial for the attractor bifurcation of \((4.2)\). The problem reads

\begin{equation}
curl^2 A + \nabla \phi = \rho A,
\end{equation}

\begin{equation}
div A = 0,
\end{equation}

\begin{equation}
A_n|_{\partial \Omega} = 0, \quad \text{curl } A \times n|_{\partial \Omega} = 0.
\end{equation}

We remark that the boundary condition in \((4.16)\), i.e., \((4.4)\), is the free boundary condition, which can be expressed as

\begin{equation}
A_n|_{\partial \Omega} = 0, \quad \frac{\partial A_n}{\partial n}
_{\partial \Omega} = 0,
\end{equation}

where \(\tau\) is the tangent vector on \(\partial \Omega\).
To see this, for a given point $x_0 \in \partial \Omega$, we take $(\tau_1, \tau_2, n)$ as an orthogonal coordinate system, where $\tau_1, \tau_2$ are unit tangent vectors and $n$ the outward unit vector at $x_0 \in \partial \Omega$. We infer then from the condition $A_n \big|_{\partial \Omega} = 0$ that

$$\text{curl} A(x_0) = -\frac{\partial A_{\tau_2}}{\partial n} \tau_1 + \frac{\partial A_{\tau_1}}{\partial n} \tau_2 + \left( \frac{\partial A_{\tau_2}}{\partial \tau_1} - \frac{\partial A_{\tau_1}}{\partial \tau_2} \right) n_{x=x_0}.$$ 

Hence we have

$$\text{curl} A(x_0) \times n = \frac{\partial A_{\tau_1}}{\partial n} \tau_1 - \frac{\partial A_{\tau_2}}{\partial n} \tau_2 \bigg|_{x=x_0},$$

which implies that (4.4) is equivalent to (4.17).

It is known that there are a real eigenvalue sequence

$$0 < \rho_1 < \rho_2 < \cdots ,$$

$$\lim_{k \to \infty} \rho_k = \infty,$$ (4.18)

and an eigenvector sequence

$$\{ a_k \in D^2(\Omega, \mathbb{R}^3) | k = 1, 2, \ldots \},$$ (4.19)

which constitutes an orthogonal basis of $\mathcal{L}^2(\Omega, \mathbb{R}^3)$.

C. Main theorems

In superconductivity, the parameter $\alpha$ cannot exceed a maximal value $\alpha(T) \leq \alpha(0)$. Hence, we have to impose a basic hypothesis:

$$\alpha_1 < \alpha(0) = \frac{2 \sqrt{bm} DN_0}{e^2 h},$$ (4.20)

where $\alpha_1$ is the first eigenvalue of (4.10), and $N_0$ the density of states at the Fermi level.

In this subsection, we consider the case where the first eigenvalue $\alpha_1$ of (4.10) has multiplicity two. We start with the introduction of a physical parameter, which determines completely the dynamic properties of the bifurcation behavior of the TDGL equations.

Let $e \in H^2(\Omega, \mathbb{C})$ be a first eigenvector of (4.10). Then there is a unique solution for

$$\text{curl}^2 A_0 + \nabla \phi = |e|^2 A_0 + \frac{\mu}{2} (e^* \nabla e - e \nabla e^*),$$

$$\text{div} A_0 = 0,$$ (4.21)

$$A_0 \cdot n\big|_{\partial \Omega} = 0, \quad \text{curl} A_0 \times n\big|_{\partial \Omega} = 0.$$

We define a physical parameter $R$ as follows

$$R = -\frac{\beta}{\gamma} + \frac{2}{\int_{\Omega} |\text{curl} A_0|^2 dx} \int_{\Omega} |e|^4 dx,$$ (4.22)
It is clear that the parameter $R$ is independent of the choice of the first eigenvectors of (4.10).

Since the first eigenvector $e$ of (4.10) and $h_0=\text{curl} A_0$ given by (4.21) depend on the applied magnetic potential $A_0$ and the geometric properties of $\Omega$, the parameter $R$ is essentially a function of $A_0, \Omega$ and physical parameters $\beta, \gamma, \mu$.

The parameter $R$ defined by (4.22) can be equivalently expressed as follows

$$R = -\frac{\beta}{\gamma} + \frac{2\sum_{k=1}^{\infty} \frac{1}{\rho_k} \left[ \int_{\Omega} (|e|^2 A_0 + 2\mu e \cdot e_1) \cdot a_k \right]^2}{\int_{\Omega} |e|^4dx},$$

where $e=e_1+ie_2, \rho_k$ are the eigenvalue of (4.16) given by (4.18), and $\{a_k\}$ are the normalized eigenvectors given by (4.19).

The main results in this section are the following theorems. Here, we always assume that the first eigenvalue $\alpha_1$ of (4.10) with one of the boundary conditions (4.5)–(4.7) is complex simple, and the condition (4.20) holds true.

**Theorem 4.1:** If the number $R$ defined by (4.22) satisfies $R<0$, then for the problem (4.2)–(4.4) with one of (4.5)–(4.7), the following assertions hold true.

1. If $\alpha \leq \alpha_1$, the steady state $(\psi, A)=0$ is locally asymptotically stable for the problem.
2. The equations bifurcate from $(\psi, A)=0$ to an attractor $\Sigma_0$ for $\alpha > \alpha_1$, which is homeomorphic to $S^1$, and consists of steady state solutions of the problem.
3. There is a neighborhood $U \subset H$ of $(\psi, A)=0$ such that the attractor $\Sigma_0$ attracts $U \cap \Gamma$ in $H$, where $\Gamma$ is the stable manifold of $(\psi, A)=0$ with co-dimension two in $H$.
4. Each $(\psi, A) \in \Sigma_0$ can be expressed as

$$\psi = \left| \frac{\alpha - \alpha_1}{R_1} \right|^{1/2} e + o \left( \left| \frac{\alpha - \alpha_1}{R_1} \right|^{1/2} \right),$$

$$\text{curl}^2 A = -\gamma \left| \frac{\alpha - \alpha_1}{R_1} \right|^2 \left[ |e|^2 A_0 + \mu \text{Im}(e \cdot e_1) \right] + o \left( \left| \frac{\alpha - \alpha_1}{R_1} \right| \right),$$

$$R_1 = \frac{\gamma R \int_{\Omega} |e|^4dx}{\int_{\Omega} |e|^2dx},$$

where $e$ is the first eigenvector of (4.10).

**Theorem 4.2:** If $R>0$, then for the problem (4.2)–(4.4) with one of (4.5)–(4.7), we have the following assertions:

1. The steady state $(\psi, A)=0$ is locally asymptotically stable at $\alpha < \alpha_1$, and unstable at $\alpha \geq \alpha_1$.
2. The equations bifurcate from $(\psi, A)=0$ to an invariant set $\Sigma_0$ on $\alpha < \alpha_1$, and have no bifurcation on $\alpha > \alpha_1$.
3. $\Sigma_0=S^1$ is a circle consisting of steady states, and has a two-dimensional unstable manifold.

Theorems 4.1 and 4.2 show that the two cases with $R<0$ and $R>0$ have completely different superconducting transition characteristics; see Sec. V for further discussion.

It is easy to check that if $\alpha=0$, $(\Psi, A)=0$ is globally asymptotically stable for (4.2)–(4.4) with one of (4.5)–(4.7); see Ref. 15. The following theorem is a direct consequence of the existence of the global attractor for the TDGL model and Theorem 4.2.
Theorem 4.3: For the case where $R > 0$, there exists a saddle-node bifurcation point $a_0(0 < a_0 < a_1)$ for the TDGL equations, such that the following statements hold true, which are described schematically by Figures 2 and 3:

1. At $a=a_0$, there is an invariant set $\Sigma_0=\Sigma_{a_0}$ with $0 \in \Sigma_0$.
2. For $a<a_0$, there is no invariant set near $\Sigma_{a_0}$.
3. For $a_0 < a < a_1$, there are two connected branches of invariant sets $\Sigma_a^1$ and $\Sigma_a^2$, and $\Sigma_a^2$ extends to $a=a_1$ and near $a_1$ as well.
4. For each $a > a_0$,
   a. $\Sigma_a^2$ is an attractor with $\text{dist}(\Sigma_a^2,0) > 0$ at $a=a_1$,
   b. $\Sigma_a^2$ consists of steady state solutions and orbits connecting them, and
   c. $\Sigma_a^2$ contains at least one cycle of steady states.
5. For $a_0 < a < a_1$,
   a. $\Sigma_a^1$ is a repeller with $0 \in \Sigma_a^1$.
   b. $\Sigma_a^1$ consists of steady state solutions and orbits connecting them,
   c. $\Sigma_a^1$ contains at least one cycle of steady states, and
   d. when $a$ is near $a_1$, $\Sigma_a^1$ is exactly the $G_a=S^1$ given in Theorem 4.2, consisting exactly of steady states.

D. Proof of Theorems 4.1 and 4.2

We proceed in several steps as follows.
1. We set the mappings $L_a=-K+B_a$ and $G:H_1 \rightarrow H$ by
   
   \[ Ku = \left( \frac{(i\mu \nabla + A_a)^2 \psi}{\xi^{-1} \text{curl}^2 A_a} \right), \]
   
   \[ B_a u = \begin{pmatrix} \alpha \psi \\ 0 \end{pmatrix}, \]
   
   \[ G(u) = \begin{pmatrix} i \psi \Phi(u) + 2 A_a \cdot A \psi + 2i \mu A \cdot \nabla \psi + |A|^2 \psi + \beta |\psi|^2 \psi \\ P \left[ \gamma \xi^{-1} A_a |\psi|^2 + \gamma \xi^{-1} A_a |\psi|^2 + \frac{\gamma \mu i}{2 \xi} (\psi^* \nabla \psi - \psi \nabla \psi^*) \right] \end{pmatrix}, \]
   
   where $u=(\psi,A)$, $\Phi(u)$ is defined by (4.9), and $P$ the Leray projection. Thus, the problem (4.2)–(4.4) with one of the boundary conditions (4.5)–(4.7) can be rewritten in the following operator form

   \[ \frac{du}{dr} = L_a u + G(u), \quad u = (\psi,A) \in H_1, \]
   
   \[ u(0) = u_0. \] (4.25)

2. We see that $L_a:H_1 \rightarrow H$ is a sectorial operator, and the eigenvalues of $L_a$ satisfy that

   \[ \beta_1(\alpha) = \beta_2(\alpha) = \alpha - \alpha_1 < 0 \quad \text{if} \quad \alpha < \alpha_1 \]
   
   \[ \beta_1(\alpha) = 0 = \beta_2(\alpha) \quad \text{if} \quad \alpha = \alpha_1 \]
   
   \[ > 0 \quad \text{if} \quad \alpha > \alpha_1, \] (4.26)

   and for $j \geq 3$,
\[
\beta_j(\alpha_1) = \alpha_1 - \alpha_k \text{ or } -\xi^{-1}\rho_i, \\
\beta_j(\alpha_1) < 0,
\]

for some \(k > 1, l \geq 1\).

It is clear that the operator \(\Phi: H_1 \rightarrow L^2(\Omega, \mathbb{C})\) defined by (4.9) is \(C^\infty\), and by the estimates proved in Ref. 15 for \(\Phi_1\), we have

\[
\int \Omega |\Phi(u)|\psi|^2 dx \leq \left[ \int \Omega |\Phi(u)|^3 dx \right]^{2/3} \left[ \int \Omega |\psi|^6 dx \right]^{1/3} \leq C\|u\|_{H_{1/2}}^{1/2} \|\psi\|^2_{L^6},
\]

where \(H_{1/2}\) is the closure of \(H_1\) for the \(H^{-1}\)-norm. Hence, it is not difficult to check that there is a number \(1/2 < \sigma < 1\) such that \(G: H_\sigma \rightarrow H\) is \(C^\infty\).

(2) It is known that the dynamic bifurcation of (4.25) is determined by its reduced equation to the center manifold.

Let

\[
\psi_0 \in E_1 = \{ze|z \in \mathbb{C} \text{ and } e \text{ the first eigenvector of (4.10)}\}.
\]

Then the reduced equation of (4.25) is given by

\[
\frac{d\psi_0}{dt} = \beta_1(\alpha)\psi_0 - P_1G(\psi_0 + \tilde{\psi}(\psi_0), \tilde{A}(\psi_0)),
\]

where \(P_1: H \rightarrow E_1\) is the canonical projection, and \(\tilde{\Phi}(\psi_0) = (\tilde{\psi}(\psi_0), \tilde{A}(\psi_0)) \in H_1\) the center manifold function.

The \(k\) multilinear operators \((k=2,3)\) in \(G\) are given by

\[
G_2(u) = -\left( \begin{array}{c} 2A_u \cdot A\psi + 2i\mu_A \cdot \nabla\psi \\ \xi^{-1}yA_v |\psi|^2 + \frac{y\mu}{2\xi} i(\psi^* \nabla \psi - \psi \nabla \psi^*) \end{array} \right),
\]

\[
G_3(u) = -\left( \begin{array}{c} i\psi \Phi_2(u) + |A|^2 \psi + \beta |\psi|^2 \psi \\ y\xi^{-1}A |\psi|^2 \end{array} \right),
\]

where \(\Phi_2(u)\) is the bilinear operator in \(\Phi(u)\).

By the first approximation of the center manifold reduction, the center manifold function \(\tilde{\Phi} = (\tilde{\psi}(\psi_0), \tilde{A}(\psi_0))\) satisfies that

\[
\text{curl}^2 \tilde{A} + \mu \nabla \phi = -yA_v |\psi_0|^2 - \frac{y\mu}{2} i(\psi^*_0 \nabla \psi_0 - \psi_0 \nabla \psi^*_0) + o(\|\psi_0\|^2, |\beta_1(\alpha)| \cdot \|\psi_0\|),
\]

\[
\tilde{\psi}(\psi_0) = O(\|\tilde{A}(\psi_0)\| \cdot \|\psi_0\|) = O(\|\psi_0\|^3).
\]

Based on (4.28) and (4.30), (4.28) can be expressed as

\[
\frac{d\psi_0}{dt} = \beta_1(\alpha)\psi_0 - g_3(\psi_0) + o(\|\psi_0\|^3) + O(\|\psi_0\|^3|\beta_1(\alpha)|),
\]

where

\[
g_3(\psi_0) = P_1[|\beta| \psi_0|^2 \psi_0 + 2A_u \cdot \tilde{A}_2 \psi_0 + 2i\mu \tilde{A}_2 \cdot \nabla \psi_0 + i\Phi_2 \psi_0],
\]
\[
\text{curl}^2 \vec{A}_2 + \nabla \phi = -\gamma A_s |\psi_0|^2 - \frac{\gamma \mu}{2} i (\psi'_0 \nabla \psi_0 - \psi_0 \nabla \psi'_0),
\]

\[
\text{div} \vec{A}_2 = 0,
\]

\[
\vec{A}_2 \cdot n|_{\partial \Omega} = 0, \quad \text{curl} \vec{A}_2 \times n|_{\partial \Omega} = 0. \quad (4.33)
\]

Equations (4.31) and (4.32) are the third-order expression of the reduction of (4.27) to the center manifold.

(3) From (4.32), we obtain

\[
\langle g_3(\psi_0), \psi_0 \rangle = \text{Re} \int_{\Omega} g_3(\psi_0) \psi_0^* dx = \int_{\Omega} [\beta |\psi_0|^4 + 2 |\psi_0|^2 A_s \cdot \vec{A}_2 + 2 \mu \vec{A}_2 \cdot (\psi'_2 - \psi_0 \nabla \psi'_2)] dx = \int_{\Omega} [\beta |\psi_0|^4 + 2 |\psi_0|^2 A_s \cdot \vec{A}_2 + 4 \mu \psi'_2 \vec{A}_2 \cdot \nabla \psi_0^2] dx, \quad (4.34)
\]

where \( \psi_0 = \psi_0^1 + i \psi_2^0 \).

Let \( \vec{A}_2 \) have the Fourier expansion for the basis (4.19) of \( \mathcal{L}^2(\Omega, \mathbb{R}^3) \) as follows:

\[
\vec{A}_2 = \sum_{k=1}^{\infty} y_k a_k.
\]

Then, for (4.33) we can derive the solution \( y_k \):

\[
y_k = -\frac{\gamma}{\rho_k} \int_{\Omega} [||\psi_0||^2 A_s \cdot a_k + 2 \mu \psi'_2 a_k \cdot \nabla \psi_0^2] dx. \quad (4.35)
\]

Inserting (4.35) into (4.34) we find

\[
\langle g_3(\psi_0), \psi_0 \rangle = \beta \int_{\Omega} |\psi_0|^4 dx - 2 \gamma \sum_{k=1}^{\infty} \frac{1}{\rho_k} \left( \int_{\Omega} |\psi_0|^2 A_s \cdot a_k dx \right)^2 + 4 \mu \left( \int_{\Omega} |\psi_0|^2 A_s \cdot a_k dx \right) \times \left( \int_{\Omega} \psi'_2 a_k \cdot \nabla \psi'_2 \ dx \right) + 4 \mu \left( \int_{\Omega} \psi'_2 a_k \cdot \nabla \psi_0^2 dx \right)^2. \quad (4.36)
\]

Let \( \psi_0 = x_1 e_1 + x_2 e_2 \), where \( (x_1, x_2) \in \mathbb{R}^2 \), and \( e_1 \) and \( e_2 \) are as in (4.14). Then we have

\[
\psi_0 = \psi_0^1 + i \psi_2^0,
\]

\[
\psi_0^1 = x_1 \psi_{11} - x_2 \psi_{12},
\]

\[
\psi_2^0 = x_1 \psi_{12} + x_2 \psi_{11}.
\]

Thus, we see that

\[
\int_{\Omega} |\psi_0|^4 dx = \int_{\Omega} (|\psi_0^1|^2 + |\psi_2^0|^2)^2 dx = (x_1^2 + x_2^2) \int_{\Omega} |e_1|^4 dx, \quad (4.37)
\]

\[
\int_{\Omega} |\psi_0|^2 A_s \cdot a_k dx = (x_1^2 + x_2^2) \int_{\Omega} |e_1|^2 A_s \cdot a_k dx, \quad (4.38)
\]
\[
\int_\Omega \psi^0 k \cdot \nabla \psi^0 \, dx = (x_1^2 + x_2^2) \int_\Omega \psi_{12} k \cdot \nabla \psi_{11} \, dx.
\]  
(4.39)

Here, in (4.39) we use the following equality:

\[
\int_\Omega \psi a_k \cdot \nabla \psi \, dx = -\frac{1}{2} \int_\Omega \psi^2 \, \text{div} \, a_k \, dx = 0,
\]

for any real function \( \psi \).

Putting (4.37)–(4.39) into (4.36) we find

\[
\langle g_3(\psi_0), \psi_0 \rangle = -\gamma R(x_1^2 + x_2^2) \int_\Omega |e_1|^4 \, dx,
\]

(4.40)

where \( R \) is as in (4.23). It is easy to see that both numbers in (4.22) and (4.23) are the same.

(4) We shall prove that the Ginzburg–Landau equations bifurcate from \((\psi, A), \alpha) = (0, \alpha_1)\) to at least one steady state solution.

Formally, there are two steady state Ginzburg–Landau systems, i.e. the stationary equations obtained directly from (4.2) which read

\[
(i \mu + A_\psi)^2 \psi + i \Phi \psi = \alpha \psi - 2A_\psi \cdot A \psi - 2i \mu A \cdot \nabla \psi - \lambda^2 \psi - \beta |\psi|^2 \psi,
\]

(4.41)

\[
\text{curl}^2 A + \zeta \mu \nabla \Phi = -\gamma (A + A_\psi)|\psi|^2 - \frac{\gamma \mu i}{2} (\psi^* \nabla \psi - \psi \nabla \psi^*),
\]

and the other one given by

\[
(i \mu \nabla + A_\psi)^2 \psi = \lambda \psi - 2A_\psi \cdot A \psi - 2i \mu A \nabla \psi - \lambda^2 \psi - \beta |\psi|^2 \psi,
\]

(4.42)

\[
\text{curl}^2 A = -\gamma (A + A_\psi)|\psi|^2 - \frac{\gamma \mu i}{2} (\psi^* \nabla \psi - \psi \nabla \psi^*).
\]

The form (4.42) was derived by Ginzburg and Landau in 1950 as the Euler-Lagrange equations of the free energy. In Ref. 15, it is proved that if \( u = (\psi, A) \in H_1 \) and \( \Phi \in H^1(\Omega) \) is a solution of (4.41) with (4.4) and (4.5), then we have that \( \Phi = 0 \). It is easy to prove in the same fashion that this assertion also holds true for the boundary conditions (4.6) and (4.7). Therefore, both steady state equations (4.41) and (4.42) are exactly the same.

The Ginzburg–Landau energy for Eq. (4.42) read

\[
E = \frac{1}{2} \int_\Omega \left[ |(i \mu \nabla + A_\psi|^2 + \frac{\beta}{2} |\psi|^4 - \alpha |\psi|^2 + \gamma^{-1} |\text{curl} A|^2 \right] \, dx.
\]

Therefore, by the classical bifurcation theorem for potential operators (see Refs. 18–20), the steady state equations (4.42) with (4.4) and one of the boundary conditions (4.5)–(4.7) must bifurcate to at least one solution from \((\psi, A), \alpha) = (0, \alpha_1)\).

(5) Proof of Theorem 4.1: When \( R < 0 \), by (4.31) and (4.40), we can obtain assertion (1), and we infer from Theorem 3.6 that the Ginzburg–Landau equations bifurcate from \((\psi, A), \alpha) = (0, \alpha_1)\) a cycle \( \Sigma_\psi \) of attractor for \( \alpha > \alpha_1 \). By Step 4, the attractor \( \Sigma_\psi \) contains a singular point. Because of the invariance of the Ginzburg–Landau equations for the gauge transformation.
\[ \psi \rightarrow \psi e^{i\theta}, \quad \theta \in \mathbb{R}^1, \]

the steady state solutions of the Ginzburg–Landau equations appear as a circle \( S^1 \). Hence the attractor \( \Sigma_\alpha = S^1 \) consists of steady state solutions. Assertion (2) is proved.

Assertion (3) follows from Theorem 3.3, and Assertion (4) can be directly derived from Eqs. (4.31) and (4.40). Thus, Theorem 4.1 is proved.

(6) Proof of Theorem 4.2: When \( R > 0 \), the time-reversed semigroup \( S_\alpha(-t) \) generated by (4.31) has the same dynamic properties as the following equation

\[
\frac{d\psi_0}{dt} = (\alpha_1 - \alpha)\psi_0 + g_3(\psi_0) + \alpha (|\psi_0|^2, |\alpha_1||\psi_0|^2).
\]

(4.43)

In the same fashion as used in Step 5, from (4.40) we infer that the semi-group \( \tilde{S}_\alpha(t) \) generated by (4.43) bifurcates from \( (\psi_0, \alpha) = (0, \alpha_1) \) to an \( S^1 \) attractor \( \Sigma_\alpha \) for \( \alpha < \alpha_1 \), which consists of singular points of (4.43). Hence, for the semi-group \( S_\alpha(t) = \tilde{S}_\alpha(-t) \) generated by (4.31) with \( R > 0 \), the Assertions (1)–(3) hold true.

Thus, Theorem 4.2 is proved.

E. Bifurcation from general eigenvalues

Although the bifurcation from general eigenvalues of (4.10) has less physical interest, we consider this problem partially for the mathematical completeness.

Before our discussion, we remark that one can prove\(^{21}\) that the first eigenvalue \( \lambda_1 \) with complex simplicity is generic. Namely, if we set

\[ \mathcal{H}^1 = \{ A \in H^1(\Omega, \mathbb{R}^3) | A \cdot n |_{\partial \Omega} = 0, \ \text{div} \ A = 0 \}, \]

then, there is an open and dense set \( U \subset \mathcal{H}^1 \) such that for any \( A_\alpha \in U \), the first eigenvalue \( \lambda_1(A_\alpha) \) of (4.10) is complex simple. However, we cannot exclude the existence of a bounded domain \( \Omega \subset \mathbb{R}^3 \) and a vector \( A_\alpha \in \mathcal{H}^1 \) such that the first eigenvalue \( \lambda_1(\Omega, A_\alpha) \) of (4.10) has higher complex multiplicity.

Let \( \alpha_k \) be an eigenvalue of (4.10) with complex multiplicity \( m \geq 1 \), and \( E_k \) be the eigenspace of \( \alpha_k \), i.e.

\[ E_k = \{ \psi \in H^2_0(\Omega, \mathbb{C}) | (i\mu \nabla + A_\alpha)^2 \psi = \alpha_k \psi \}. \]

It is clear that \( \dim E_k = 2 \dim \nu, E_k = 2m \).

Let \( F \) be function defined on \( E_k \):

\[ F(\psi) = 2 \sum_{n=1}^{\infty} \frac{1}{\rho_n} \left[ \int_{\Omega} (|\psi|^2 A_n + 2\mu \psi_2 \nabla \psi_1) \alpha_n dx \right]^2, \]

where \( \psi = \psi_1 + i\psi_2 \in E_k \), and \( \rho_n, \alpha_n \) are as in (4.18) and (4.19).

Set

\[ R_1 = \sup_{\psi \in E_k, \psi \neq 0} \frac{F(\psi)}{\int_{\Omega} |\psi|^4 dx}, \]

\[ R_2 = \inf_{\psi \in E_k, \psi \neq 0} \frac{F(\psi)}{\int_{\Omega} |\psi|^4 dx}. \]

(4.44)

Then, we have the following theorem.
Theorem 4.4: For the problem (4.2)–(4.4) with one of (4.5)–(4.7), we have the following assertions.

1. If the physical parameter \( \beta \gamma > R_1 \), then this problem bifurcates from \( (\psi, A) = (0, \alpha_0) \) to an invariant set \( \Sigma_\alpha \) for \( \alpha > \alpha_0 \).
2. If \( \beta \gamma < R_2 \), then this problem bifurcates from \( (\psi, A) = (0, \alpha_0) \) to an invariant set \( \Sigma_\alpha \) for \( \alpha < \alpha_0 \).
3. The invariant set \( \Sigma_\alpha \) is a \((2m-1)\)-dimensional homological sphere, i.e. \( 2m-1 \leq \text{dim} \Sigma_\alpha \leq 2m \), and \( \Sigma_\alpha \) has the same homology as a \((2m-1)\)-dimensional sphere.
4. \( \Sigma_\alpha \) contains at least a circle of singular points of the equations.
5. When \( m=1 \), \( \Sigma_\alpha \) is a circle \( S^1 \).
6. When \( \alpha_0 = \alpha_1 \) and \( \beta \gamma > R_1 \), \( \Sigma_\alpha \) is an attractor, which attracts an open set \( U \setminus \Gamma \), where \( U \subset H \) is a neighborhood of \( (\psi, A) = 0 \), and \( \Gamma \) is the stable manifold of \( (\psi, A) = 0 \) with co-dimension \( 2m \) in \( H \).

Remark 4.5: We conjecture that the bifurcated invariant set \( \Sigma_\alpha \) in Theorem 4.4 is homeomorphic to a \((2m-1)\)-dimensional sphere \( S^{2m-1} \), and \( \Sigma_\alpha \) contains at least \( m \) circles consisting of singular points.

When \( m=1 \), these two numbers \( R_1 \) and \( R_2 \) are the same: \( R_1 = R_2 \), and \( R = R_1 - \beta \gamma \) is as in (4.22).

The proof of Theorem 4.4 is the same as that of Theorems 4.1 and 4.2; we omit the details.

V. CONCLUSIONS AND REMARKS

A. General remarks

The permanent current, called supercurrent, is expressed in the Ginzburg–Landau equations by (2.2). In the steady state case, the supercurrent in the second type of nondimensional form is written as

\[
J_s = J_s(\psi, A) = -\gamma(A_n + A)|\psi|^2 - \frac{\gamma \mu}{2} i(\psi^* \nabla \psi - \psi \nabla \psi^*).
\]

To take the Meissner effect into account in the Ginzburg–Landau equation. Mathematically speaking, in the normal state, the magnetic field \( H \) in a sample should be \( H = H_n + H_a \), \( H_n = \text{curl} A_n \) is the applied field and \( H_a = \text{curl} A \) the nonequilibrium fluctuation, and in the superconducting state \( H = \text{curl} A \). In the both cases, \( A \) satisfy the Ginzburg–Landau equation (4.2) and boundary condition (4.4). Namely, we can express the magnetic field \( H \) in a sample \( \Omega \) in the following form

\[
H = \text{curl} A, \quad \forall x \in \Omega,
\]

\[
A = \begin{cases} 
A_n + A & \text{in the normal state}, \\
A & \text{in the superconducting state},
\end{cases}
\]

and the supercurrent \( J_s \) in the nondimensional form also is given by

\[
J_s = \text{curl}^2 A.
\]

Here \( A \) satisfies (4.2) and (4.4).

An equilibrium state \( (\tilde{\psi}, \tilde{A}) \) of the TDGL equations (4.2) is called in the normal state if \( \tilde{\psi} = 0 \), and \( (\tilde{\psi}, \tilde{A}) \) is called in the superconducting state if \( \tilde{\psi} \neq 0 \). A solution \( (\psi, A) \) of (4.2) is said in the normal state if \( (\psi, A) \) is in a domain of attraction of a normal equilibrium state, otherwise \( (\psi, A) \) is said in the superconducting state.

We consider the simplest case where the applied field vanishes \( A_n = 0 \). In this case, the eigenvalue equation (4.10) becomes
\[-\mu \Delta \psi = \alpha \psi.\] (5.4)

The first eigenvalue \(\alpha_1\) of (5.4) with one of the boundary conditions (4.5)–(4.7) is simple, and the eigenvector is real. Therefore, the parameter \(R\) defined by (4.22) reads

\[R = -\frac{\beta}{\gamma} < 0.\]

By Theorem 4.1, when the parameter \(\alpha(T) < \alpha_1\) the solutions \((\psi, A)\) of (4.2) is in the normal state, and when \(\alpha(T) > \alpha_1\), \((\psi, A)\) with the initial \((\psi_0, A_0)\) in \(U \cap \Gamma\) is in the superconducting states.

When \(A_n = 0\), the steady state solutions \((\tilde{\psi}, \tilde{A})\) of (4.2) are real, i.e. \(\tilde{\psi} = e^{i\theta} \psi, \tilde{J}_m \psi = 0\). Hence

\[\tilde{\psi} \nabla \tilde{\psi} - \psi \nabla \psi^* = 0,\]

which implies that \(\tilde{A} = 0\). Thus, the supercurrent (5.3) (or (5.1)) vanishes

\[J_s = 0.\]

This shows that with zero applied field \(H_a = 0\), there is no current in a superconductor.

**Implications of (4.20):** For the Neumann boundary condition (4.5), i.e. the sample is enclosed by an insulator, the first eigenvalue \(\alpha_1 = 0\) for (5.4), which is independent of \(\Omega\), the geometry of sample. Therefore, the condition (4.20) always holds true.

However, for the Dirichlet and the Robin boundary conditions (4.6) and (4.7), the situation is different. It is known that the first eigenvalue \(\alpha_1\) of (5.4) depends on \(\Omega\). In particular,

\[\alpha_1 = \alpha_1(\Omega) \to \infty \quad \text{if } |\Omega| \to 0.\]

The condition (4.20) implies that for the cases where the samples are enclosed by a magnetic material or a normal metal, the volume of a sample must be greater than some critical value \(|\Omega| > V_c > 0\). Otherwise no superconducting state occurs at any temperature. This property also holds true for the case where there is an applied magnetic field \(H_a\) present. Of course, in this case, the critical volume \(V_c\) depends on \(H_a\) as well.

**B. Transitions in the case with \(R<0\)**

As mentioned earlier, there are two transitions determined by a simple parameter \(R\) defined by (4.22) for transitions near a complex simple eigenvalue of the linearized problem (respectively by two parameters \(R_1\) and \(R_2\) defined by (4.44) for transitions near an eigenvalues with higher multiplicity). The parameter \(R\) links the superconducting behavior with the geometry of the material, the applied field and the physical parameters. For simplicity, we address here only the case near a complex simple eigenvalue of the linearized problem, and physical conclusions can be derived in the same fashion.

Let a magnetic field \(H_a = \text{curl} A_n\) be applied. By the bifurcation theorems, the critical temperature \(T_c^1\) of superconducting transition satisfies then \(T_c^1 < T_c^2\), where \(T_c^2\) is given in (2.4) and \(T_c^1\) satisfies that

\[\alpha(T_c^1) = \alpha_1 > 0,\] (5.5)

where \(\alpha_1 = \alpha_1(A_n)\) is the first eigenvalue of (4.10). It is known that

\[\alpha_1(A_n) \to \infty \quad \text{if } |A_n| \to \infty.\]

It implies that the applied magnetic field \(H_a\) can not be very strong for superconductivity as required by the condition (4.20).

From Theorems 4.1 and 4.2, we see that the number \(R\) defined by (4.22) is an important parameter to distinguish two different types of superconducting transitions. We first examine the case where \(R<0\).
By Theorem 4.1, when \( \alpha > \alpha_1 \), Eq. (4.2) bifurcate from \((\psi_n, A_n) = (0, \alpha_1)\) to a steady state solution \((\psi_n, A_n)\) which is an attractor attracting an open set \(U \setminus \Gamma \subset H\). Physically speaking, this theorem leads to the following properties of superconducting transitions in the case where \( R < 0 \); see Fig. 1.

**Theorem 5.1:** Let \( R < 0 \) and \( m = 1 \) as in Theorem 4.1, and \( T^1_c \) be given by (5.5). Then the following physical properties hold true

1. **When the control temperature decreases (resp. increases) and crosses the critical temperature \( T^1_c \), there will be a phase transition of the sample from the normal to superconducting states (respectively from superconducting to normal state).**

2. **(Stability) When the control temperature \( T \geq T^1_c \), under a fluctuation deviating the normal state, the sample will soon be restored to the normal state. In addition, when \( T < T^1_c \), under a fluctuation deviating both the normal and superconducting states, the sample will soon be restored to the superconducting states.**

3. **In general, the supercurrent given by**

\[
J_\alpha = -\gamma (A_n + A_n) |\psi_n|^2 - \frac{\nu_1}{2} (\nabla \psi_n \cdot \nabla \psi_n - |\psi_n|^2)
\]

is nonzero, i.e., \( J_\alpha \neq 0 \) for \( \alpha_1 < \alpha (T < T^1) \).

4. **(Continuity) The order parameter \( \Psi_n \) and supercurrent \( J_\alpha \) depend continuously on the parameter \( \alpha \) (or the control temperature \( T \)), namely**

\[
\psi_n \to 0, \quad J_\alpha \to 0, \quad \text{if} \quad \alpha \to \alpha_1 + 0 \quad \text{or} \quad T \to T^1_c - 0. \quad (5.6)
\]

5. **The superconducting state of the system is dominated by the lowest-energy eigenfunction of (4.10) in the sense given by (4.24).**

### C. Transitions with \( R > 0 \)

Transitions in this case are precisely described by Theorems 4.2 and 4.3, as shown in Figs. 2 and 3. In particular, we have the following theorem, which recapitulates some phase transition properties obtained in Theorems 4.2 and 4.3 in physical terms.

**Theorem 5.2:** Consider a material described by the TDGL with \( R > 0 \). There are two critical temperatures \( T^0_c \) and \( T^1_c \) \((T^0_c > T^1_c)\) such that

\[
\alpha(T^i_c) = \alpha_i \quad (i = 0, 1),
\]

and the following phase transition properties hold true:

1. **When the control temperature \( T \) decreases and crosses \( T^1_c \), or equivalently \( \alpha \) increases and crosses \( \alpha_1 \), the stability of the normal state changes from stable to unstable.**

2. **When \( T^1_c < T \) (or \( \alpha < \alpha_1 \)), physically observable states consists of the normal state, and the superconducting states in \( \Sigma^2_n \). When \( T^1_c > T \) (or \( \alpha > \alpha_1 \)), physically observable states are in \( \Sigma^2_n \).**

3. **(Instability) When the control temperature \( T \) is in the interval: \( T^1_c < T < T^0_c \) (or \( \alpha_0 < \alpha < \alpha_1 \)), the superconducting states given \( \Sigma^1_n \) are unstable, i.e., with a fluctuation deviating a superconducting state in \( \Sigma^1_n \), transition to either the normal state or a superconducting state in \( \Sigma^2_n \) will occur.**

4. **(Discontinuity) At the critical temperature \( T^0_c \) (respectively at \( T^1_c \)) of the phase transitions, there is a jump from the superconducting states to the normal state (respectively from the normal state to superconducting states).**

5. **The other lower-energy eigenfunctions possibly have a stronger influence for the superconducting states.**

It is noteworthy to remark that the phase transitions in the case where \( R > 0 \) is very different from the transitions for the case where \( R < 0 \) as described.
Physical significance of $R$: We note that the parameter $\beta/\gamma$ can be characterized by the Ginzburg–Landau parameter $\kappa$ and the parameter $\mu$:

$$\frac{\beta}{\gamma} = \kappa^2 \mu^2, \quad \kappa^2 = \frac{m^2 c^2}{2\pi e^2 h^2}, \quad \mu^2 = \frac{h D}{es \sqrt{b}}. \quad (5.7)$$

In the Ginzburg–Landau energy, the term

$$E_0 = \int_{\Omega} |e|^4 dx$$

represents the nonlinear part of the energy of the superconducting electrons in the lowest-energy state, and the term

$$E_m = \int_{\Omega} H_0^4 dx, \quad h_0 \text{satisfies (4.21)}, \quad (5.9)$$

is the energy contributed by the magnetic field associated with the supercurrent

$$\text{curl } h_0 = |e|^2 A_n + \frac{\mu}{2} i(e^* \nabla e - e \nabla e^*),$$

which is generated by the applied magnetic potential $A_0$ and the superconducting electrons in the lowest-energy states.

By (4.2) and (4.22), we obtain from (5.7)–(5.9) that

$$R = -\kappa^2 \mu^2 + \frac{2E_m}{E_0}. \quad (5.10)$$

Hence, the type of superconducting phase transitions among the two described above for a given material depends on the “competition” between the two energies $E_0$ and $E_m$:

$$R = \begin{cases} <0 & \text{if and only if } \frac{\kappa^2}{2} E_0 > \frac{1}{\mu^2} E_m, \\ >0 & \text{if and only if } \frac{\kappa^2}{2} E_0 < \frac{1}{\mu^2} E_m. \end{cases} \quad (5.11)$$

According to the Abrikosov theory, the materials with $\kappa^2 < 2$ and $\kappa^2 > 2$ are of types I and II, respectively. From (5.11), we infer that for given geometrical shape of sample and applied magnetic field, a type I material favors more to the jumped phase transition (i.e., the case $(R > 0)$), and a type II material favors the continuous phase transition (i.e the case $(R < 0)$).

D. Topological structure of supercurrents

In 1957, Abrikosov predicted that in the mixed state of type II superconductors, there is a square array of vortices of supercurrents, and this vortex array was confirmed by experiments later.

In the two-dimensional case, by Theorems 4.1 and 4.2, the bifurcated attractor contains superconducting states $(\psi, A)$ such that the supercurrent $J_s = J_s(\psi, A)$ given by (5.1) is in general not identically zero and

$$\text{div } J_s = 0, \quad J_s \cdot n|_{\partial \Omega} = 0. \quad (5.12)$$

Thanks to (5.12), the geometric theory for two-dimensional incompressible flows developed recently by the authors can be applied to study the structure of the supercurrent $J_s$ in the physical space. For this purpose, we first recall briefly some basic results and concepts directly related to the study of the structure of $J_s$.  


Structure of 2D Incompressible flows: First, let $C'(\Omega, \mathbb{R}^2)$ be the space of all $C'$ vector fields on $\Omega$, and

$$D'(\Omega) = \{ v \in C'(\Omega, \mathbb{R}^2) | v \cdot n|_{\partial \Omega} = 0, \text{div} v = 0 \}.$$ 

Let $v \in D'(\Omega)$. A point $p \in \Omega$ is called a singular point of $v$ if $v(p) = 0$; a singular point $p$ of $v$ is called nondegenerate if the Jacobian matrix $Dv(p)$ is invertible; $v$ is called regular if all singular points of $v$ are non-degenerate. An interior nondegenerate singular point of $v$ can be either a center or a saddle, and a nondegenerate boundary singularity must be a saddle. Saddles of $v$ must be connected to saddles. An interior saddle $p \in \Omega$ is called self-connected if $p$ is connected only to itself, i.e., $p$ occurs in a graph whose topological form is that of the number 8.

Second, let $v \in D'(\Omega)$ be regular, and $\Omega$ be a connected domain with $k$ holes. Let $C$ be the number of centers of $v$, $S$ the number of interior saddles, and $B$ the number of boundary saddles. Then\(^5,22\)

$$C - S - B = 1 - k. \quad (5.13)$$

Third, again, let $v \in D'(\Omega)$, and $p \in M$ be a center. Then there is an open neighborhood $C$ of $p$, such that for any $x \in C(x \neq p)$, the orbit $\{ \Phi(x, t) \}_{t \in \mathbb{R}}$ is closed. The largest such neighborhood $C$ of $p$ is called a circle cell of $v$. Let $B \subset M$ be an open set, such that for any $x \in B$, the orbit $\{ \Phi(x, t) \}_{t \in \mathbb{R}}$ is closed, and each connected component $\Sigma$ of $\partial B$ is not a single point. Then $B$ is called a circle band of $v$.

Then it is proved that\(^5,23\) for a regular divergence-free vector field $v \in D'(\Omega)(r = 1)$, the topological structure of $v$ consists of finite number of circle cells, circle bands, and saddle connections.

Fourth, two vector fields $u, v \in D'(\Omega, \mathbb{R}^2)$ are called topologically equivalent if there exists a homeomorphism of $\varphi: \Omega \rightarrow \Omega$, which takes the orbits of $u$ to orbits of $v$ and preserves their orientation. A vector field $v \in D'(\Omega, \mathbb{R}^2)$ is called structurally stable in $D'(\Omega, \mathbb{R}^2)$ if there exists a neighborhood $U \subset D'(\Omega, \mathbb{R}^2)$ of $v$ such that for any $u \in U$, $u$ and $v$ are topologically equivalent.

Then it is proved that $v \in D'(\Omega, \mathbb{R}^2)(r = 1)$ is structurally stable in $D'(\Omega, \mathbb{R}^2)$ if and only if $v$ is regular, b) all interior saddles of $v$ are self-connected, and c) each boundary saddle point is connected to boundary saddle points on the same connected component of the boundary. Moreover, the set of all structurally stable vector fields is open and dense in $D'(\Omega, \mathbb{R}^2)$.

Structure of $J_s$: Now we are in position to study the structure of the supercurrent $J_s$ in the physical space. First observe that in the context of superconductivity, the centers of the supercurrent correspond to vortices. Hence the following is a direct consequence of (5.13), which predicts the existence of vortices.

**Theorem 5.3.** Let the domain be simply connected. If the supercurrent $J_s$ for a given superconducting state is regular, then there is at least one vortex for this superconducting state.

When $R < 0$, the superconducting states in the bifurcated attractor is dictated by the first eigenfunction $e$ of (4.10). By Theorem 4.1 and the structural stability theorem mentioned above, the structure of the supercurrent $J_s$ for the superconducting states in the bifurcated attractor is determined by the structure of the following vector field:

$$J_0 = -A_i d\lambda^2 - \frac{\mu}{2} \epsilon^* \nabla e - e \nabla e^* + \nabla \phi, \quad (5.14)$$

which satisfies that $\text{div} J_0 = 0$ and $J_0 \cdot n|_{\partial \Omega} = 0$. Then it is easy to obtain the following result.

**Theorem 5.4.** Assume that the vector field $J_0$ given by (5.14) is structurally stable in $D'(\Omega, \mathbb{R}^2)(r = 1)$. Then there are an $\epsilon > 0$ and a time $t_0 > 0$ such that if $\alpha_t \leq \alpha < \alpha_t + \epsilon$ and $t > t_0$, then for any initial data $(\psi_0, A_0) \in U \setminus \Gamma$ where $U \setminus \Gamma \subset H$ is the open set given in Theorem 4.1, the supercurrent...
\[ J_s(\psi_a, A_a) = -\gamma (A_a + A_a)|\psi_a|^2 - \frac{\gamma \mu}{2} i(\psi^* \nabla_a \psi_a - \psi_a \nabla \psi_a^*), \]

corresponding to the solution \((\psi_a, A_a)\) is structurally stable, and is topologically equivalent to \(J_0\).

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