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Stability and Bifurcation of the Taylor Problem

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Abstract

The main objective of this paper is to address the stability and bifurcation of the Couette flow between two concentric rotating cylinders, and to verify rigorously TAYLOR'S observation in his experiments [13]. A nonlinear theory is obtained for the Taylor problem, leading in particular to rigorous justifications of the linear theory used by physicists, and the Taylor vortex structure. The main technical tools are the dynamic bifurcation theory and the geometric theory for incompressible flows, both developed recently by MA & WANG [10, 12].

1. Introduction

In his celebrated paper [13] in 1923, G.I. TAYLOR observed and studied the instability of laminar flow, now known as the Couette flow. Taylor studied the case where the gap between the two cylinders is small in comparison with the mean radius, and the two cylinders rotate in the same direction. He found that when the Taylor number T is smaller than a critical value $T_c > 0$, called the critical Taylor number, the Couette flow is stable, and when the Taylor number crosses the critical value, the Couette flow breaks out into a cellular pattern that is radially symmetric.

Since Taylor's pioneering work, there have been intensive studies into this problem; see, among others, CHANDRASEKHAR [1] and DRAZIN & REID [2] for linear theories, and KIRCHGÄSSNER [6], VELTE [15], and YUDOVICH [16], and the references therein, for nonlinear theories. Most, if not all, known results on bifurcation and stability of the Taylor problem are restricted to the analysis when the Taylor number crosses a simple eigenvalue in certain subspaces of the entire phase space obtained by imposing a certain symmetry.

The main objective of this paper is to address the stability and bifurcation of the Couette flow between concentric rotating cylinders, and to verify rigorously Taylor's observation. This work forms part of a research program to study the connections between Lagrangian dynamics and Eulerian dynamics. This field is both difficult and important, yet still very little is known. Our previous works on the boundary layer separation [12] and on the structure in the physical spaces of the bifurcated solutions of the Rayleigh-Bénard (RB) problem [9, 10] are two such examples in this direction.

The main tool is the new notion of bifurcation, called attractor bifurcation (and its corresponding theory) recently developed by the authors. The theory was introduced in [10, 11], and was applied to the RB convection in [9]. For the Rayleigh-Bénard convection problem, owing to the fact that its linearized problem is symmetric, we were able to derive a nonlinear bifurcation and stability theory including (i) bifurcation theorem when the Rayleigh number crosses the first critical number for all physically sound boundary conditions, (ii) asymptotic stability of bifurcated solutions, and (iii) the structure/patterns and their stability and transitions in the physical space.

As just mentioned, one key ingredient for the RB convection problem is the fact that its linearized problem is symmetric, leading to asymptotic stability of the basic flow at the critical Rayleigh number. However, for the Taylor problem, the linearization is a perturbation of a symmetric problem with a small perturbation parameter ε. Therefore, we apply the corresponding perturbation theory on attractor bifurcations to the Taylor problem.

For the unperturbed problem, the main theorem of this paper, Theorem 2,which is associated with the attractor bifurcation, states that when the control parameter λ crosses certain critical value λ_0, there are $m+1$ eigenvalues cross the imaginary axis. The system thus bifurcates from the trivial steady state solution to an attractor Σ_λ (with dimension between m and $m+1$) provided the critical state is locally asymptotically stable.

For the perturbed problem, the main theorems, Theorems 3 and 4, show that when the control parameter λ crosses certain critical value $\widetilde{\lambda}_0$, which approximates λ_0 and is an eigenvalue of the perturbed linear problem, we obtain an attractor $\Sigma_\lambda^\varepsilon$ of the perturbed problem. The attractor $\Sigma_\lambda^\varepsilon$ posesses the following properties:

(i) it approximates the attractor Σ_λ for the unperturbed system,
(ii) it has dimension less than or equal to $m+1$,
(iii) it excludes the trivial solution,
(iv) any element u^λ in $\Sigma_\lambda^\varepsilon$ is a small perturbation of the eigenvectors of the **unperturbed** linear eigenvalue problem corresponding to zero eigenvalue at $\lambda = \lambda_0$,
(v) when $m = 0$, i.e. the eigenvalue of the unperturbed system is simple, the perturbed attractor contains exactly two asymptotically stable steady states, and
(vi) $\Sigma_\lambda^\varepsilon$, as an attractor, has a basin of attraction, leading to its asymptotic stability.

This theory is directly applied to the Taylor problem, and the main results are given in Theorems 5 and 6. In combination with structural stability theorems derived by the authors in [7, 8, 12], our main results lead to some physically significant conclusions. First, the linear theory used by physicists is rigorously justified by the nonlinear theory. Second, thanks to properties (iv) and (vi) above, starting from any initial condition, the structure of the solution of the Taylor problem in the physical space is dictated by the structure of the eigenvectors of the unperturbed linear eigenvalue problem corresponding to zero eigenvalue at $\lambda = \lambda_0$. The latter can be explicitly examined. This result leads to a much needed mathematical justification of the Taylor vortices observed in [13]; see Theorem 9.

This paper is organized as follows. Section 2 introduces the Taylor problem, its governing equations and their mathematical backgrounds. In addition, a time uniform bound for the H^k norm of the solutions of the Taylor problem is proved. Section 3 recalls the attractor bifurcation theory, and its applications to the Taylor problem are given in Section 4. Section 5 establishes structural stability of solutions in the perturbed attractor, leading to rigorous justification of the Taylor vortex structure.

2. The Taylor problem

2.1. Taylor's experiments and Taylor vortices

Let r_1 and r_2 $(r_2 > r_1)$ be the radii of the two coaxial cylinders, Ω_1 and Ω_2 the angular velocities with which the inner and the outer cylinders rotate, and

$$\mu = \frac{\Omega_2}{\Omega_1}, \qquad \eta = \frac{r_1}{r_2}. \tag{1}$$

The nondimensional Taylor number is defined by

$$T = \frac{4\Omega_1^2}{\nu^2} L^4, \tag{2}$$

where ν is the kinematic viscosity, and L a length scale.

Based on the Rayleigh criterion, if $\mu > \eta^2$ the Couette flow is always stable at a distribution of angular velocities

$$\Omega(r) = \frac{a+b}{r^2}, \qquad r_1 \leqq r \leqq r_2, \tag{3}$$

where a and b are constants depending on μ, ν and Ω_1.

However, if $\mu < \eta^2$ the situation is different. In his experiments, Taylor studied the case where the gap $r_2 - r_1$ between the two cylinders is small in comparison with the mean radius $r_0 = \frac{1}{2}(r_1 + r_2)$, and the two cylinders are rotating in the same direction:

$$\begin{aligned} &r_2 - r_1 \ll \frac{1}{2}(r_1 + r_2), \\ &\mu > 0. \end{aligned}$$

He found that when the Taylor number T is smaller than a critical value $T_c > 0$, the critical Taylor number, the Couette flow with the angular velocity (3) is stable, and when $T_c < T < T_c + \varepsilon$ for some $\varepsilon > 0$ small, the basic flow breaks out into a cellular pattern which is radially symmetric: see Fig. 1.

When the cylinders rotate in opposite directions, the phenomena that can be seen are much more complex: see [1] for details.

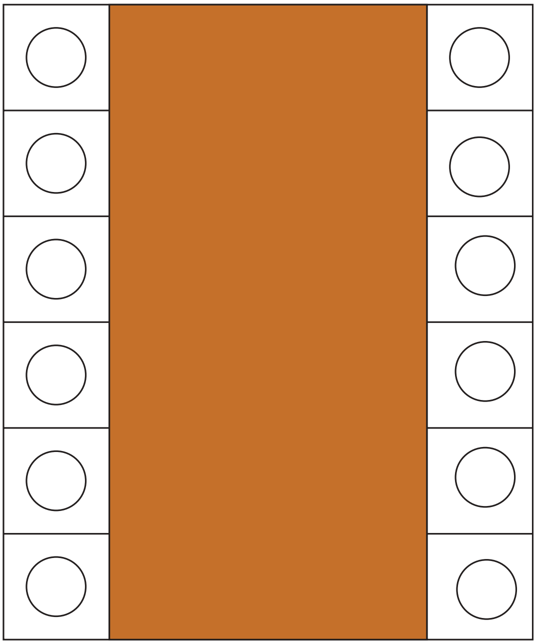

Fig. 1. Taylor's vortex pattern of flow between two cylinders rotating in the same direction.

2.2. Governing equations

The hydrodynamical equations governing an incompressible viscous fluid between two coaxial cylinders are the Navier-Stokes equations in the cylindrical coordinates (r, θ, z). They are given by

$$
\begin{aligned}
&\frac{\partial u_r}{\partial t} + (u \cdot \nabla_3) u_r - \frac{u_\theta^2}{r} = -\frac{\partial}{\partial r}\left(\frac{p}{\rho}\right) + \nu\left(\Delta_3 u_r - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} - \frac{u_r}{r^2}\right),\\
&\frac{\partial u_\theta}{\partial t} + (u \cdot \nabla_3) u_\theta + \frac{u_r u_\theta}{r} = -\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{p}{\rho}\right) + \nu\left(\Delta_3 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right),\\
&\frac{\partial u_z}{\partial t} + (u \cdot \nabla_3) u_z = -\frac{\partial}{\partial z}\left(\frac{p}{\rho}\right) + \nu\, \Delta_3 u_z,\\
&\frac{\partial (r u_r)}{\partial r} + \frac{\partial u_\theta}{\partial \theta} + \frac{\partial (r u_z)}{\partial z} = 0,
\end{aligned}
\tag{4}
$$

where ν is the kinematic viscosity, ρ the density, $u = (u_r, u_\theta, u_z)$ the velocity field, p the pressure function, and

$$
\begin{aligned}
u \cdot \nabla_3 &= u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial \theta} + u_z \frac{\partial}{\partial z},\\
\Delta_3 &= \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}.
\end{aligned}
$$

The basic flow for (4) is the Couette flow, a steady state solution defined by

$$\begin{aligned} u_r &= u_z = 0, \quad u_\theta = V(r), \\ p &= \rho \int \frac{1}{r} V^2(r) dr, \\ V(r) &= ar + \frac{b}{r}. \end{aligned} \tag{5}$$

By the boundary conditions:

$$V(r_1) = \Omega_1 r_1, \qquad V(r_2) = \Omega_2 r_2, \tag{6}$$

the constants a and b in (5) are given by

$$a = -\Omega_1 \eta^2 \, \frac{1 - \mu/\eta^2}{1 - \eta^2}, \qquad b = \Omega_1 \frac{r_1^2 (1 - \mu)}{1 - \eta^2}\,, \tag{7}$$

where μ and η are given by (1). We always assume that

$$\eta^2 > \mu \geqq 0. \tag{8}$$

In order to investigate the stability of the flow described by (5), we need to consider the perturbed state

$$u_r, \; V + u_\theta, \; u_z, \; \text{and} \;\; p + \rho \int \frac{1}{r} V^2(r) dr.$$

Assuming that the perturbations are axisymmetric and are independent of θ, we obtain from (4):

$$\begin{aligned} \frac{\partial u_z}{\partial t} + (u \cdot \nabla) u_z &= \nu \Delta u_z - \frac{\partial p}{\partial z}, \\ \frac{\partial u_r}{\partial t} + (u \cdot \nabla) u_r - \frac{u_\theta^2}{r} &= \nu \left(\Delta u_r - \frac{u_r}{r^2} \right) - \frac{\partial p}{\partial r} + \frac{2V}{r} u_\theta, \\ \frac{\partial u_\theta}{\partial t} + (u \cdot \nabla) u_\theta + \frac{u_r u_\theta}{r} &= \nu \left(\Delta u_\theta - \frac{u_\theta}{r^2} \right) - \left(\frac{dV}{dr} + \frac{V}{r} \right) u_r, \\ \frac{\partial (r u_r)}{\partial r} + \frac{\partial (r u_z)}{\partial z} &= 0, \end{aligned} \tag{9}$$

where

$$\begin{aligned} \Delta &= \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}, \\ (u \cdot \nabla) &= u_r \frac{\partial}{\partial r} + u_2 \frac{\partial}{\partial z}\,. \end{aligned}$$

The spatial domain for (9) is $M = (r_1, r_2) \times (0, L) \subset \mathbb{R}^2$, where L is the height of the fluid between the two cylinders. There are different physically-sound boundary conditions. At the top and bottom in the z direction ($z = 0, L$), either the free boundary conditions, the rigid boundary conditions, or the periodic boundary conditions can be used:

Free slip boundary condition:

$$u_z = 0, \quad \frac{\partial u_r}{\partial z} = \frac{\partial u_\theta}{\partial z} = 0 \quad \text{at } z = 0, L; \tag{10}$$

Dirichlet boundary condition (or rigid condition):

$$u_z = u_r = u_\theta = 0 \quad \text{at} \quad z = 0, L; \tag{11}$$

Free rigid boundary condition:

$$\begin{aligned} &u_z = 0, \ \frac{\partial u_r}{\partial z} = \frac{\partial u_\theta}{\partial z} = 0 && \text{at } z = L, \\ &u_z = u_r = u_\theta = 0 && \text{at } z = 0; \end{aligned} \tag{12}$$

Periodic condition:

$$u = (u_z, u_r, u_\theta) \ \text{ is } L \text{ periodic in the } z \text{ direction.} \tag{13}$$

In the radial direction, there are two kinds of boundary condition:

Free boundary condition:

$$u_r = 0, \quad \frac{\partial u_z}{\partial r} = 0, \quad u_\theta = 0 \qquad \text{at } \ r = r_1, r_2; \tag{14}$$

Rigid boundary condition:

$$u_z = u_r = u_\theta = 0 \quad \text{at } r = r_1, r_2. \tag{15}$$

In this paper, our analysis will be conducted using one set of boundary conditions. The other physically sound boundary conditions can be analyzed in the same fashion.

2.3. Narrow-gap approximation

We shall investigate the stability of the flow described by equations (9) in a narrow gap with $\mu > 0$. By (5) and (7), the equations (9) are given in the following form

$$\begin{aligned} &\frac{Du_z}{Dt} = \nu \Delta_2 u_z - \frac{\partial p}{\partial z}, \\ &\frac{Du_r}{Dt} - \frac{u_\theta^2}{r} = \nu\left(\Delta_2 u_r - \frac{u_r}{r^2}\right) - \frac{\partial p}{\partial r} + 2\Omega_1 \left(\frac{r_1^2}{r^2}\frac{(1-\mu)}{1-\eta^2} - \frac{\eta^2-\mu}{1-\eta^2}\right) u_\theta, \\ &\frac{Du_\theta}{Dt} + \frac{u_r u_\theta}{r} = \nu\left(\Delta_2 u_\theta - \frac{u_\theta}{r^2}\right) + 2\Omega_1 \frac{\eta^2-\mu}{1-\eta^2} u_r, \\ &\frac{\partial (r u_r)}{\partial r} + \frac{\partial (r u_z)}{\partial z} = 0, \end{aligned} \tag{16}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_r \frac{\partial}{\partial r} + u_z \frac{\partial}{\partial z}$$

$$\Delta_2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2} .$$

With proper scaling, without loss of generality, the gap $r_2 - r_1$ can be chosen to be 1:

$$r_2 - r_1 = 1. \tag{17}$$

As mentioned before, in this paper we study the case in which the gap between the two cylinders is small in comparison to the mean radius. Equivalently, the radii of the two cylinders are sufficiently large in comparsion to the gap $r_2 - r_1$:

$$1 = r_2 - r_1 \ll \frac{1}{2}(r_2 + r_1). \tag{18}$$

This is the case studied by Taylor in his experiments.

Under the assumptions (17) and (18), we can neglect the terms having the factors r^{-n} $(n \geqq 1)$. Let

$$\alpha = \frac{\eta^2 - \mu}{1 - \eta^2}.$$

By assumption (8), it is easy to see that

$$\alpha > 0.$$

Replace u_θ by $\sqrt{\alpha}\, u_\theta$ in (16), and note that

$$\begin{aligned} 2\Omega_1 \left(\frac{1-\mu}{1-\eta^2}\frac{r_1^2}{r^2} - \frac{\eta^2-\mu}{1-\eta^2} \right) &= 2\Omega_1 \left(1 - \frac{1-\mu}{1-\eta^2}\frac{r^2 - r_1^2}{r^2} \right) \\ &\simeq 2\Omega_1 \left(1 - (1-\mu)(r - r_1)\right). \end{aligned}$$

We then derive the following approximate equations describing the flow between two cylinders with a narrow gap:

$$\begin{aligned} \frac{\partial u_z}{\partial t} + (\widetilde{u}\cdot\nabla)u_z &= \nu\Delta u_z - \frac{\partial p}{\partial z}, \\ \frac{\partial u_r}{\partial t} + (\widetilde{u}\cdot\nabla)u_r &= \nu\Delta u_r - \frac{\partial p}{\partial r} + 2\sqrt{\alpha}\,\Omega_1\left(1 - (1-\mu)(r - r_1)\right)u_\theta, \\ \frac{\partial u_\theta}{\partial t} + (\widetilde{u}\cdot\nabla)u_\theta &= \nu\Delta u_\theta + 2\sqrt{\alpha}\,\Omega_1 u_r, \\ \frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z} &= 0, \end{aligned} \tag{19}$$

where

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial z^2}, \quad (\widetilde{u} \cdot \nabla) = u_r \frac{\partial}{\partial r} + u_z \frac{\partial}{\partial z}.$$

For convenience, hereafter we always consider either the following Dirichlet boundary conditions:

$$\begin{cases} u_z = 0, \quad \frac{\partial u_r}{\partial z} = 0, \quad \frac{\partial u_\theta}{\partial z} = 0 & \text{at } z = 0, L, \\ u_z = u_r = u_\theta = 0 & \text{at } r = r_1, r_1 + 1; \end{cases} \tag{20}$$

or the following free slip boundary conditions:

$$\begin{cases} u_z = 0, \quad \frac{\partial u_r}{\partial z} = 0, \quad \frac{\partial u_\theta}{\partial z} = 0 & \text{at } z = 0, L, \\ u_r = 0, \quad \frac{\partial u_z}{\partial r} = 0, \quad u_\theta = 0 & \text{at } r = r_1, r_1 + 1. \end{cases} \tag{21}$$

We note, however, that all results presented in this article hold true as well for all physically-sound boundary conditions.

2.4. Functional setting and properties of solutions

Here, we recall the functional setting of equations (19) with boundary conditions given by (20). For simplicity, let the spatial domain be $M = (r_1, r_1 + 1) \times (0, L)$, coordinate system (r, z), the velocity field $u = (u_z, u_r, u_\theta)$, and $\widetilde{u} = (u_z, u_r)$. Let

$$H = \{u = (\widetilde{u}, u_\theta) \in L^2(M)^3 \quad | \quad \operatorname{div} \widetilde{u} = 0, \quad \widetilde{u} \cdot n|_{\partial M} = 0\}, \tag{22}$$

$$\begin{aligned} V = \{u = (\widetilde{u}, u_\theta) \in H^1(M)^3 \quad | \quad & \operatorname{div} \widetilde{u} = 0, \\ u_z = 0 \text{ at } z = 0, L, \quad & u = 0 \text{ at } r = r_1, r_1 + 1\}, \end{aligned} \tag{23}$$

Here, $H^1(M)$ is the usual Sobolev space.

Thus the results concerning the existence of a solution for (19) with (20) are classical. For every $u_0 \in H$, (19) with Dirichlet boundary conditions (20) possesses a weak solution:

$$u \in L^\infty([0, \tau]; H) \cap L^2(0, \tau; V) \quad \forall \tau > 0. \tag{24}$$

If $u_0 \in V$, (19) with Dirichlet boundary conditions (20) possesses a unique solution:

$$u \in C([0, \tau]; V) \cap L^2(0, \tau_1; H^2(M)^3 \cap V). \tag{25}$$

Due to these existence results, we can define a semigroup

$$S(t) : u_0 \to u(t),$$

which possesses the semigroup properties.

2.5. Uniform boundedness of H^knorm

In order to obtain Theorem 9 in Section 5, we need the following uniform boundedness theorem of H^knorm for solutions of (19), supplemented with either (20) or (21).

For any $k \geqq 1$, let

$$D_f^k = \left\{(u_z, u_r, u_\theta) \in H^k(M)^3 \ \text{ satisfies (21)}\right\},$$
$$D_d^k = \left\{(u_z, u_r, u_\theta) \in H^k(M)^3 \ \text{ satisfies (20)}\right\},$$

where $H^k(M)$ is the usual Sobolev space, the subscript f stands for free slip boundary condition, and the subscript d for Dirichlet boundary conditions.

Theorem 1. *Let $k \geqq 1$ and $\varphi \in D_f^k$ (resp. $\varphi \in D_d^k$). Then there exists a number $C > 0$ depending on φ such that for any $t \geqq 0$,*

$$\|u(t, \varphi)\|_{H^k} \leqq C,$$

where $u(t, \varphi)$ is the solution of (19) *with either* (21) *or* (20) *and with $u(0, \varphi) = \varphi$.*

Proof. We only prove the result for (19) supplemented with boundary conditions (20), and the result for (19) with boundary conditions (21) can be proved in the same fashion.

Let ψ be the stream function given by

$$u_z = \frac{\partial \psi}{\partial r}, \qquad u_r = -\frac{\partial \psi}{\partial z}.$$

Then (19) can be rewritten as

$$\begin{cases} \frac{\partial \Delta\psi}{\partial t} + J[\psi, \Delta\psi] + \nu\Delta^2\psi - 2\sqrt{\alpha}\,\Omega_1(1 - (1-\mu)(r - r_1))\frac{\partial u_\theta}{\partial z} = 0, \\ \frac{\partial u_\theta}{\partial t} + J[u_\theta, \psi] - \nu\Delta u_\theta + 2\sqrt{\alpha}\,\Omega_1\frac{\partial \psi}{\partial z} = 0, \end{cases} \tag{26}$$

where the Jacobian is defined by

$$J[g, f] = \frac{\partial g}{\partial z}\frac{\partial f}{\partial r} - \frac{\partial g}{\partial r}\frac{\partial f}{\partial z}.$$

The boundary conditions (20) are transformed into

$$\begin{cases} \psi = 0, & \frac{\partial \psi}{\partial r} = 0, & u_\theta = 0 & \text{at } r = r_1, r_2, \\ \psi = 0, & \frac{\partial^2 \psi}{\partial z^2} = 0, & \frac{\partial u_\theta}{\partial z} = 0 & \text{at } z = 0, L. \end{cases} \tag{27}$$

It then suffices to prove the uniform boundedness of solutions of (26) and (27) in the following function space:

$$E^k = \{(\psi, u_\theta) \in H^{k+1}(M) \times H^k(M) \text{ satisfies (27)}\}.$$

We know that the eigenvectors $\{\psi_n, v_n\}$ of the following eigenvalue problem constitute an orthogonal basis of E^k:

$$\begin{aligned} &\Delta^2\psi = \lambda\psi, \\ &-\Delta v = \lambda v, \\ &\qquad (\psi, \nu) \text{ satisfies the boundary condition (27),} \end{aligned} \tag{28}$$

and they are given by

$$(\psi_n, v_n) = (\sin\frac{n\pi}{L} z\varphi_n(r), \cos\frac{n\pi}{L} zW_n(r)), \tag{29}$$

where φ_n and W_n satisfy

$$\begin{cases} \left(\frac{d^2}{dr^2} - \frac{n^2\pi^2}{L^2}\right)^2 \varphi_n = \lambda_n\varphi_n & \text{in } (r_1, r_2), \\ \varphi_n = \varphi_n' = 0 & \text{at } r = r_1, r_2, \end{cases}$$

$$\begin{cases} -\left(\frac{d^2}{dr^2} - \frac{n^2\pi^2}{L^2}\right) W_n = \lambda_n W_n & \text{in } (r_1, r_2), \\ W_n = 0 & \text{at } r = r_1, r_2. \end{cases}$$

In addition, we consider the z-periodic boundary condition

$$\psi(z, r) = \psi(z + 2L, r) \quad \text{and} \quad \psi = \psi' = 0 \quad \text{at } r = r_1, r_2, \tag{30}$$

$$u_\theta(z, r) = u_\theta(z + 2L, r) \quad \text{and} \quad u_\theta = 0 \quad \text{at } r = r_1, r_2, \tag{31}$$

Let $Q = \mathbb{R} \times (r_1, r_2) \subset \mathbb{R}^2$, and

$$\begin{aligned} H_p^k(Q) &= \left\{\psi \in H^k(Q) \mid \psi \quad \text{satisfies (30)}\right\}, \\ \mathcal{H}_p^k(Q) &= \left\{u_\theta \in H^k(Q) \mid u_\theta \quad \text{satisfies (31)}\right\}. \end{aligned}$$

Then for any $(\psi, v) \in H_p^{k+1}(Q) \times \mathcal{H}_p^k(Q)$, we have

$$\begin{aligned} \psi &= \sum_{n=1}^{\infty} \psi_n \sin\frac{n\pi z}{L} \varphi_n(r) + \sum_{n=1}^{\infty} \widetilde{\psi}_n \cos\frac{n\pi z}{L} \varphi_n(r) \\ v &= \sum_{n=1}^{\infty} v_n \cos\frac{n\pi z}{L} W_n(r) + \sum_{n=1}^{\infty} \widetilde{v}_n \sin\frac{n\pi z}{L} W_n(r). \end{aligned}$$

We shall consider the subspace $E^k(Q)$ of $H_p^{k+1}(Q) \times \mathcal{H}_p^k(Q)$ containing all the pairs (ψ, v) such that

$$\psi(-z, r) = -\psi(z, r), \quad v(-z, r) = v(z, r).$$

These functions ψ and v admit Fourier expansions:

$$\psi = \sum_{n=1}^{\infty} \psi_n \sin\frac{n\pi z}{L} \varphi_n(r), \qquad v = \sum_{n=1}^{\infty} \widetilde{v}_n \sin\frac{n\pi z}{L} W_n(r).$$

It is clear that the subspace $E^k(Q)$ is closed in $H_p^{k+1}(Q) \times \mathcal{H}_p^k(Q)$. Let $E(Q)$ be the closure of $E^k(Q)$ in the L^2norm. We find that for $(\psi, u_\theta) \in E^k(Q)$ $(k \geqq 2)$,

$$(\Delta\psi, u_\theta), (\Delta^2\psi, \Delta u_\theta), \left(\frac{\partial u_\theta}{\partial z}, \frac{\partial \psi}{\partial z}\right), ([\psi, \Delta\psi], [u_\theta, \psi]) \in E(Q).$$

Therefore, the subspace $E^k(Q)$ is invariant for the equations (26) with the z-periodic conditions (30) and (31). More precisely, if the initial value $\varphi \in E^k(Q)$, the solution of (26), (30) and (31) will remain in $E^k(Q)$ for all time:

$$(\psi(t, \varphi), u_\theta(t, \varphi)) \in E^k(Q), \qquad \forall\, t \geqq 0,$$

where $(\psi(0, \varphi), u_\theta(0, \varphi)) = \varphi$.

It is known that the solution $(\psi(t, \varphi), u_\theta(t, \varphi))$ of (26), (30) and (31) is uniformly bounded in H^k norm, provided that $\varphi \in E^k(Q)$; see Remark 3.3 of [3]. We note that the functions of (29) form an orthogonal basis in E^k, and

$$E^k = E^k(Q)|_M, \tag{32}$$

where $E^k(Q)|_M$ is the space consisting of all functions in $E^k(Q)$ restricted on M. Thus, we prove that for any $\varphi \in E^k$, the solution $(\psi(t, \varphi), u_\theta(t, \varphi))$ of (26) and (27) with $(\psi(0, \varphi), u_\theta(0, \varphi)) = \varphi$ is uniformly bounded in the Sobolev $H^{k+1} \times H^k$ norm. Hence, the proof of the theorem is complete. □

3. Attractor bifurcation theory

We recapitulate in this section basic theorems in the attractor bifurcation theory developed recently by MA & WANG [10, 11].

3.1. Attractor bifurcation

Let H_1 and H be two Hilbert spaces, and $H_1 \to H$ be a dense and compact inclusion. Consider the following nonlinear evolution equations

$$\frac{du}{dt} = L_\lambda u + G(u, \lambda), \qquad u \in H_1,\ \lambda \in \mathbb{R}, \tag{33}$$

$$u(0) = u_0, \tag{34}$$

where $L_\lambda : H_1 \to H$ are parametrized, linear and completely continuous fields, depending continuously on $\lambda \in \mathbb{R}^1$, such that

$$\begin{cases} L_\lambda = -A + B_\lambda, \\ A : H_1 \to H \quad \text{is a linear homeomorphism,} \\ B_\lambda : H_1 \to H \quad \text{are the parameterized linear compact operators.} \end{cases} \tag{35}$$

We assume that there exist a real eigenvalue sequence $\{\rho_k\} \subset \mathbb{R}^1$ and an eigenvector sequence $\{e_k\} \subset H_1$ of A:

$$\begin{aligned} Ae_k &= \rho_k e_k, \\ 0 < \rho_1 &\leqq \rho_2 \leqq \cdots, \\ \rho_k &\to \infty \ (k \to \infty), \end{aligned} \tag{36}$$

such that $\{e_k\}$ is an orthogonal basis of H. Using A, we can define naturally fractional spaces H_α such that $H_{\alpha_1} \subset H_{\alpha_2}$ if $\alpha_1 > \alpha_2$, and $H_0 = H$.

For the other part of the linear operator, we assume that for some $\sigma < 1$,

$$B_\lambda : H_\sigma \to H \text{ is bounded }, \forall \lambda \in \mathbb{R}^1. \tag{37}$$

Furthermore, we assume that the nonlinear term $G(\cdot, \lambda) : H_\sigma \to H$ for some $1 > \sigma \geqq 0$ are a family of parametrized C^r bounded operators ($r \geqq 1$) depending continuously on the parameter $\lambda \in \mathbb{R}^1$, such that

$$\begin{cases} G(\cdot, \lambda) : H_\sigma \to H \text{ is } C^r & (r \geqq 1), \\ G(u, \lambda) = o(\|u\|_{H_\sigma}), & \forall\ \lambda \in \mathbb{R}^1. \end{cases} \tag{38}$$

Let $\{S_\lambda(t)\}_{t \geqq 0}$ be an operator semigroup generated by the equation (33). Then, the solution of (33) and (34) may be expressed as

$$u(t) = S_\lambda(t)u_0, \qquad t \geqq 0.$$

Definition 1. A set $\Sigma \subset H$ is called an invariant set of (33), if for any $t \geqq 0$, $S_\lambda(t)\Sigma = \Sigma$. An invariant set $\Sigma \subset H$ of (33) is called an attractor, if Σ is compact, and there exists a neighborhood $U \subset H$ of Σ such that for any $\varphi \in U$

$$\operatorname{dist}(u(t, \varphi), \Sigma) \longrightarrow 0 \quad \text{in } H - \text{norm as } t \to \infty,$$

where $u(t, \varphi)$ is the solution of (33) and (34). If U can be taken as any bounded open set of H, then Σ is called a global attractor.

Definition 2. We say that the system (33) bifurcates from $(u, \lambda) = (0, \lambda_0)$ attractors Σ_λ, if there exists a sequence of attractors $\{\Sigma_{\lambda_n}\}$ of (33) such that

$$\begin{aligned} 0 &\notin \Sigma_{\lambda_n}, \\ \lim_{n\to\infty} \lambda_n &= \lambda_0, \\ \lim_{n\to\infty} \max_{x \in \Sigma_{\lambda_n}} \|x\| &= 0. \end{aligned}$$

Definition 3. A number $\beta = \alpha + i\gamma \in \mathbb{C}$ is called an eigenvalue of a linear operator $L : H_1 \to H$, if there exist $u, v \in H_1$ with $u \neq 0$ such that

$$Lw = \beta w, \qquad w = u + iv.$$

The space

$$E_\beta = \left\{u, v \in H_1 \mid (L - \beta)^n w = 0,\ \ w = u + iv,\ n = 1, 2, \cdots\right\}$$

is called the eigenspace of L_λ corresponding to β, and the elements $u, v \in E_\beta$ are called eigenvectors of L_λ.

Let $\{\beta_k(\lambda) \in \mathbb{C} \mid k = 1, 2, \cdots\}$ be the eigenvalues of $L_\lambda = -A + B_\lambda$ (counting the multiplicities). Suppose that

$$Re\beta_i(\lambda) \begin{cases} < 0 & \text{if } \lambda < \lambda_0 \\ = 0 & \text{if } \lambda = \lambda_0 \\ > 0 & \text{if } \lambda > \lambda_0 \end{cases} \quad \text{for } 1 \leqq i \leqq m + 1, \tag{39}$$

$$Re\beta_j(\lambda_0) < 0 \qquad\qquad \text{for } j \geqq m + 2. \tag{40}$$

Let E_0 be the following eigenspace of L_λ at λ_0:

$$E_0 = \cup_{1 \leqq i \leqq m+1} \left\{ u \in H \mid (L_{\lambda_0} - \beta_i(\lambda_0))^k u = 0, \ k = 1, 2, \cdots \right\}. \tag{41}$$

By (39) we know that $\dim E_0 = m + 1$.

The following theorem was obtained by the authors in [10, 11].

Theorem 2 (Ma & Wang [10, 11])**.** *Assume that* (35)–(38) *hold true and* $u = 0$ *is a locally asymptotically stable equilibrium point of* (33) *at* $\lambda = \lambda_0$*. Then the following assertions hold true:*

(i) Equation (33) *bifurcates from* $(u, \lambda) = (0, \lambda_0)$ *to attractors* Σ_λ *for* $\lambda > \lambda_0$*, with* $m \leqq \dim \Sigma_\lambda \leqq m + 1$*, which is connected if* $m > 0$.

(ii) Σ_λ *is a limit of a sequence of* $(m+1)$*-dimensional annulus* A_k *with* $A_k \subset A_{k+1}$*. In particular, if* Σ_λ *is a finite simplicial complex, then* Σ_λ *has the homotopy type of* m*-dimensional sphere* S^m*.*

(iii) For any $u_\lambda \in \Sigma_\lambda$*,* u_λ *can be expressed as*

$$u_\lambda = v_\lambda + o(\|v_\lambda\|), \qquad v_\lambda \in E_0.$$

(iv) If the equilibrium points of (33) *in* Σ_λ *are finite, then we have the index formula (condition* (38) *implies that* $G : H_1 \to H_0$ *is compact):*

$$\sum_{u_i \in \Sigma_\lambda} \operatorname{ind}[-(L_\lambda + G), u_i] = \begin{cases} 2 & \textit{if } m = \textit{odd}, \\ 0 & \textit{if } m = \textit{even}. \end{cases}$$

(v) If $u = 0$ *is globally asymptotically stable for* (33) *at* $\lambda = \lambda_0$*, then for any bounded open set* $U \subset H$ *with* $\Sigma_\lambda \subset U$*, there is an* $\varepsilon > 0$ *such that as* $\lambda_0 < \lambda < \lambda_0 + \varepsilon$*, the attractor* Σ_λ *of* (33) *attracts* U/Γ*, where* $\Gamma \subset H$ *is the stable manifold of* $u = 0$*, having codimension* $m+1$ *in* H*. In particular, if* (33) *has a global attractor for any* λ *near* λ_0*, then* ε *can be taken independently of* U*.*

3.2. Perturbation theory

In this section, we shall recall two theorems on the bifurcated attractor given in Theorem 2, which will be used in studying the bifurcation phenomena for the Taylor problem. One theorem is related to the general case and the other to the case with simple eigenvalues.

3.2.1. General case. Consider the perturbed equation of (33) given by

$$\frac{dv}{dt} = (L_\lambda + S_\lambda^\varepsilon)v + G(v, \lambda), \tag{42}$$

where L_λ and G are as in (33), and $S_\lambda^\varepsilon \in B_\sigma$ is a perturbation operator of L_λ, depending continuously on $\lambda \in \mathbb{R}$, such that

$$\|S_{\lambda_0}^\varepsilon\|_{\mathcal{B}_\sigma} < \varepsilon. \tag{43}$$

Here $B_\sigma = \mathcal{L}(H_\sigma, H)$ is the space of all linear bounded operators from H_σ to H, i.e.

$$B_\sigma = \mathcal{L}(H_\sigma, H) = \{B : H_\sigma \to H \text{ linear bounded }\}. \tag{44}$$

By the spectral theory of linear complete continuous fields [5], as $\varepsilon > 0$ sufficiently small, there exists a parameter λ_0^ε approximating to λ_0 such that the eigenvalues $\{\beta_k^\varepsilon(\lambda)\}$ of $L_\lambda + S_\lambda^\varepsilon$ satisfy

$$\operatorname{Re} \beta_i^\varepsilon(\lambda) \begin{cases} < 0, & \lambda < \lambda_0^\varepsilon \\ = 0, & \lambda = \lambda_0^\varepsilon \\ > 0, & \lambda > \lambda_0^\varepsilon \end{cases} \quad \text{for } 1 \leqq i \leqq m_1 + 1, \tag{45}$$

$$\operatorname{Re} \beta_j^\varepsilon(\lambda_0^\varepsilon) < 0 \qquad \text{for } j \geqq m_1 + 2, \tag{46}$$

where $0 \leqq m_1 \leqq m$, and m is as in (39).

The following is the attractor stability theorem for the perturbed equation (42); see [10].

Theorem 3. *Let the conditions in* Theorem 2 *hold true. Then there are* $\varepsilon > 0$ *and* $\delta > 0$ *such that if* $S_\lambda^\varepsilon \in \mathcal{B}_\sigma$ *satisfies* (43) *and* $0 < \lambda - \lambda_0^\varepsilon < \delta$, *then the following assertions hold true:*

(i) There exists a neighborhood $U \subset H$ *of* $u = 0$ *such that* (42) *has an attractor* $\Sigma_\lambda^\varepsilon \subset U$ *satisfying*

$$\dim \Sigma_\lambda^\varepsilon \leqq m + 1, \quad 0 \notin \Sigma_\lambda^\varepsilon,$$

and $\Sigma_\lambda^\varepsilon$ *attracts an open and dense set* U_0 *of* U.

(ii) Each element $u_\lambda \in \Sigma_\lambda^\varepsilon$ *can be expressed as*

$$\begin{aligned} u_\lambda &= v_\lambda + w_\lambda^\varepsilon, \\ v_\lambda &\in E_0, \\ \lim_{\substack{\lambda \to \lambda_0 \\ \varepsilon \to 0}} \|w_\lambda^\varepsilon\| / \|v_\lambda\| &= 0. \end{aligned} \tag{47}$$

(iii) If $u = 0$ *is globally asymptotically stable for the unperturbed equation* (33), *and* (42) *has a global attractor for any* $\lambda \in \mathbb{R}$, *then* $\Sigma_\lambda^\varepsilon$ *attracts each bounded open set of* $H \setminus \Gamma$, *where* Γ *is the stable manifold of* $u = 0$ *with codimension* $m + 1$.

3.2.2. Perturbation at simple eigenvalues. We now consider the case where $m = 0$ in (39), i.e. the first eigenvalue $\beta_1(\lambda)$ of L_λ is simple. Let $v_0 \in H_1$ be the eigenvector of L_λ at $\lambda = \lambda_0$:

$$-Av_0 + B_{\lambda_0} v_0 = 0, \qquad \|v_0\| = 1. \tag{48}$$

We assume that

$$\begin{aligned} &L_\lambda \text{ is symmetric },\\ &m = 0 \text{ in (39) and (40)},\\ &G(u) \text{ is bilinear},\\ &(G(u, v), v) = 0, \ \forall u, v \in H_\sigma. \end{aligned} \tag{49}$$

Here, because G is bilinear, we can write G as $G(\cdot, \cdot)$, which is linear for each argument.

Theorem 4 (Ma & Wang [10]). *Let the conditions in* Theorem 3 *and the conditions* (49) *hold true. Then we have the following assertions:*

(i) The attractor $\Sigma_\lambda^\varepsilon$ of (42) *obtained in* Theorem 3 *consists of exactly two equilibrium points of* (42)*, i.e. $\Sigma_\lambda^\varepsilon = \{u_1^\lambda, u_2^\lambda\}$, such that*

$$\begin{aligned} u_1^\lambda &= \alpha_1(\lambda, \varepsilon) v_0 + w_1(\lambda, \varepsilon),\\ u_2^\lambda &= -\alpha_2(\lambda, \varepsilon) v_0 + w_2(\lambda, \varepsilon),\\ w_i(\lambda, \varepsilon) &= o(|\alpha_i(\lambda, \varepsilon)|) \in H_1 \quad i = 1, 2, \end{aligned} \tag{50}$$

where for $i = 1, 2$,

$$\begin{aligned} &\alpha_i > 0,\\ &\lim_{\substack{\lambda \to \lambda_0\\ \varepsilon \to 0}} \alpha_i(\lambda, \varepsilon) = 0. \end{aligned}$$

(ii) If (42) *has a global attractor for any $\lambda \in \mathbb{R}$, then H can be decomposed into two open sets U_1^λ and U_2^λ:*

$$H = \overline{U}_1^\lambda + \overline{U}_2^\lambda,$$

such that

$$\begin{aligned} &U_1^\lambda \cap U_2^\lambda = \emptyset,\\ &0 \in \partial U_1^\lambda \cap \partial U_2^\lambda,\\ &u_i^\lambda \in U_i^\lambda \qquad i = 1, 2,\\ &\lim_{t \to \infty} \|v(t, \varphi) - u_i^\lambda\| = 0 \qquad \textit{for } \varphi \in U_i^\lambda \ (i = 1, 2), \end{aligned}$$

where $v(t, \varphi)$ is the solution of (42) *with $v(0, \varphi) = \varphi$.*

4. Bifurcation of the Couette flow and stability of secondary flows

4.1. The main theorems

In this section we consider equations (19) constrained by boundary conditions (20). We recall that the function spaces H and V are defined by (22) and (23), and let

$$H_1 = V \cap H^2(M)^3.$$

First, the linearized equations of (19) read as follows:

$$\begin{aligned} -\Delta u_z + \frac{\partial p}{\partial z} &= 0, \\ -\Delta u_r + \frac{\partial p}{\partial r} &= \lambda u_\theta - \lambda(1-\mu)(r-r_1)u_\theta, \\ -\Delta u_\theta &= \lambda u_r, \\ \frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z} &= 0, \end{aligned} \tag{51}$$

where

$$\lambda^2 = T = 4\alpha\Omega_1^2/\nu^2$$

is the Taylor number. Let $\widetilde{\lambda}_0 > 0$ be the first eigenvalue of (51), and we call

$$T_c = \widetilde{\lambda}_0^2,$$

the critical Taylor number.

As $\mu \to 1$, (51) reduces to the following symmetric linear equations:

$$\begin{aligned} -\Delta u_z + \frac{\partial p}{\partial z} &= 0, \\ -\Delta u_r + \frac{\partial p}{\partial r} &= \lambda u_\theta, \\ -\Delta u_\theta &= \lambda u_r, \\ \frac{\partial u_z}{\partial z} + \frac{\partial u_r}{\partial r} &= 0. \end{aligned} \tag{52}$$

Let the first eigenvalue $\lambda_0 > 0$ for (52) with (20) have multiplicity $m+1$ ($m \geqq 0$), the corresponding eigenvectors be v_i ($i = 1, 2, \cdots, m+1$), and the corresponding eigenspace be

$$E_0 = \text{ span } \{v_i \mid 1 \leqq i \leqq m+1\}. \tag{53}$$

Hereafter, we shall see that λ_0 for (52) with (20), or with the free boundary conditions (10), is simple for almost all $L > 0$. But, in general, $m \geqq 0$, and λ_0 may not be simple.

We remark here that under conditions (8), (17) and (18), the condition $\mu \to 1$ can be replaced by

$$(1-\mu)r_1 = 2+\delta, \tag{54}$$

for some $\delta > 0$. Note that this condition implies that $\alpha = \delta/2$.

The main results of this section are the following theorems:

Theorem 5. *Assume* (17), (18) *and* (54) *hold true. Then, there is a* $\kappa > 0$ *such that if the Taylor number* $T = \lambda^2$ *satisfies* $0 < T - T_c < \kappa$, *with* $T_c = \widetilde{\lambda}_0^2$ *the critical Taylor number, or equivalently* $0 < \lambda - \widetilde{\lambda}_0 < \varepsilon$ *for some* $\varepsilon > 0$, *then the following assertions hold true:*

(i) The problems (19) *and* (20) *have an attractor* $\Sigma_\lambda \subset H_1$, *also denoted by* $\Sigma_T = \Sigma_\lambda$, *such that*
  (a) $\dim \Sigma_\lambda \leqq m+1$,
  (b) $0 \notin \Sigma_\lambda$,
  (c) Σ_λ *attracts every bounded and open set of* $H \setminus \Gamma$, *where* Γ *is the stable manifold of* $u = 0$ *with codimension* $m+1$.

(ii) Each $u_\lambda \in \Sigma_\lambda$ *can be expressed as*

$$\begin{aligned} u_\lambda &= v_\lambda + w_\lambda^\mu, \\ v_\lambda &\in E_0, \\ \lim_{\mu\to 1, \lambda\to\lambda_0} \|w_\lambda^\mu\| \big/ \|v_\lambda\| &= 0, \end{aligned} \tag{55}$$

where λ_0 *is the first eigenvalue of* (52) *with* (20).

Theorem 6. *Let* (52), (53) *and* (54) *hold true, and let the first eigenvalue* λ_0 *of* (52) *with* (20) *be simple. We then have the following assertions:*

(i) The attractor Σ_λ *in Theorem* 5 *consists of exactly two equilibrium points of* (19) *with* (20), *i.e.* $\Sigma_\lambda = \{u_1^\lambda, u_2^\lambda\}$, *such that*

$$\begin{aligned} u_1^\lambda &= \alpha_1(\lambda,\mu)v_0 + w_1(\lambda,\mu), \\ u_2^\lambda &= -\alpha_2(\lambda,\mu)v_0 + w_2(\lambda,\mu), \\ w_i(\lambda,\mu) &= o(|\alpha_i(\lambda,\mu)|) \in H_1, \quad i = 1,2, \end{aligned} \tag{56}$$

where $\alpha_i > 0$ *and* $\alpha_i(\lambda,\mu) \to 0$ *as* $\lambda \to \lambda_0$ *and* $\mu \to 1$.

(ii) Moreover, H *can be decomposed into two open sets* U_1^λ *and* U_2^λ:

$$H = \overline{U}_1^\lambda + \overline{U}_2^\lambda, \quad U_1^\lambda \cap U_2^\lambda = \emptyset, \quad 0 \in \partial U_1^\lambda \cap \partial U_2^\lambda,$$

with $u_i^\lambda \in U_i^\lambda$ $(i = 1,2)$ *such that*

$$\lim_{t\to\infty} \|u(t,\varphi) - u_i^\lambda\| = 0, \qquad \text{as } \varphi \in U_i^\lambda (i = 1,2),$$

where $u(t,\varphi)$ *is the solution of* (19) *with* (20) *and with* $u(0,\varphi) = \varphi$.

Remark 1. In fact, the intersection $\partial U_1^\lambda \cap \partial U_2^\lambda$ of both open sets U_1^λ and U_2^λ in Theorem 6 is the stable manifold of $u = 0$, which has the codimension manifold in H.

Remark 2. The expressions (55) and (56) are very useful for the Taylor problem, which show that the asymptotic topological structure of equations (19) with (20) governed by the first eigenvectors of the symmetric linearized equations (52). In the next section we shall see that the first eigenvectors of (52), with the free boundary condition have the Taylor vortex type of structure.

Remark 3. For (19) with other boundary conditions, given in Section 2, the results of Theorems 5 and 6 are also valid.

4.2. Proof of Theorem 5

We shall apply Theorem 3 to prove this theorem.

Step 1. Let

$$\begin{aligned} &G : H_1 \to H, \\ &L_\lambda = -A + B_\lambda : H_1 \to H, \\ &L_\lambda^\mu = -A + B_\lambda + S_\lambda^\mu : H_1 \to H, \end{aligned}$$

be mappings defined by

$$\begin{aligned} Au &= \{\nu P \Delta \widetilde{u}, \nu \Delta u_\theta\}, \\ B_\lambda u &= \{\nu \lambda P(0, u_\theta), \nu \lambda u_r\}, \\ S_\lambda^\mu u &= \{\nu \lambda P(0, -(1-\mu)(r - r_1)u_\theta), 0\}, \\ G(u) &= \{P(\widetilde{u} \cdot \nabla)\widetilde{u}, (\widetilde{u} \cdot \nabla)u_\theta\}, \end{aligned} \tag{57}$$

where $u = \{u_z, u_r, u_\theta\} \in H_1$, and the operator P is the Leray projection. Thus, (19) can be written in the abstract form:

$$\frac{du}{dt} = L_\lambda u + S_\lambda^\mu u + G(u). \tag{58}$$

It is well known that the conditions (35)–(38) are satisfied by the operators (57). In particular,

$$\begin{aligned} B_\lambda, S_\lambda^\mu : H_\sigma \to H_0 &\text{ is bounded } \forall \sigma \geqq 0, \\ G : H_\sigma \to H_0 &\text{ is } C^\infty \text{ for } \frac{1}{2} < \sigma, \end{aligned}$$

where H_α are the fractional Sobolev spaces defined by the interpolation between $H_0 = H$ and H_1. Thus (37) and (38) are fulfilled, and (39) is obviously valid for the nonlinear operator G defined by (57).

It is clear that L_λ is symmetric, and $S_\lambda^\mu : H_\sigma \to H_0$ ($\sigma \geqq 0$) satisfies that

$$\|S_\lambda^\mu\|_{\mathcal{B}_\sigma} \leqq \nu\lambda(1-\mu), \qquad \forall\, \sigma \geqq 0.$$

By (54) and (58), $(1-\mu) \to 0$ as $r_1 \to \infty$. Thus the condition (43) is verified.

Step 2. Consider the eigenvalue problem

$$L_\lambda u = \beta(\lambda) u, \qquad u = (u_z, u_r, u_\theta) \in H_1. \tag{59}$$

By (57), the abstract form (59) can be referred to the following eigenvalue equations in H_1:

$$\begin{aligned} \Delta u_z - \frac{\partial p}{\partial z} &= \beta(\lambda) u_z, \\ \Delta u_r - \frac{\partial p}{\partial r} + \lambda u_\theta &= \beta(\lambda) u_r, \\ \Delta u_\theta + \lambda u_r &= \beta(\lambda) u_\theta, \\ \frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z} &= 0. \end{aligned} \tag{60}$$

It is known that the eigenvalues β_k $(k = 1, 2, \cdots)$ of (60) in H_1 are real numbers satisfying

$$\beta_1(\lambda) \geqq \beta_2(\lambda) \geqq \cdots \geqq \beta_k(\lambda) \geqq \cdots; \qquad \beta_k \to -\infty \ (k \to \infty). \tag{61}$$

The first eigenvalue $\beta_1(\lambda)$ and the first eigenvalue $\lambda_0 > 0$ of (51) have the relation

$$\beta_i(\lambda) \begin{cases} < 0 & \text{if } 0 \leqq \lambda < \lambda_0, \\ = 0 & \text{if } \lambda = \lambda_0, \end{cases} \qquad (1 \leqq i \leqq m+1), \tag{62}$$

where $m+1$ is the multiplicity of $\beta_1(\lambda_0)$. We need to prove that

$$\beta_i(\lambda) > 0 \quad \text{as } \lambda > \lambda_0 \quad (1 \leqq i \leqq m+1). \tag{63}$$

Since L_λ is symmetric, the first eigenvalues $\beta_i(\lambda)$ of (60) display the minimal property

$$-\beta_i(\lambda) = \min_{u \in H_1} \frac{\int_\Omega [\,|\nabla u_z|^2 + |\nabla u_r|^2 + |\nabla u_\theta|^2 - 2\lambda u_r u_\theta]\, dx}{\int_\Omega [\,|u_z|^2 + |u_r|^2 + |u_\theta|^2]\, dx}, \tag{64}$$

and the first eigenvectors $u_i \in H_1$ achieve the minimum. Hence, from (64) we find

$$\int_\Omega u_{ir} u_{i\theta} dx > 0, \qquad \text{for the first eigenvector } u_i. \tag{65}$$

It follows from (62), (64) and (65) that

$$\int_\Omega \left[|\nabla u_i|^2 - 2\lambda u_{ir} u_{i\theta} \right] dx \begin{cases} = 0 & \text{if } \lambda = \lambda_0, \\ < 0 & \text{if } \lambda > \lambda_0. \end{cases} \tag{66}$$

Thus (63) follows from (64) and (66). Therefore, the conditions (39) and (40) are verified.

Step 3. Next, we need to prove that $u = 0$ is globally asymptotically stable for the following equation at $\lambda = \lambda_0$:

$$\frac{du}{dt} = L_\lambda u + G(u), \qquad u \in H_1. \tag{67}$$

The equations corresponding to (67) are as follows

$$\begin{aligned}
\frac{\partial u_z}{\partial t} + (\widetilde{u} \cdot \nabla)u_z &= \nu \Delta u_z - \frac{\partial p}{\partial z}, \\
\frac{\partial u_r}{\partial t} + (\widetilde{u} \cdot \nabla)u_r &= \nu \Delta u_r - \frac{\partial p}{\partial r} + \nu \lambda u_\theta, \\
\frac{\partial u_\theta}{\partial t} + (\widetilde{u} \cdot \nabla)u_\theta &= \nu \Delta u_\theta + \nu \lambda u_r, \\
\frac{\partial u_r}{\partial r} + \frac{\partial u_z}{\partial z} &= 0.
\end{aligned} \tag{68}$$

Equations (68) have the same form as that of the two dimensional Boussinesq equations. Hence, for any $\nu, \lambda \in R$ the equations (68) have a global attractor; see [3]. As in Step 4 in the proof of Theorem 4.1 in [9], $u = 0$ is globally asymptotically stable for (67) at $\lambda = \lambda_0$.

Step 4. Finally, by using the same fashion as used in [3], we can prove that the equations (58), i.e. equations (20), have a global attractor for any $\lambda, \mu \in R$. Thus, this theorem follows from Theorem 3. The proof is complete. □

4.3. Proof of Theorem 6

The proof follows directly from Theorem 4 and the proof of Theorem 5, which has verified the conditions that appeared in Theorem 4 and were used in Theorem 6. □

5. Structure of the secondary flows and the Taylor vortices

The main objective of this section is to study the structure and its stability in the physical space of the solutions corresponding to the secondary flows for the Taylor problem obtained in the previous section. In particular, we shall provide a rigorous confirmation of the presence of the Taylor vortices structure as observed by TAYLOR [13].

This is part of a research program that the authors initiated to study the connections between Eulerian and the Lagrangian dynamics. As we mentioned in the Introduction, the connection made here in this article adds to a few other connections that we have previously established, including the boundary layer separation of incompressible flows, and the role structure of the bifurcated solutions of the Rayleigh-Bénard problem [9].

5.1. Structural stability of two-dimensional incompressible flows

We start with a brief introduction to the structural stability theory obtained by MA & WANG [7, 8].

Let $C^r(M,\mathbb{R}^2)$ be the space of all C^r $(r \geqq 1)$ vector fields on a C^r open set $M \subset \mathbb{R}^2$. We consider the spaces of all divergence-free vector fields with either the free slip boundary conditions or the Dirichlet boundary conditions:

$$\begin{aligned} B^r(M,\mathbb{R}^2) &= \{v \in C^r(M,\mathbb{R}^2) \mid \operatorname{div} v = 0, \quad v_n = \partial v_\tau/\partial n = 0 \text{ on } \partial M\}, \\ B^r_0(M,\mathbb{R}^2) &= \{v \in C^r(M,\mathbb{R}^2) \mid \operatorname{div} v = 0, \quad v \ = 0 \text{ on } \partial M\}, \end{aligned}$$

where $v_n = v \cdot n$, $v_\tau = v \cdot \tau$, n the unit outward normal vector and τ the unit tangent vector on ∂M.

Let $X = B^r(M,\mathbb{R}^2)$ or $X = B^r_0(M,\mathbb{R}^2)$ in the following definitions.

Definition 4. Two vector fields $u, v \in X$ are called topologically equivalent if there exists a homeomorphism of $\varphi : M \to M$, which takes the orbits of u to orbits of v and preserves their orientation.

Definition 5. A vector field $v \in X$ is called structurally stable in X if there exists a neighborhood $\mathcal{O} \subset X$ of v such that for any $u \in \mathcal{O}$, u and v are topologically equivalent.

Next we recall the structural stability theorems obtained in [8, 7]. For this purpose, consider a vector field $v \in B^r(M,\mathbb{R}^2)$. A point $p \in M$ is called a singular point of v if $v(p) = 0$; a singular point p of v is called nondegenerate if the Jacobian matrix $Dv(p)$ is invertible; v is called regular if all singular points of v are nondegenerate.

Theorem 7 (MA & WANG [8]). *A divergence-free vector field $u \in B^r(M,\mathbb{R}^2)$ is structurally stable in $B^r(M,\mathbb{R}^2)$ if and only if,*

(i) u is regular,
(ii) all interior saddle points of u are self-connected,
(iii) each boundary saddle point is connected only to a boundary saddle point on the same connected component of the boundary.

Moreover, all structurally stable vector fields in $B^r(M,\mathbb{R}^2)$ form an open and dense set of $B^r(M,\mathbb{R}^2)$.

For $u \in B^r_0(M,\mathbb{R}^2)$ $(r \geqq 2)$, a different singularity concept for points on the boundary was introduced in [7]. We proceed as follows:

(i) A point $p \in \partial M$ is called a ∂-regular point of u if $\partial u_\tau(p)/\partial n \neq 0$; otherwise, $p \in \partial M$ is called a ∂-singular point of u.
(ii) A ∂-singular point $p \in \partial M$ of u is called nondegenerate if

$$\det \begin{pmatrix} \dfrac{\partial^2 u_\tau(p)}{\partial \tau \partial n} & \dfrac{\partial^2 u_\tau(p)}{\partial n^2} \\[2ex] \dfrac{\partial^2 u_n(p)}{\partial \tau \partial n} & \dfrac{\partial^2 u_n(p)}{\partial n^2} \end{pmatrix} \neq 0.$$

A nondegenerate ∂-singular point of u is also called a ∂-saddle point of u.

(iii) A vector field $u \in B_0^r(M, \mathbb{R}^2)$ $(r \geqq 2)$ is called D-regular if u is regular in M, and all ∂-singular points of u on ∂M are nondegenerate.

The following theorem provides necessary, and sufficient, conditions for structural stability in $B_0^r(M, \mathbb{R}^2)$.

Theorem 8 (MA & WANG [7]). *Let $u \in B_0^r(M, \mathbb{R}^2)$ $(r \geqq 2)$. Then u is structurally stable in $B_0^r(M, \mathbb{R}^2)$ if and only if,*

(i) u is D-regular;
(ii) all interior saddle points of u are self-connected;
(iii) each ∂-saddle point of u on ∂M is connected to a ∂–saddle point on the same connected component of ∂M.

Moreover, the set of all structurally stable vector fields is open and dense in $B_0^r(M, \mathbb{R}^2)$.

5.2. Eigenvalue problem of the Taylor problem

In the following, we shall compute the first eigenvalue and the first eigenvectors of (52) with either Dirichlet boundary condition (20) or (21). The results and methods in this subsection are known; see [1]. We introduce them for convenience.

For the eigenvalue equation (52), we take the separation of variables as follows:

$$\begin{aligned} u_z &= \frac{1}{a^2}\frac{dh(z)}{dz}\frac{dR(r)}{dr}, \\ u_r &= h(z)R(r), \\ u_\theta &= h(z)\varphi(r), \end{aligned} \tag{69}$$

where $a^2 > 0$ is an arbitrary constant.

By (52) and the boundary conditions $u_z = 0$ at $z = 0, L$, we find

$$\begin{aligned} \frac{d^2h}{dz^2} &= -a^2 h, \\ h'(0) &= h'(L) = 0. \end{aligned} \tag{70}$$

Furthermore, the functions R and φ satisfy

$$\begin{aligned} \left(\frac{d^2}{dr^2} - a^2\right)^2 R &= a^2\lambda\varphi, \\ \left(\frac{d^2}{dr^2} - a^2\right)\varphi &= -\lambda R. \end{aligned} \tag{71}$$

With the free slip boundary condition (21) we obtain

$$\begin{aligned} \varphi(r_1) &= \varphi(r_1 + 1) = 0, \\ R(r_1) &= R(r_1 + 1) = 0, \\ R''(r_1) &= R''(r_1 + 1) = 0 \end{aligned} \tag{72}$$

While the Dirichlet boundary condition (20) yields

$$\begin{aligned}\varphi(r_1) &= \varphi(r_1+1) = 0,\\ R(r_1) &= R(r_1+1) = 0,\\ R'(r_1) &= R'(r_1+1) = 0.\end{aligned} \tag{73}$$

The solutions of (70) are given by

$$h(z) = \cos az, \qquad a^2 = \frac{k^2\pi^2}{L^2} \quad k = 1, 2, \cdots . \tag{74}$$

We now discuss the eigenvalue problem of (71) in two cases.

The free boundary condition. The first eigenvalue $\lambda_0(a)$ and eigenvectors of (71) and (72) for each given $a^2 = k^1\pi^2/L^2$ are obtained as follows

$$\begin{aligned}&\lambda_0(a) = (\pi^2 + a^2)^{3/2}\big/a,\\ &(R, \varphi) = \left(\sin\pi(r - r_1),\ \frac{1}{a}\sqrt{\pi^2 + a^2}\ \sin\pi(r - r_1)\right).\end{aligned} \tag{75}$$

It is clear that the first eigenvalue λ_0 of (52) with boundary conditions (21) is the minimum of $\lambda_0(a)$:

$$\lambda_0^2 = \min_a \lambda_0^2(a) = \min_{k\in\mathbb{N}} \left[\pi^4 L^2 \left(1 + \frac{k^2}{L^2}\right)^3 \Big/ k^2\right]. \tag{76}$$

The corresponding first eigenvectors of (52) with (21) are derived from (69), (75) and (76) as follows:

$$\begin{aligned}u_z &= -\frac{\pi}{a}\ \sin az \cos\pi(r - r_1),\\ u_r &= \cos az \sin\pi(r - r_1),\\ u_\theta &= \frac{1}{a}\sqrt{\pi^2 + a^2}\ \cos az \sin\pi(r - r_1),\end{aligned} \tag{77}$$

where $a = k\pi/L$ satisfies (76).

The Dirichlet boundary condition. The eigenvalue problem of (71) with boundary conditions (73) is equivalent to

$$\begin{aligned}\left(\frac{d^2}{dr^2} - a^2\right)^3 R &= -a^2\lambda^2 R,\\ R &= 0,\, R' = 0,\\ (\frac{d^2}{dr^2} - a^2)^2 R &= 0 \text{ at } r = r_1, r_1 + 1.\end{aligned} \tag{78}$$

Because the height L is sufficiently large, the first eigenvalue and eigenvectors of (78) is given in Chapter II–15 of [1] by $\lambda_0^2 \simeq 1700$, and

$$R(r) \simeq \cos\alpha_0\xi - 0.06\cos h\alpha_1\xi\cos\alpha_2\xi + 0.1\sin h\alpha_1\xi\sin\alpha_2\xi, \tag{79}$$

where $\xi = r - r_1 - \frac{1}{2}$, $\alpha_0 \simeq 3.97$, $\alpha_1 \simeq 5.2$, and $\alpha_2 \simeq 2.1$. This function is illustrated by Fig. 2; see [1].

Thus, the first eigenvectors of (52) with boundary conditions (20) are given by

$$\begin{aligned} u_z &= -\frac{1}{a}\sin azR'(r), \\ u_r &= \cos azR(r), \\ u_\theta &= \frac{1}{a^2\lambda_0}\cos az\left(\frac{d^2}{dr^2} - a^2\right)R(r), \end{aligned} \tag{80}$$

where $R(r)$ solves (78).

Remark 4. We can use another method to obtain the first eigenvalue and eigenvectors of (57) with (21). Let $u_z = \frac{\partial\psi}{\partial r}$ and $u_r = -\frac{\partial\psi}{\partial z}$, then the equations (57) are equivalent to

$$\begin{aligned} \Delta^2\psi &= \lambda\frac{\partial u_\theta}{\partial z}, \\ \Delta u_\theta &= \lambda\frac{\partial\psi}{\partial z}. \end{aligned} \tag{81}$$

The free boundary condition (21) deduces that

$$\begin{array}{lll} \psi = 0, & & D^2\psi = 0 \text{ on } \partial M, \\ u_\theta = 0 & \text{at} & r = r_1, r_1 + 1, \\ \frac{\partial u_\theta}{\partial z} = 0 & \text{at} & z = 0, L. \end{array} \tag{82}$$

The functions (ψ, u_θ) satisfying the boundary condition (82) must have the Fourier expansion as follows:

$$\begin{aligned} \psi &= \sum \alpha_{k_j}\sin\frac{k\pi}{L}z\sin j\pi(r - r_1), \\ u_\theta &= \sum \beta_{k_j}\cos\frac{k\pi}{L}z\sin j\pi(r - r_1). \end{aligned} \tag{83}$$

Replacing ψ, u_θ in (81) by (82) and by comparing the coefficients, we can obtain the eigenvalue (76) and eigenvectors (77).

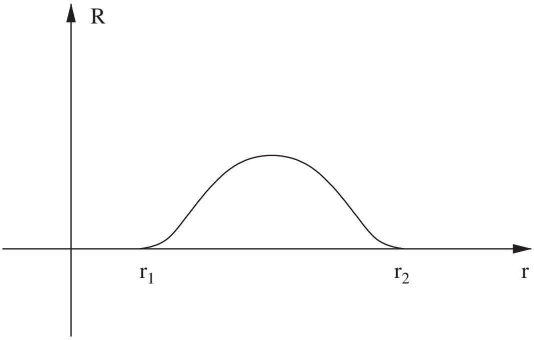

Fig. 2. Eigenfunction of (78).

5.3. Structural stability of the first eigenvectors

It is easy to see that the first eigenvectors $\widetilde{u} = (u_z, u_r)$ given by (77) and (80) have the structure of the Taylor vortices, and by Theorems 7 and 8, we readily verify the structural stability of the vector fields $\widetilde{u} = (u_z, u_r)$ of (77) and (80) in $B^r(M, \mathbb{R}^2)$ $(r \geqq 1)$ and $B_0^r(M, \mathbb{R}^2)$ $(r \geqq 2)$, respectively.

The following two lemmas establish the structure of of first eigenvalue vectors.

Lemma 1. *The vector fields of* (77)

$$(u_z, u_r) = \alpha \left[-\frac{L}{k} \sin \frac{k\pi z}{L} \cos \pi(r - r_1), \cos \frac{k\pi z}{L} \sin \pi(r - r_1) \right] \tag{84}$$

are structurally stable in $B^k(M, \mathbb{R}^2)$ $(k \geqq 1)$. *In addition,* (u_z, u_r) *is topologically equivalent to the flow structure as shown in* Fig. 3.

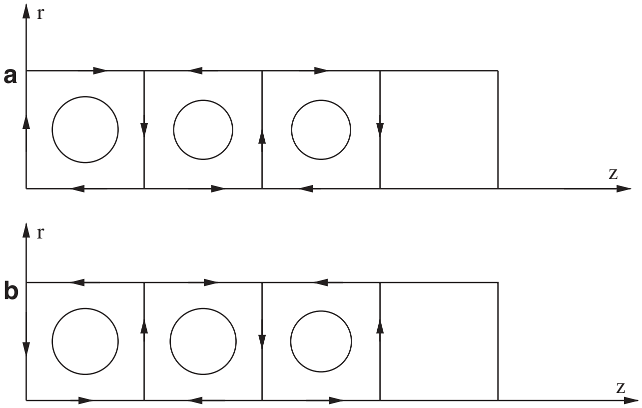

Fig. 3. Schematic flow structure of (u_z, u_r) with the free slip boundary conditions as given by (84): (a) $\alpha > 0$, and (b) $\alpha < 0$.

Proof. It is easy to see that as $\alpha \neq 0$, (u_z, u_r) is regular. Hence the result follows from Theorem 7, Theorem 1 and its proof below. □

Lemma 2. *The vector fields of* (80)

$$\widetilde{u} = (u_z, u_r) = \left(-\frac{\alpha L}{k\pi} \sin \frac{k\pi z}{L} R'(r), \ \alpha \cos \frac{k\pi z}{L} R(r) \right) \tag{85}$$

are structurally stable in the space $B_0^k(M, \mathbb{R}^2)$. *Furthermore,* (u_z, u_r) *is topologically equivalent to the flow structure as shown in* Fig. 4.

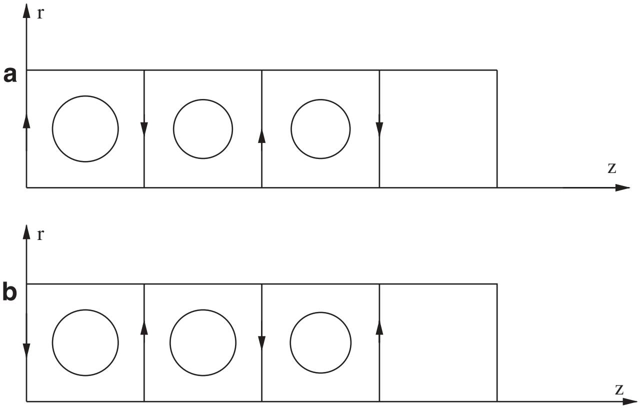

Fig. 4. Schematic flow structure of (u_z, u_r) with the Dirichlet boundary conditions as given by (85): (a) $\alpha > 0$, and (b) $\alpha < 0$.

Proof. By (79) we see that

$$R''(r) \neq 0 \quad \text{at}\, r = r_1 \text{and} r_2, \text{and} R''(r_0) \neq 0,$$

where $r_0 = \frac{1}{2}(r_1 + r_2)$ satisfies that $R'(r_0) = 0$. Hence we have

$$\det D\widetilde{u}(r, z) \neq 0 \quad \text{at } (r, z) = \left(r_0, \frac{jL}{2k}\right), \qquad j = 1, \cdots, k, \text{ and}$$

$$\det \begin{pmatrix} \frac{\partial^2 u_z}{\partial z \partial r} & \frac{\partial^2 u_z}{\partial r^2} \\ \frac{\partial^2 u_r}{\partial z \partial r} & \frac{\partial^2 u_r}{\partial r^2} \end{pmatrix} = -\alpha^2 \cos^2 \frac{k\pi z}{L} R''^2(r) \neq 0$$

for all $(r, z) = (r_i, \frac{jL}{k})$, $i = 1, 2$, $j = 1, \cdots, k$. Hence, the vector fields (85) are D-regular. Thus, Lemma 2 follows from Theorem 8, Theorem 1 and its proof below. □

5.4. Taylor vortex structure of solutions of the Taylor problem

The following is the main theorem in this section which shows that the solutions of the Taylor problem with the axisymmetric perturbation have the Taylor vortices as their asymptotic structure.

Theorem 9. *Let* (17), (18) *and* (54) *hold true, and let L be sufficiently large. Then as the Taylor number* $T = \lambda^2$ *satisfies* $0 < T - T_c < \kappa$ *for some* $\kappa > 0$, *with* $T_c = \widetilde{\lambda}_0^2$ *the critical Taylor number, or equivalently* $0 < \lambda - \widetilde{\lambda}_0 < \varepsilon$ *for some* $\varepsilon > 0$, *the space* D^k_f *(resp.* D^k_d*) can be decomposed into two open sets* U^λ_1 *and* U^λ_2*:*

$$D^k_f = \overline{U}^\lambda_1 + \overline{U}^\lambda_2 \qquad (\textit{resp.} D^k_d = \overline{U}^\lambda_1 + \overline{U}^\lambda_2),$$
$$U^\lambda_1 \cap U^\lambda_2 = \emptyset, \qquad 0 \in \partial U^\lambda_1 \cap \partial U^\lambda_2,$$

such that the following assertions hold true.

(i) *For any* $\varphi \in U_1^\lambda$, *there is a time* $t_\varphi \geqq 0$, *for the solution* $u(t, \varphi) = (\widetilde{u}, u_\theta)$ *of* (19) *and* (21) (*resp. of* (19) *and* (20)) *with* $u(0, \varphi) = \varphi$, $\widetilde{u}(t, \varphi) = (u_z, u_r)$ *is topologically equivalent to the structure as shown in* Fig. 3(a) (*resp. in* Fig. 4(a)) *for any* $t > t_\varphi$. *Namely, the solution* $u(t, \varphi)$ *of* (19) *and* (21) (*resp. of* (19) *and* (20)) *has the Taylor vortex structure as* $t > t_\varphi$.

(ii) *For any* $\varphi \in U_2^\lambda$, *there is a* $t_\varphi \geqq 0$, *the solution* $\widetilde{u}(t, \varphi) = (u_z, u_r)$ *of* (19) *and* (21) (*resp. of* (19) *and* (20)) *is topologically equivalent to the structure as shown in* Fig. 3(b) (*resp. in* Fig. 4(b)) *for any* $t > t_\varphi$.

Proof. As the height $L > 0$ is sufficiently large, it is easy to see from [1] that the first eigenvalue of (52) and (20) is simple. When $L^2 \neq \ell^2$ with $\ell^2 \simeq 5$ satisfying

$$(\ell^2 + 1)^3 = \frac{(\ell^2 + 4)^3}{4}$$

we infer from (76) that the first eigenvalue of (52) with boundary conditions (21) is also simple. Thus, the theorem follows immediately from Theorem 6, Remark 3, the structural stability theorems (Lemmas 1 and 2), and the uniform boundedness of the H^k norm for the solutions of the equations (19) with either the boundary condition (21) or the boundary condition (19). Thus, this proof is complete. □

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