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Stratified rotating Boussinesq equations in geophysical fluid dynamics: Dynamic bifurcation and periodic solutions

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The main objective of this article is to study the dynamics of the stratified rotating Boussinesq equations, which are a basic model in geophysical fluid dynamics. First, for the case where the Prandtl number is greater than 1, a complete stability and bifurcation analysis near the first critical Rayleigh number is carried out. Second, for the case where the Prandtl number is smaller than 1, the onset of the Hopf bifurcation near the first critical Rayleigh number is established, leading to the existence of nontrivial periodic solutions. The analysis is based on a newly developed bifurcation and stability theory for nonlinear dynamical systems (both finite and infinite dimensional) by two of the authors [T. Ma and S. Wang, *Bifurcation Theory and Applications*, World Scientific Series on Nonlinear Sciences Vol. 53 (World Scientific, Singapore, 2005)].
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I. INTRODUCTION

The phenomena of the atmosphere and oceans are extremely rich in their organization and complexity, and a lot of them cannot be produced by experiments. These phenomena involve a broad range of temporal and spatial scales. As we know, both the atmospheric and oceanic flows are flows under the rotation of the earth. In fact, fast rotation and small aspect ratio are two main characteristics of the large-scale atmospheric and oceanic flows. The small aspect ratio characteristic leads to the primitive equations, and the fast rotation leads to the quasigeostrophic equations. These are fundamental equations in the study of atmospheric and oceanic flows; see Ghil and Childress,[6] Lions *et al.*,[12,13] and Pedlosky.[22] Furthermore, convection occurs in many regimes of the atmospheric and oceanic flows.

Key problems in the study of climate dynamics and in geophysical fluid dynamics are the understanding and the prediction of the periodic, quasiperiodic, aperiodic, and fully turbulent characteristics of large-scale atmospheric and oceanic flows. Stability/bifurcation theory enables one to determine how different flow regimes appear and disappear as control parameters, such as the Reynolds number, vary. It, therefore, provides one with a powerful tool to explore the theoretical capability in the predictability problem. Most studies so far have only considered systems of ordinary differential equations (ODEs) that are obtained by projecting the partial differential equations (PDEs) onto a finite-dimensional solution space, either by finite differencing or by

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truncating a Fourier expansion (see Ghil and Childress[6] and further references therein). These were pioneered by Lorenz,[14,15] Stommel,[25] and Veronis[27,28] among others, who explored the bifurcation structure of low-order models of atmospheric and oceanic flows. More recently, pseudo-arc-length continuation methods have been applied to atmospheric (Legras and Ghil[11]) and oceanic (Speich *et al.*[24] and Dijkstra[5]) models with increasing horizontal resolution. These numerical bifurcation studies have produced so far fairly reliable results for two classes of geophysical flows: (i) atmospheric flows in a periodic midlatitude channel, in the presence of bottom topography and a forcing jet, and (ii) oceanic flows in a rectangular midlatitude basin, subject to wind stress on its upper surface; see among others Charney and DeVore,[2] Pedlosky,[23] Legras and Ghil,[11] and Jin and Ghil[10] for saddle-node and Hopf bifurcations in the atmospheric channel, and Refs. 20, 1, 8, 9, 19, and 24 for saddle-node, pitchfork, or Hopf bifurcations in the oceanic basin.

The main objective of this article is to conduct bifurcation and stability analysis for the original PDEs that govern geophysical flows. This approach should allow us to overcome some of the inherent limitations of the numerical bifurcation results that dominate the climate dynamics literature up to this point, and to capture the essential dynamics of the governing PDE systems.

The present article addresses the stability and transitions of basic flows for the stratified rotating Boussinesq equations. These equations are fundamental equations in the geophysical fluid dynamics; see among others Pedlosky.[22] We obtain two main results in this article. The first is to conduct a rigorous and complete bifurcation and stability analysis near the first eigenvalue of the linearized problem. The second is the onset of the Hopf bifurcation, leading to the existence of periodic solutions of the model.

The detailed analysis is carried out in two steps. The first is a detailed study of the eigenvalue problem for the linearized problem around the basic state. In comparison to the classical Bénard convection problem, the linearized problem here is non-self-adjoint, leading to much more complicated spectrum, and more complicated dynamics. We derive, in particular, two critical Rayleigh numbers R_{c_1} and R_{c_2}. Here R_{c_1} is the first critical Rayleigh number for the case where the Prandtl number is greater than 1, and R_{c_2} is the first critical Rayleigh number for the case where the Prandtl number is less than 1. Moreover, R_{c_1} leads to the onset of the steady state bifurcation, while R_{c_2} leads to the onset of the Hopf bifurcation. Both parameters are explicitly given in terms of the physical parameters. The crucial issues here include (1) a complete understanding of the spectrum, (2) identification of the critical Rayleigh numbers, and most importantly (3) the verification of the principle of exchange of stabilities near these critical Rayleigh numbers.

The second step is to conduct a rigorous nonlinear analysis to derive the bifurcations at both critical Rayleigh numbers based on the classical Hopf bifurcation theory and on a newly developed dynamic bifurcation theory by two of the authors. This new dynamic bifurcation theory is centered at a new notion of bifurcation, called attractor bifurcation for dynamical systems, both finite dimensional and infinite dimensional, together with new strategies for the Lyapunov-Schmidt reduction and the center manifold reduction procedures. The bifurcation theory has been applied to various problems from science and engineering, including, in particular, the Kuramoto-Sivashinshy equation, the Cahn-Hillard equation, the Ginzburg-Landau equation, reaction-diffusion equations in biology and chemistry, the Bénard convection problem, and the Taylor problem; see Refs. 16 and 17 and references therein.

We remark that the non-self-adjointness of the linearized problem gives rise to the onset of the Hopf bifurcation. We prove that the Hopf bifurcation appears at the Rayleigh number R_{c_2}. As mentioned earlier, the understanding and prediction of the periodic, quasiperiodic, aperiodic, and fully turbulent characteristics of large-scale atmospheric and oceanic flows are key issues in the study of climate dynamics and in geophysical fluid dynamics. It is hoped that the study carried out in this article will provide some insights in these important issues.

Also, we would like to mention that rigorous proof of the existence of periodic solutions for a fluid system is normally a very difficult task from the mathematical point of view. For instance, with a highly involved analysis, Chen *et al.*[3] proved the existence of a Hopf bifurcation in an idealized Fourier space.

The paper is organized as follows. Section II gives the basic setting of the problem. Section III states the main results. The proofs of the main results occupy the remaining part of the paper: Sec. IV recapitulates the essentials of the attractor bifurcation theory, Sec. V is on the eigenanalysis, and Sec. VI is on the central manifold reduction and the completion of the proofs.

II. STRATIFIED ROTATING BOUSSINESQ EQUATIONS IN GEOPHYSICAL FLUID DYNAMICS

The stratified rotating Boussinesq equations are basic equations in the geophysical fluid dynamics, and their nondimensional form is given by

$$\frac{\partial U}{\partial t} = \sigma(\Delta U - \nabla p) + \sigma R T e - \frac{1}{Ro} e \times U - (U \cdot \nabla) U,$$

$$\frac{\partial T}{\partial t} = \Delta T + w - (U \cdot \nabla) T, \tag{2.1}$$

$$\text{div } U = 0,$$

for (x,y,z) in the nondimensional domain $\Omega = \mathbb{R}^2 \times (0,1)$, where $U=(u,v,w)$ are the velocity fields, $e=(0,0,1)$ is the unit vector in the z direction, σ is the Prandtl number, R is the thermal Rayleigh number, Ro is the Rossby number, T is the temperature function, and p is the pressure function. We refer the interested readers to Pedlosky[22] and Lions *et al.*[13] for the derivation of this model and the related parameters. In particular, the term $(1/\mathrm{Ro}) e \times U$ represents the Coriolis force, the w term in the temperature equation is derived using the stratification, and the definition of the Rayleigh number R is as follows:

$$R = \frac{g\alpha\beta}{\kappa\nu} h^4. \tag{2.2}$$

We consider the periodic boundary condition in the x and y directions,

$$(U,T)(x,y,z,t) = (U,T)(x + 2j\pi/\alpha_1, y, z, t) = (U,T)(x, y + 2k\pi/\alpha_2, z, t), \tag{2.3}$$

for any $j,k \in \mathbb{Z}$. At the top and bottom boundaries, we impose the free-free boundary conditions:

$$(T,w) = 0, \quad \frac{\partial u}{\partial z} = 0, \quad \frac{\partial v}{\partial z} = 0, \quad z = 0,1. \tag{2.4}$$

It is natural to put the constraint

$$\int_\Omega u \mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_\Omega v \mathrm{d}x\mathrm{d}y\mathrm{d}z = 0. \tag{2.5}$$

The initial value conditions are given by

$$(U,T) = (\tilde{U}, \tilde{T}) \quad \text{at } t = 0. \tag{2.6}$$

Let

$$H = \{(U,T) \in L^2(\Omega)^4 | \text{div } U = 0, w|_{z=0,1} = 0, (u,v) \text{ satisfies Eqs. (2.3) and (2.5)}\},$$

$$H_1 = \{(U,T) \in H^2(\Omega)^4 \cap H | (U,T) \text{ satisfies Eqs. (2.3) – (2.5)}\},$$

$$\tilde{H} = \{(U,T) \in H | (u,v,w,T)(-x,-y,z) = (-u,-v,w,T)(x,y,z)\},$$

$$\tilde{H}_1 = H_1 \cap \tilde{H}.$$

Let $L_R = -A - B_R : H_1 \to H$ $(\tilde{H}_1 \to \tilde{H})$ and $G : H_1 \to H$ $(\tilde{H}_1 \to \tilde{H})$ be defined by

$$A\psi = \left(-P\left[\sigma\Delta U - \frac{1}{Ro} e \times U\right], -\Delta T\right),$$

$$B_R\psi = (-P[\sigma RTe], -w),$$

$$G(\psi) = G(\psi, \psi),$$

for any $\psi = (U, T) \in H_1$ $(\tilde{H}_1)$, where

$$G(\psi_1, \psi_2) = (-P[(U_1 \cdot \nabla)U_2], -(U_1 \cdot \nabla)T_2),$$

for any $\psi_1 = (U_1, T_1)$, $\psi_2 = (U_2, T_2) \in H_1$. Here P is the Leray projection to L^2 fields; for a detailed account of the function spaces, see among many others Ref. 26.

Remark 2.1: Note that $\tilde{H}_1$ and $\tilde{H}$ are invariant under the bilinear operator G in the sense that

$$G(\psi_1, \psi_2) \in \tilde{H} \quad \text{for } \psi_1, \psi_2 \in \tilde{H}_1.$$

Hence, $\tilde{H}_1$ and $\tilde{H}$ are invariant under the operator $L_R + G$.

Then, the Boussinesq equations (2.1)–(2.5) can be written in the following operator form:

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = L_R\psi + G(\psi), \quad \psi = (U, T). \tag{2.7}$$

III. MAIN RESULTS

A. Definition of attractor bifurcation

To state the main theorems of this article, we proceed with the definition of attractor bifurcation, first introduced by Ma and Wang in Refs. 16 and 17.

Let H and H_1 be two Hilbert spaces, and $H_1 \hookrightarrow H$ be a dense and compact inclusion. We consider the following nonlinear evolution equations:

$$\frac{\mathrm{d}u}{\mathrm{d}t} = L_\lambda u + G(u, \lambda),$$

$$u(0) = u_0, \tag{3.1}$$

where $u : [0, \infty) \to H$ is the unknown function, $\lambda \in \mathbb{R}$ is the system parameter, and $L_\lambda : H_1 \to H$ are parametrized linear completely continuous fields depending continuously on $\lambda \in \mathbb{R}^1$, which satisfy

$$-L_\lambda = A + B_\lambda \quad \text{a sectorial operator},$$

$$A : H_1 \to H \quad \text{a linear homeomorphism}, \tag{3.2}$$

$$B_\lambda : H_1 \to H \quad \text{parametrized linear compact operators}.$$

It is easy to see[7] that L_λ generates an analytic semigroup $\{e^{tL_\lambda}\}_{t \geq 0}$. Then, we can define fractional power operators $(-L_\lambda)^\mu$ for any $0 \leq \mu \leq 1$ with domain $H_\mu = D((-L_\lambda)^\mu)$ such that $H_{\mu_1} \subset H_{\mu_2}$ if $\mu_1 > \mu_2$, and $H_0 = H$.

Furthermore, we assume that the nonlinear terms $G(\cdot,\lambda):H_\mu \to H$ for some $1>\mu\geq 0$ are a family of parametrized C^r bounded operators $(r\geq 1)$ continuously depending on the parameter $\lambda\in\mathbb{R}^1$, such that

$$G(u,\lambda)=o(\|u\|_{H_\mu}), \quad \forall\,\lambda\in\mathbb{R}^1. \tag{3.3}$$

In this paper, we are interested in the sectorial operator $-L_\lambda=A+B_\lambda$ such that there exists an eigenvalue sequence $\{\rho_k\}\subset\mathbb{C}^1$ and an eigenvector sequence $\{e_k,h_k\}\subset H_1$ of A:

$$Az_k=\rho_k z_k, \quad z_k=e_k+ih_k,$$

$$\text{Re}\,\rho_k\to\infty \quad (k\to\infty), \tag{3.4}$$

$$|\text{Im}\,\rho_k/(a+\text{Re}\,\rho_k)|\leq c,$$

for some $a,c>0$, such that $\{e_k,h_k\}$ is a basis of H. Also we assume that there is a constant $0<\theta<1$ such that

$$B_\lambda:H_\theta\to H \text{ bounded}, \quad \forall\,\lambda\in\mathbb{R}^1. \tag{3.5}$$

Under conditions (3.4) and (3.5), the operator $-L_\lambda=A+B_\lambda$ is a sectorial operator.

Let $\{S_\lambda(t)\}_{t\geq 0}$ be an operator semigroup generated by Eq. (3.1). Then, the solution of Eq. (3.1) can be expressed as $\psi(t,\psi_0)=S_\lambda(t)\psi_0$, for any $t\geq 0$.

Definition 3.1: A set $\Sigma\subset H$ is called an invariant set of Eqs. (3.1) if $S(t)\Sigma=\Sigma$ for any $t\geq 0$. An invariant set $\Sigma\subset H$ of Eqs. (3.1) is called an attractor if Σ is compact, and there exists a neighborhood $W\subset H$ of Σ such that for any $\psi_0\in W$ we have

$$\lim_{t\to\infty}\text{dist}_H(\psi(t,\psi_0),\Sigma)=0.$$

Definition 3.2:

(1) We say that the solution of Eqs. (3.1) bifurcates from $(\psi,\lambda)=(0,\lambda_0)$ to an invariant set Ω_λ if there exists a sequence of invariant sets $\{\Omega_{\lambda_n}\}$ of Eqs. (3.1) such that $0\notin\Omega_{\lambda_n}$, $\lim_{n\to\infty}\lambda_n=\lambda_0$, and

$$\lim_{n\to\infty}\max_{x\in\Omega_{\lambda_n}}|x|=0.$$

(2) If the invariant sets Ω_λ are attractors of Eqs. (3.1), then the bifurcation is called an attractor bifurcation.

B. Main theorems

In this article, we consider two cases:

$$\sigma>1 \quad \text{and} \quad R_{c_1} \text{ is obtained only at } (j,k,l)=(j_1,0,1), \tag{3.6}$$

$$\sigma<1 \quad \text{and} \quad R_{c_2} \text{ is obtained only at } (j,k,l)=(j_2,0,1), \tag{3.7}$$

for some j_1, $j_2\in\mathbb{N}$, where R_{c_1} and R_{c_2} are defined in Eqs. (5.18) and (5.22), respectively. In the above cases, R_{c_1} and R_{c_2} are given by the following formulas:

$$R_{c_1}=\frac{(j_1^2\alpha_1^2+\pi^2)^3}{j_1^2\alpha_1^2}+\frac{\pi^2}{\sigma^2\text{Ro}^2 j_1^2\alpha_1^2},$$

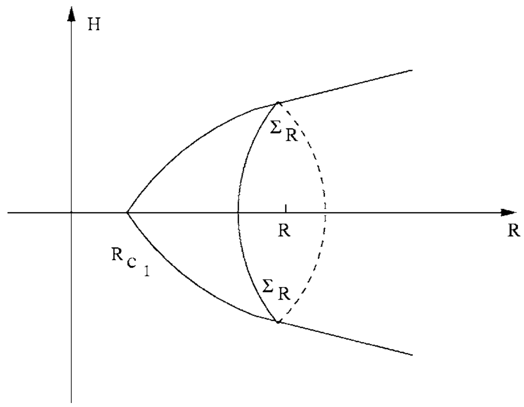

FIG. 1. Bifurcation from $(0,R_{c_1})$ to an attractor Σ_R for $R>R_{c_1}$.

$$R_{c_2}=\frac{2(\sigma+1)(j_2^2\alpha_1^2+\pi^2)^3}{j_2^2\alpha_1^2}+\frac{2\pi^2}{(\sigma+1)\mathrm{Ro}^2 j_2^2\alpha_1^2}.$$

Remark 3.1:

(1) Condition (3.6) guarantees that for $R\approx R_{c_1}$, the first eigenvalue of $L_R|_{H_1}$ ($L_R|_{\tilde{H}_1}$) is real and of multiplicity 2 (1); see Remark (5.1).
(2) Condition (3.7) guarantees that, for $R\approx R_{c_2}$, there exists only one simple pair of conjugate complex eigenvalues of $L_R|_{\tilde{H}_1}$ crossing the imaginary axis; see Lemma (5.5)
(3) Condition (3.6) or (3.7) can be satisfied easily; see Lemmas (5.3) and (5.4).

Theorem 3.1: *Assume Eq. (3.6). Then, the following assertions for problems (2.1)–(2.5) defined in H hold true.*

(1) If $R\leq R_{c_1}$, the steady state $(U,T)=0$ is locally asymptotically stable.
(2) For $R>R_{c_1}$, the problem bifurcates from $((U,T),R)=(0,R_{c_1})$ to an attractor $\Sigma_R=S^1$, consisting of only steady state solutions (Fig. 1).

Theorem 3.2: *Assume Eq. (3.7) and*

$$\mathrm{Ro}^2<\frac{(1-\sigma)\pi^2}{\sigma^2(1+\sigma)(j_2^2\alpha_1^2+\pi^2)^3}.$$

The following statements are true.

(1) For problems (2.1)–(2.5) defined in H, the steady state $(U,T)=0$ is locally asymptotically stable if $R<R_{c_2}$.
(2) For problems (2.1)–(2.5) defined in $\tilde{H}$, a Hopf bifurcation occurs generically when R crosses R_{c_2}.

IV. PRELIMINARIES

A. Attractor bifurcation theory

Consider Eq. (3.1) satisfying Eqs. (3.2) and (3.3). We start with the principle of exchange of stabilities (PES). Let the eigenvalues (counting the multiplicity) of L_λ be given by $\beta_1(\lambda)$, $\beta_2(\lambda)$, …. Suppose that

$$\mathrm{Re}\,\beta_i(\lambda)\begin{cases}<0 & \text{if } \lambda<\lambda_0\\ =0 & \text{if } \lambda=\lambda_0,\\ >0 & \text{if } \lambda>\lambda_0,\end{cases}\qquad \text{if } 1\leq i\leq m \tag{4.1}$$

$$\mathrm{Re}\,\beta_j(\lambda_0)<0 \quad \text{if } m+1\leq j. \tag{4.2}$$

Let the eigenspace of L_λ at λ_0 be

$$E_0=\bigcup_{1\leq j\leq m}\bigcup_{k=1}^{\infty}\{u,v\in H_1|(L_{\lambda_0}-\beta_j(\lambda_0))^k w=0, w=u+iv\}.$$

It is known that $\dim E_0=m$.

Theorem 4.1: *(Ma and Wang[16,17]) Assume that the conditions (3.2)–(3.5), (4.1), and (4.2) hold true, and $u=0$ is locally asymptotically stable for Eq. (3.1) at $\lambda=\lambda_0$. Then, the following assertions hold true.*

(1) For $\lambda>\lambda_0$, Eq. (3.1) bifurcates from $(u,\lambda)=(0,\lambda_0)$ to attractors Σ_λ, having the same homology as S^{m-1}, with $m-1\leq\dim\Sigma_\lambda\leq m$, which is connected if $m>1$.

(2) For any $u_\lambda\in\Sigma_\lambda$, u_λ can be expressed as

$$u_\lambda=v_\lambda+o(\|v_\lambda\|_{H_1}), \quad v_\lambda\in E_0.$$

(3) There is an open set $U\subset H$ with $0\in U$ such that the attractor Σ_λ bifurcated from $(0,\lambda_0)$ attracts $U\backslash\Gamma$ in H, where Γ is the stable manifold of $u=0$ with codimension m.

B. Center manifold theory

A crucial ingredient for the proof of the main theorems using the above attractor bifurcation theorem is an approximation formula for center manifold functions; see Ref. 16.

Let H_1 and H be decomposed into

$$H_1=E_1^\lambda\oplus E_2^\lambda, \quad H=\widetilde{E}_1^\lambda\oplus\widetilde{E}_2^\lambda, \tag{4.3}$$

for λ near $\lambda_0\in\mathbb{R}^1$, where E_1^λ, E_2^λ are invariant subspaces of L_λ, such that $\dim E_1^\lambda<\infty$, $\widetilde{E}_1^\lambda=E_1^\lambda$, $\widetilde{E}_2^\lambda=$ closure of E_2^λ in H. In addition, L_λ can be decomposed into $L_\lambda=\mathcal{L}_1^\lambda\oplus\mathcal{L}_2^\lambda$ such that for any λ near λ_0,

$$\mathcal{L}_1^\lambda=L_\lambda|_{E_1^\lambda}:E_1^\lambda\to\widetilde{E}_1^\lambda,$$

$$\mathcal{L}_2^\lambda=L_\lambda|_{E_2^\lambda}:E_2^\lambda\to\widetilde{E}_2^\lambda, \tag{4.4}$$

where all eigenvalues of $\mathcal{L}_2^\lambda$ possess negative real parts, and the eigenvalues of $\mathcal{L}_1^\lambda$ possess nonnegative real parts at $\lambda=\lambda_0$. Furthermore, with $\mu<1$ given by Eq. (3.3), let

$$E_2^\lambda(\mu)=\text{closure of } E_2^\lambda \text{ in } H_\mu.$$

By the classical center manifold theorem (see among others Refs. 7 and 26), there exists a neighborhood of λ_0 given by $|\lambda-\lambda_0|<\delta$ for some $\delta>0$, a neighborhood $B_\lambda\subset E_1^\lambda$ of $x=0$, and a C^1 center manifold function $\Phi(\cdot,\lambda):B_\lambda\to E_2^\lambda(\theta)$, called the center manifold function, depending continuously on λ. Then, to investigate the dynamic bifurcation of Eq. (3.1) it suffices to consider the finite-dimensional system as follows:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mathcal{L}_1^{\lambda}x + g_1(x,\Phi_\lambda(x),\lambda), \quad x \in B_\lambda \subset E_1^{\lambda}. \tag{4.5}$$

Hence, an approximation formula for the center manifold function Φ_λ is crucial for the bifurcation and stability study.

Let the nonlinear operator G be in the following form:

$$G(u,\lambda) = G_n(u,\lambda) + o(\|u\|^n), \tag{4.6}$$

for some integer $n \geqslant 2$. Here $G_n : H_1 \times \cdots \times H_1 \to H$ is an n-multilinear operator, and $G_n(u,\lambda) = G_n(u,\ldots,u,\lambda)$.

Theorem 4.2: *(Ma and Wang[16]) Under the conditions (4.3), (4.4), and (4.6), the center manifold function $\Phi(x,\lambda)$ can be expressed as*

$$\Phi(x,\lambda) = (-\mathcal{L}_2^{\lambda})^{-1}P_2G_n(x,\lambda) + o(\|x\|^n) + O(|\mathrm{Re}\,\beta|\|x\|^n), \tag{4.7}$$

where $\mathcal{L}_2^{\lambda}$ is as in Eq. (4.4), $P_2 : H \to \widetilde{E}_2$ the canonical projection, $x \in E_1^{\lambda}$, and $\beta = (\beta_1(\lambda),\ldots,\beta_m(\lambda))$ the eigenvectors of $\mathcal{L}_1^{\lambda}$.

V. EIGENVALUE PROBLEM

The eigenvalue problem of the linearized problem of Eqs. (2.1)–(2.4) is given by

$$\begin{gathered} \sigma(\Delta U - \nabla p) + \sigma RTe - \frac{1}{\mathrm{Ro}}e \times U = \beta U, \\ \Delta T + w = \beta T, \\ \mathrm{div}\, U = 0, \end{gathered} \tag{5.1}$$

supplemented with Eqs. (2.3) and (2.4). For $\psi=(U,T)$ satisfying Eqs. (2.3) and (2.4), we expand the field ψ in Fourier series,

$$\psi(x,y,z) = \sum_{j,k=-\infty}^{\infty} \psi_{jk}(z)\mathrm{e}^{i(j\alpha_1 x + k\alpha_2 y)}. \tag{5.2}$$

Plugging Eq. (5.2) into Eq. (5.1), we obtain the following system of ordinary differential equations:

$$\begin{gathered} \sigma(D_{jk}u_{jk} - ij\alpha_1 p_{jk}) + \frac{1}{\mathrm{Ro}}v_{jk} = \beta u_{jk}, \\ \sigma(D_{jk}v_{jk} - ik\alpha_2 p_{jk}) - \frac{1}{\mathrm{Ro}}u_{jk} = \beta v_{jk}, \\ D_{jk}w_{jk} - p'_{jk} + RT_{jk} = \sigma^{-1}\beta w_{jk}, \\ D_{jk}T_{jk} + w_{jk} = \beta T_{jk}, \\ ij\alpha_1 u_{jk} + ik\alpha_2 v_{jk} + w'_{jk} = 0, \end{gathered} \tag{5.3}$$

$$u'_{jk}|_{z=0,1} = v'_{jk}|_{z=0,1} = w_{jk}|_{z=0,1} = T_{jk}|_{z=0,1} = 0,$$

for $j,k \in \mathbb{Z}$, where $' = \mathrm{d}/\mathrm{d}z$, $D_{jk} = \mathrm{d}^2/\mathrm{d}z^2 - \alpha_{jk}^2$, and $\alpha_{jk}^2 = j^2\alpha_1^2 + k^2\alpha_2^2$. If $w_{jk} \neq 0$, Eq. (5.3) can be reduced to a single equation for $w_{jk}(z)$:

$$\left\{(D_{jk}-\beta)(\sigma D_{jk}-\beta)^2 D_{jk} + \frac{1}{\mathrm{Ro}^2}(D_{jk}-\beta)(D_{jk}+\alpha_{jk}^2) + \sigma R\alpha_{jk}^2(\sigma D_{jk}-\beta)\right\} w_{jk} = 0, \tag{5.4}$$

$$w_{jk} = w''_{jk} = w^{(4)}_{jk} = w^{(6)}_{jk} = 0 \quad \text{at} \quad z = 0,1, \tag{5.5}$$

for $j,k \in \mathbb{Z}$. Thanks to Eq. (5.5), w_{jk} can be expanded in a Fourier sine series,

$$w_{jk}(z) = \sum_{l=1}^{\infty} w_{jkl} \sin l\pi z, \tag{5.6}$$

for $(j,k) \in \mathbb{Z} \times \mathbb{Z}$. Substituting Eq. (5.6) into Eq. (5.4), we see that the eigenvalues β of the problem (5.1) satisfy the cubic equations

$$\beta^3 + (2\sigma+1)\gamma_{jkl}^2\beta^2 + \left[(\sigma^2+2\sigma)\gamma_{jkl}^4 + \frac{l^2\pi^2}{\mathrm{Ro}^2\gamma_{jkl}^2} - \sigma R\frac{\alpha_{jk}^2}{\gamma_{jkl}^2}\right]\beta + \sigma^2\gamma_{jkl}^6 - \sigma^2 R\alpha_{jk}^2 + \frac{l^2\pi^2}{\mathrm{Ro}^2} = 0, \tag{5.7}$$

for $j,k \in \mathbb{Z}$ and $l \in \mathbb{N}$, where $\gamma_{jkl}^2 = \alpha_{jk}^2 + l^2\pi^2$. In the following discussions, we let

$$g_{jkl}(\beta) = (\beta + \gamma_{jkl}^2)[(\beta + \sigma\gamma_{jkl}^2)^2 + l^2\pi^2\mathrm{Ro}^{-2}\gamma_{jkl}^{-2}],$$

$$h_{jkl}(\beta) = \sigma R\alpha_{jk}^2\gamma_{jkl}^{-2}(\beta + \sigma\gamma_{jkl}^2), \tag{5.8}$$

$$f_{jkl}(\beta) = g_{jkl}(\beta) - h_{jkl}(\beta),$$

and $\beta_{jkl1}(R)$, $\beta_{jkl2}(R)$, and $\beta_{jkl3}(R)$ be the zeros of f_{jkl} with

$$\mathrm{Re}(\beta_{jkl1}) \geqslant \mathrm{Re}(\beta_{jkl2}) \geqslant \mathrm{Re}(\beta_{jkl3}).$$

A. Eigenvectors

In the following discussions, we consider the following index sets:

$$\Lambda_1 = \{(j,k,l) \in \mathbb{Z}^2 \times \mathbb{N} \,|\, j \geqslant 0, (j,k) \neq (0,0)\},$$

$$\Lambda_2 = \{(j,k,l) \in \mathbb{Z}^2 \times \{0\} \,|\, j \geqslant 0, (j,k) \neq (0,0)\},$$

$$\Lambda_3 = \{(j,k,l) \in \{(0,0)\} \times \mathbb{N}\},$$

$$\Lambda = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3.$$

1. For $(j,k,0) \in \Lambda_2$, we define

$$\psi_1^{\beta_{jk0}} = (k\alpha_2 \sin(j\alpha_1 x + k\alpha_2 y), -j\alpha_1 \sin(j\alpha_1 x + k\alpha_2 y), 0, 0)^t,$$

$$\psi_2^{\beta_{jk0}} = (-k\alpha_2 \cos(j\alpha_1 x + k\alpha_2 y), j\alpha_1 \cos(j\alpha_1 x + k\alpha_2 y), 0, 0)^t,$$

$$E_{jk0} = \text{span}\{\psi_1^{\beta_{jk0}}, \psi_2^{\beta_{jk0}}\},$$

$$\beta_{\Lambda_2} = \cup_{(j,k,0)\in\Lambda_2}\{\beta_{jk0}\},$$

where $\beta_{jk0} = -\sigma\gamma_{jk0}^2 = -\sigma\alpha_{jk}^2 = -\sigma(j^2\alpha_1^2 + k^2\alpha_2^2)$. It is not hard to see that $L_R(\psi_1^{\beta_{jk0}}) = \beta_{jk0}\psi_1^{\beta_{jk0}}$ and $L_R(\psi_2^{\beta_{jk0}}) = \beta_{jk0}\psi_2^{\beta_{jk0}}$.

2. For $(0,0,l)\in\Lambda_3$, we define

$$\psi^{\beta_{00l1}} = (0,0,0,\sin l\pi z)^t, \quad \psi^{\beta_{00l2}} = (\cos l\pi z,0,0,0)^t,$$

$$\psi^{\beta_{00l3}} = (0,\cos l\pi z,0,0)^t, \quad E_{00l} = \text{span}\{\psi^{\beta_{00l1}}, \psi^{\beta_{00l2}}, \psi^{\beta_{00l3}}\},$$

$$\beta_{\Lambda_3} = \cup_{l=1}^{\infty}\cup_{q=1}^{3}\{\beta_{00lq}\}, \quad \beta_{\tilde{\Lambda}_3} = \cup_{l=1}^{\infty}\{\beta_{00l1}\},$$

where $\beta_{00l1} = -\gamma_{00l}^2 = -l^2\pi^2$, $\beta_{00l2} = -\sigma\gamma_{00l}^2 - (1/\text{Ro})^i$, and $\beta_{00l3} = -\sigma\gamma_{00l}^2 + (1/\text{Ro})^i$. It is easy to check that

$$L_R(\psi^{\beta_{00l1}}) = \beta_{0011}\psi^{\beta_{00l1}},$$

$$L_R(\psi^{\beta_{00l2}}) = -\sigma\gamma_{00l}^2\psi^{\beta_{00l2}} - \frac{1}{\text{Ro}}\psi^{\beta_{00l3}},$$

$$L_R(\psi^{\beta_{00l3}}) = \frac{1}{\text{Ro}}\psi^{\beta_{00l2}} - \sigma\gamma_{00l}^2\psi^{\beta_{00l3}}.$$

3. For $(j,k,l)\in\Lambda_1$, we define

$$\phi_{jkl}^1 = \left(-\frac{j\alpha_1 l\pi}{\alpha_{jk}^2}\sin(j\alpha_1 x + k\alpha_2 y)\cos l\pi z, -\frac{k\alpha_2 l\pi}{\alpha_{jk}^2}\sin(j\alpha_1 x + k\alpha_2 y)\cos l\pi z,\right.$$
$$\left.\cos(j\alpha_1 x + k\alpha_2 y)\sin l\pi z, 0\right)^t,$$

$$\phi_{jkl}^2 = \left(\frac{k\alpha_2 l\pi}{\alpha_{jk}^2}\sin(j\alpha_1 x + k\alpha_2 y)\cos l\pi z, -\frac{j\alpha_1 l\pi}{\alpha_{jk}^2}\sin(j\alpha_1 x + k\alpha_2 y)\cos l\pi z, 0, 0\right),$$

$$\phi_{jkl}^3 = (0,0,0,\cos(j\alpha_1 x + k\alpha_2 y)\sin l\pi z)^t,$$

$$\phi_{jkl}^4 = \left(\frac{j\alpha_1 l\pi}{\alpha_{jk}^2}\cos(j\alpha_1 x + k\alpha_2 y)\cos l\pi z, \frac{k\alpha_2 l\pi}{\alpha_{jk}^2}\cos(j\alpha_1 x + k\alpha_2 y)\cos l\pi z,\right.$$
$$\left.\sin(j\alpha_1 x + k\alpha_2 y)\sin l\pi z, 0\right)^t,$$

$$\phi_{jkl}^5 = \left(-\frac{k\alpha_2 l\pi}{\alpha_{jk}^2}\cos(j\alpha_1 x + k\alpha_2 y)\cos l\pi z, \frac{j\alpha_1 l\pi}{\alpha_{jk}^2}\cos(j\alpha_1 x + k\alpha_2 y)\cos l\pi z, 0, 0\right)^t,$$

$$\phi_{jkl}^6 = (0,0,0,\sin(j\alpha_1 x + k\alpha_2 y)\sin l\pi z)^t,$$

$$E_{jkl}^1 = \text{span}\{\phi_{jkl}^1, \phi_{jkl}^2, \phi_{jkl}^3\}, \quad E_{jkl}^2 = \text{span}\{\phi_{jkl}^4, \phi_{jkl}^5, \phi_{jkl}^6\},$$

$$E_{jkl} = E^1_{jkl} \oplus E^2_{jkl}, \quad \beta_{\Lambda_1} = \cup_{(j,k,l)\in\Lambda_1} \cup_{q=1}^3 \{\beta_{jklq}\}.$$

It is easy to check that E^1_{jkl} and E^2_{jkl} are invariant subspaces of the linear operator L_R, i.e., $L_R(E^1_{jkl}) \subset E^1_{jkl}$ and $L_R(E^2_{jkl}) \subset E^2_{jkl}$. The characteristic polynomial of $L_R|_{E^1_{jkl}}$ ($L_R|_{E^2_{jkl}}$) is given by f_{jkl} as defined in Eq. (5.8). Since E^1_{jkl} (E^2_{jkl}) is of dimension 3, the (generalized) eigenvectors of $L_R|_{E^1_{jkl}}$, $\cup_{q=1}^3\{\psi_1^{\beta_{jklq}}\}$ ($\cup_{q=1}^3\{\psi_2^{\beta_{jklq}}\}$), form a basis of E^1_{jkl} (E^2_{jkl}), i.e., span $\times\{\cup_{q=1}^3\{\psi_1^{\beta_{jklq}}\}\} = E^1_{jkl}$ (span$\{\cup_{q=1}^3\{\psi_2^{\beta_{jklq}}\}\} = E^2_{jkl}$). If β_{jklq} is a real zero of f_{jkl}, the eigenvector corresponding to β_{jklq} in E^1_{jkl} (E^2_{jkl}) is given by

$$\psi_1^{\beta_{jklq}} = \phi^1_{jkl} + A_1(\beta_{jklq})\phi^2_{jkl} + A_2(\beta_{jklq})\phi^3_{jkl},$$

$$(\psi_2^{\beta_{jklq}} = \phi^4_{jkl} + A_1(\beta_{jklq})\phi^5_{jkl} + A_2(\beta_{jklq})\phi^6_{jkl}), \tag{5.9}$$

where

$$A_1(\beta) = \frac{-1}{\mathrm{Ro}(\beta + \sigma\gamma^2_{jkl})}, \quad A_2(\beta) = \frac{1}{\beta + \gamma^2_{jkl}}. \tag{5.10}$$

If $\beta_{jklq_1} = \bar{\beta}_{jklq_2}$ (imaginary numbers) are zeros of f_{jkl}, the (generalized) eigenvectors corresponding to β_{jklq_1} and β_{jklq_2} in E^1_{jkl} (E^2_{jkl}) are given by

$$\psi_1^{\beta_{jklq_1}} = \phi^1_{jkl} + R_1(\beta_{jklq_1})\phi^2_{jkl} + R_2(\beta_{jklq_1})\phi^3_{jkl},$$

$$\psi_1^{\beta_{jklq_2}} = I_1(\beta_{jklq_1})\phi^2_{jkl} + I_2(\beta_{jklq_1})\phi^3_{jkl}, \tag{5.11}$$

$$\begin{pmatrix} \psi_2^{\beta_{jklq_1}} = \phi^4_{jkl} + R_1(\beta_{jklq_1})\phi^5_{jkl} + R_2(\beta_{jklq_1})\phi^6_{jkl}, \\ \psi_2^{\beta_{jklq_2}} = I_1(\beta_{jklq_1})\phi^5_{jkl} + I_2(\beta_{jklq_1})\phi^6_{jkl}, \end{pmatrix},$$

where

$$R_1(\beta) = \mathrm{Re}(A_1(\beta)), \quad R_2(\beta) = \mathrm{Re}(A_2(\beta)),$$

$$I_1(\beta) = \mathrm{Im}(A_1(\beta)), \quad I_2(\beta) = \mathrm{Im}(A_2(\beta)). \tag{5.12}$$

The dual vector corresponding to $\psi_1^{\beta_{jklq}}$ ($\psi_2^{\beta_{jklq}}$) is given by

$$\Psi_1^{\beta_{jklq}} = \phi^1_{jkl} + C_1(\beta_{jklq})\phi^2_{jkl} + C_2(\beta_{jklq})\phi^3_{jkl},$$

$$(\Psi_2^{\beta_{jklq}} = \phi^4_{jkl} + C_1(\beta_{jklq})\phi^5_{jkl} + C_2(\beta_{jklq})\phi^6_{jkl}), \tag{5.13}$$

where

$$C_1(\beta) = \frac{1}{\mathrm{Ro}(\beta + \sigma\gamma^2_{jkl})}, \quad C_2(\beta) = \frac{\sigma R}{\beta + \gamma^2_{jkl}}. \tag{5.14}$$

The dual vector $\Psi_1^{\beta_{jklq}}$ ($\Psi_2^{\beta_{jklq}}$) satisfies

$$\langle \psi_1^{\beta_{jklq^*}}, \Psi_1^{\beta_{jklq}} \rangle_H = 0 \quad (\langle \psi_2^{\beta_{jklq^*}}, \Psi_2^{\beta_{jklq}} \rangle_H = 0), \tag{5.15}$$

for $q^* \neq q$.

We note that $E_{j_1k_1l_1}$ is orthogonal to $E_{j_2k_2l_2}$ for $(j_1,k_1,l_1) \neq (j_2,k_2,l_2)$ and E^1_{jkl} is orthogonal to E^2_{jkl} for $(j,k,l) \in \Lambda_1$. Hence, the dual vector $\Psi_1^{\beta_{jklq}}$ ($\Psi_2^{\beta_{jklq}}$) satisfies

$$\langle \psi, \Psi_1^{\beta_{jklq}} \rangle_H = 0 \quad \text{for } \psi \in (\cup_{(j^*,k^*,l^*)\neq(j,k,l)} E_{j^*k^*l^*}) \cup E_{jkl}^2,$$

$$(\langle \psi, \Psi_2^{\beta_{jklq}} \rangle_H = 0 \quad \text{for } \psi \in (\cup_{(j^*,k^*,l^*)\neq(j,k,l)} E_{j^*k^*l^*}) \cup E_{jkl}^1). \tag{5.16}$$

In view of the Fourier expansion, we see that $\cup_{(j,k,l)\in\Lambda} E_{jkl}$ is a basis of H_1 and $(\cup_{(j,k,l)\in\Lambda_1} E_{jkl}^1) \cup (\cup_{(j,k,0)\in\Lambda_2}\{\psi_1^{\beta_{jk0}}\}) \cup (\cup_{(0,0,l)\in\Lambda_3}\{\psi^{\beta_{00l1}}\})$ is a basis of $\widetilde{H}_1$. Hence, by the discussion above, we have the following conclusions.

(a) The set $\beta_{H_1} = \beta_{\Lambda_1} \cup \beta_{\Lambda_2} \cup \beta_{\Lambda_3}$ consists of all eigenvalues of $L_R|_{H_1}$, and the (generalized) eigenvectors of $L_R|_{H_1}$ form a basis of H_1.
(b) The set $\beta_{\widetilde{H}_1} = \beta_{\Lambda_1} \cup \beta_{\Lambda_2} \cup \beta_{\widetilde{\Lambda}_3}$ consists of all eigenvalues of $L_R|_{\widetilde{H}_1}$, and the (generalized) eigenvectors of $L_R|_{\widetilde{H}_1}$ form a basis of $\widetilde{H}_1$.
(c) $\text{Re}(\beta) < 0$ for each $\beta \in \beta_{\Lambda_2} \cup \beta_{\Lambda_3}$.

Lemma 5.1: If R is small, then $\text{Re}(\beta_{jklq}(R)) < 0$ *for each* $\beta_{jklq} \in \beta_{\Lambda_1}$.

Proof: Plugging $\beta = \gamma_{jkl}^2 \beta^*$ into f_{jkl}, we get $f_{jkl}(\beta) = \gamma_{jkl}^6 \widetilde{f}_{jkl}(\beta^*)$, where

$$\widetilde{f}_{jkl}(\beta^*) = (\beta^* + 1)(\beta^* + \sigma)^2 + \frac{l^2\pi^2}{\gamma_{jkl}^6 \text{Ro}^2}(\beta^* + 1) - \sigma R \frac{\alpha_{jk}^2}{\gamma_{jkl}^6}(\beta^* + \sigma).$$

Hence, we only need to show that the real part of each zero of $\widetilde{f}_{jkl}$ is strictly negative when R is small. We observe that $\widetilde{f}_{jkl}(\beta^*) > 0$ for all $\beta^* \geq 0$ provided $R < 1 + \sigma^{-1}$. Therefore, if all zeros of $\widetilde{f}_{jkl}$ are real numbers, we are done.

For the case where only one of the zeros of $\widetilde{f}_{jkl}$ is real, this real zero, β_1^*, is a perturbation of -1. There exists an ϵ (depending on σ only) such that $-(1+2\sigma) < \beta_1^* < 0$ provided $R < \epsilon$. This makes the real part of the other two zeros of $\widetilde{f}_{jkl}$ strictly negative and the proof is complete. □

B. Characterization of critical Rayleigh numbers

Based on the above discussion, we know that only the eigenvalues in β_{Λ_1} depend on the Rayleigh number R. Hence, to study the principle of exchange of stabilities for problem (5.1), it suffices to focus the problem on the set β_{Λ_1}. We proceed with the following two cases.

Case 1: $\beta=0$ is a zero of f_{jkl} if and only if the constant term of the polynomial f_{jkl} is 0. In this case, we have

$$R = \frac{\gamma_{jkl}^6}{\alpha_{jk}^2} + \frac{l^2\pi^2}{\sigma^2 \text{Ro}^2 \alpha_{jk}^2} \geq \frac{(\alpha_{jk}^2 + \pi^2)^3}{\alpha_{jk}^2} + \frac{\pi^2}{\sigma^2 \text{Ro}^2 \alpha_{jk}^2}. \tag{5.17}$$

Hence, the critical Rayleigh number R_{c_1} is given by

$$R_{c_1} = \min_{(j,k,l)\in\Lambda_1} \left\{ \frac{\gamma_{jkl}^6}{\alpha_{jk}^2} + \frac{l^2\pi^2}{\sigma^2 \text{Ro}^2 \alpha_{jk}^2} \right\} = \frac{\gamma_{j_1k_11}^6}{\alpha_{j_1k_1}^2} + \frac{\pi^2}{\sigma^2 \text{Ro}^2 \alpha_{j_1k_1}^2}, \tag{5.18}$$

for some $(j_1, k_1, 1) \in \Lambda_1$.

Case 2: A careful analysis on Eq. (5.7) shows that $\beta = ai$ $(a \neq 0)$, a purely imaginary number, is a zero of f_{jkl} if and only if the following two equations hold true:

$$(\sigma^2 + 2\sigma)\gamma_{jkl}^4 + \frac{l^2\pi^2}{\text{Ro}^2 \gamma_{jkl}^2} - \sigma R \frac{\alpha_{jk}^2}{\gamma_{jkl}^2} > 0,$$

$$(2\sigma+1)\gamma_{jkl}^2\left[(\sigma^2+2\sigma)\gamma_{jkl}^4+\frac{l^2\pi^2}{\mathrm{Ro}^2\gamma_{jkl}^2}-\sigma R\frac{\alpha_{jk}^2}{\gamma_{jkl}^2}\right]=\sigma^2\gamma_{jkl}^6-\sigma^2R\alpha_{jk}^2+\frac{l^2\pi^2}{\mathrm{Ro}^2}.$$

In this case, we have

$$R=\frac{2(\sigma+1)\gamma_{jkl}^6}{\alpha_{jk}^2}+\frac{2l^2\pi^2}{(\sigma+1)\mathrm{Ro}^2\alpha_{jk}^2}, \tag{5.19}$$

$$R<\frac{(\sigma+2)\gamma_{jkl}^6}{\alpha_{jk}^2}+\frac{l^2\pi^2}{\sigma\mathrm{Ro}^2\alpha_{jk}^2}. \tag{5.20}$$

Plugging Eq. (5.20) into Eq. (5.19), we derive an upper bound for Ro^2,

$$\mathrm{Ro}^2<\frac{(1-\sigma)l^2\pi^2}{\sigma^2(1+\sigma)\gamma_{jkl}^6}, \tag{5.21}$$

which could only hold true when $\sigma<1$.

As in Case 1, the minimum of the right hand side of Eq. (5.19) is always obtained at $l=1$. Hence, the critical Rayleigh number R_{c_2} is given by

$$R_{c_2}=\min_{(j,k,l)\in\Lambda_1}\left\{\frac{2(\sigma+1)\gamma_{jkl}^6}{\alpha_{jk}^2}+\frac{2l^2\pi^2}{(\sigma+1)\mathrm{Ro}^2\alpha_{jk}^2}\right\}=\frac{2(\sigma+1)\gamma_{j_2k_21}^6}{\alpha_{j_2k_2}^2}+\frac{2\pi^2}{(\sigma+1)\mathrm{Ro}^2\alpha_{j_2k_2}^2}, \tag{5.22}$$

for some $(j_2,k_2,1)\in\Lambda_1$. In the case of $\sigma<1$, Eq. (5.21) with $l=1$ implies R_{c_2} is smaller than R_{c_1}. Hence, for problems (2.1)–(2.5), R_{c_1} is the first critical Rayleigh number if $\sigma>1$ and R_{c_2} is the first critical Rayleigh number if $\sigma<1$. Therefore, the principle of exchange of stabilities is given by Lemmas 5.2 and 5.5.

Lemma 5.2: For fixed $\sigma>1$ *and* $\mathrm{Ro}>0$, *suppose that* $(\alpha_{jk}^2,l)=(\alpha_{j_1k_1}^2,1)$ *minimizes the right hand side of Eq. (5.17), then*

$$\beta_{j_1k_111}(R)\begin{cases}<0 & \text{if } R<R_{c_1}\\ =0 & \text{if } R=R_{c_1}\\ >0 & \text{if } R>R_{c_1},\end{cases} \tag{5.23}$$

$$\mathrm{Re}\,\beta_{jklq}(R)<0 \quad \text{for } (\alpha_{jk}^2,l)\neq(\alpha_{j_1k_1}^2,1),\quad q=1,2,3,\quad R \text{ near } R_{c_1}. \tag{5.24}$$

Proof: By the above discussion, we only need to show that the first eigenvalue crosses the imaginary axis. We note that $f_{j_1k_11}(\beta)=0$ is equivalent to $g_{j_1k_11}(\beta)=h_{j_1k_11}(\beta)$, i.e.,

$$(\beta+\gamma_{j_1k_11}^2)[(\beta+\sigma\gamma_{j_1k_11}^2)^2+l^2\pi^2\mathrm{Ro}^{-2}\gamma_{j_1k_11}^{-2}]=\sigma R\alpha_{j_1k_1}^2\gamma_{j_1k_11}^{-2}(\beta+\sigma\gamma_{j_1k_11}^2). \tag{5.25}$$

We see that both $g_{j_1k_11}$ and $h_{j_1k_11}$ are strictly increasing for $\beta>-\gamma_{j_1k_11}^2$ (since $\sigma>1$). Let Γ_1 be the graph of $\eta=g_{j_1k_11}(\beta)$ and Γ_2 be the graph of $\eta=h_{j_1k_11}(\beta)$, as shown in Fig. 2. When $R=R_{c_1}$, point S_0, the intersecting point of Γ_1 and Γ_2 corresponding to $\beta_{j_1k_11}(R)$ [i.e., the β coordinate of S_0 is $\beta_{j_1k_11}(R)$], is on the η axis. When R increases (decreases), S_0 becomes S_1 (S_2). This proves Eq. (5.23) and the proof is complete. □

Remark 5.1:

(1) In the proof of Lemma 5.2, as shown by Eq. (5.25) and Fig. 2, we see that, for $R\approx R_{c_1}$, the first eigenvalue $\beta_{j_1k_111}$ is a simple zero of $f_{j_1k_11}(\beta)$. We have seen in Sec. V A that there are eigenvectors $\psi_1^{\beta_{j_1k_111}}\in E_{j_1k_1l}^1$ and $\psi_2^{\beta_{j_1k_111}}\in E_{j_1k_1l}^2$ corresponding to $\beta_{j_1k_111}$. Therefore, the

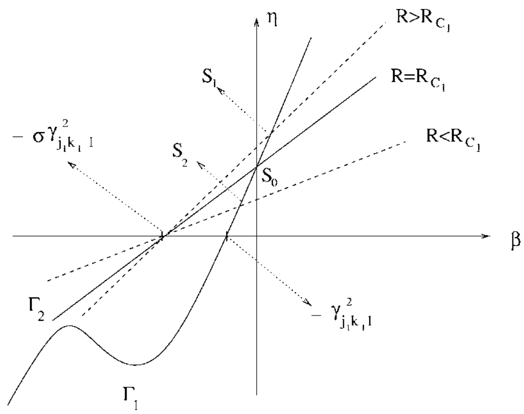

FIG. 2. Exchange of stability regions.

multiplicity of the first eigenvalue of $L|_{H_1}$ ($L|_{\tilde{H}_1}$) is $m_{H_1}=2m$ ($m_{\tilde{H}_1}=m$), where m is the number of $(j,k,1)$'s ($\in \Lambda_1$) satisfying $\alpha^2_{jk}=\alpha^2_{j_1k_1}$. Hence, condition (3.6) guarantees that, for $R\approx R_{c_1}$, the first eigenvalue of $L_R|_{H_1}$ ($L_R|_{\tilde{H}_1}$) is real and of multiplicity 2 (1).

(2) For the classical Bénard problem without rotation, the second term on the right hand side of Eq. (5.17), hence the second term on the right hand side of Eq. (5.18), is not presented. Therefore, the first critical Rayleigh number of the classical Bénard problem depends only on the aspect of ratio, while the first critical Rayleigh number of the rotating problem depends on the aspect of ratio, the Prandtl number, and the Rossby number. It is clear that the first critical Rayleigh number of fast rotating flows is remarkably larger than the first critical Rayleigh number of the classical Bénard problem. This indicates that the rotating flows are much more stable than the nonrotating flows.

(3) R_{c_1} is the first critical Rayleigh number if the Prandtl number is greater than 1. For the case where the Prandtl number is smaller than 1, R_{c_2} is the first critical Rayleigh number and, in general, there are a few critical values between R_{c_2} and R_{c_1}.

For $x>0$, $b\geqslant 0$, we define

$$f_b(x) = \frac{(x+\pi^2)^3 + b}{x}. \tag{5.26}$$

Let $x=\alpha^2_{jk}$, then the right hand side of Eq. (5.18) could be expressed as $f_{b_1}(x)$, where $b_1=\pi^2/\sigma^2\mathrm{Ro}^2$, and the second line of Eq. (5.22) could be expressed as $2(\sigma+1)f_{b_2}(x)$, where $b_2=\pi^2/(\sigma+1)^2\mathrm{Ro}^2$. Consider

$$f'_b(x) = \frac{(2x-\pi^2)(x+\pi^2)^2 - b}{x^2}. \tag{5.27}$$

As shown in Fig. 3, it is easy to see that

(a) for $x\in(0,\infty)$, $f_b(x)$ has only one critical number x_b;
(b) $f'_b(x)<0$ if $x<x_b$;
(c) $f'_b(x)>0$ if $x>x_b$;
(d) $f_b(x_b)$ is the global minimum of $f_b(x)$; and
(e) x_b is strictly increasing in b, hence, $x_{b_1}>x_{b_2}>\pi^2/2$.

In Lemmas 5.3 and 5.4, we consider the following different conditions:

$$x_{b_1} \leqslant \alpha_1^2 < \alpha_2^2, \tag{5.28}$$

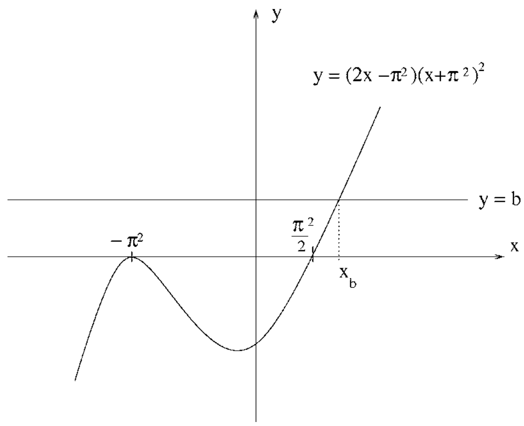

FIG. 3. Schematic showing properties of the function in (5.27).

$$\alpha_1^2 \leqslant \frac{1}{5}x_{b_1} < 2x_{b_1} < \alpha_2^2, \tag{5.29}$$

$$x_{b_2} \leqslant \alpha_1^2 < \alpha_2^2, \tag{5.30}$$

$$\alpha_1^2 \leqslant \frac{1}{5}x_{b_2} < 2x_{b_2} < \alpha_2^2. \tag{5.31}$$

Lemma 5.3:

(1) *Condition (3.6) holds true under the assumption (5.28).*
(2) *Generically, condition (3.6) holds true under the assumption (5.29).*

Proof:

(1) Under the assumption (5.28), by (c), we conclude that R_{c_1} is only obtained at $(j,k,l) = (1,0,1)$, i.e., $j_1 = 1$.
(2) Under the assumption (5.29), there exists $j^* \geqslant 2$ such that $j^{*2}\alpha_1^2 \leqslant x_{b_1} < (j^*+1)^2\alpha_1^2$. We note that

$$(j^*+1)^2\alpha_1^2 \begin{cases} < 2j^{*2}\alpha_1^2 < 2x_{b_1} < \alpha_2^2 & \text{if } j^* \geqslant 3, \\ = 9\alpha_1^2 < \frac{9}{5}x_{b_1} < 2x_{b_1} < \alpha_2^2 & \text{if } j^* = 2. \end{cases}$$

Hence, by (b) and (c), we conclude that

$$R_{c_1} = \min\{f_{b_1}(j^{*2}\alpha_1^2), f_{b_1}((j^*+1)^2\alpha_1^2)\},$$

i.e., $j_1 = j^*$ or $j_1 = j^*+1$. Note that, by (b) and (c), generically $f_{b_1}(j^{*2}\alpha_1^2) \neq f_{b_1}((j^*+1)^2\alpha_1^2)$. The proof is complete. □

Lemma 5.4:

(1) *Condition (3.7) holds true under the assumption (5.30).*
(2) *Generically, condition (3.7) holds true under the assumption (5.31).*

Proof: Consider

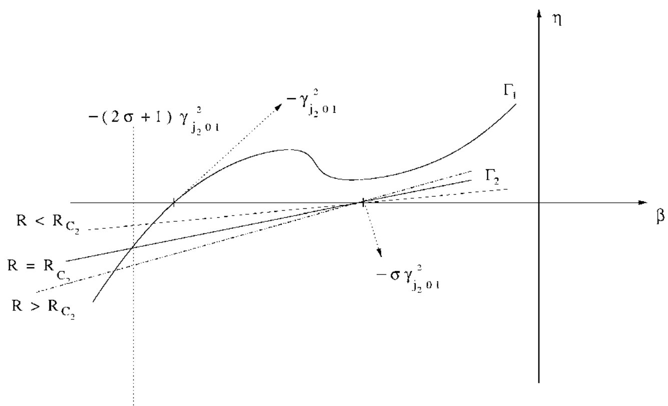

FIG. 4. Schematic illustrating the proof of (5.34).

$$R_{c_2} = \min_{(j,k,1)\in\Lambda_1} \{2(\sigma+1) f_{b_2}(\alpha^2_{jk})\}.$$

The rest part of the proof is the same as the proof of Lemma 5.3. □

Lemma 5.5: Assume Eq. (3.7), $R \approx R_{c_2}$, and Ro^2 satisfies Eq. (5.21) for $(j,k,l)=(j_2,0,1)$, i.e., $\mathrm{Ro}^2 < (1-\sigma)\pi^2/\sigma^2(1+\sigma)\gamma^6_{j_2 01}$, then $\{\beta_{j_2 011}(R), \beta_{j_2 012}(R)\}$ $(\beta_{j_2 011}(R) = \bar{\beta}_{j_2 012}(R))$ is the only simple pair of complex eigenvalues of the problem (5.1) in space $\tilde{H}_1$ satisfying

$$\mathrm{Re}(\beta_{j_2 011}(R)) \begin{cases} <0 & \text{if } R < R_{c_2}, \\ =0 & \text{if } R = R_{c_2}, \\ >0 & \text{if } R > R_{c_2}, \end{cases} \tag{5.32}$$

$$\mathrm{Re}\, \beta_{jklq}(R) < 0 \quad \text{for } (\alpha^2_{jk}, l) \neq (\alpha^2_{j_2 0}, 1),\ q = 1,2,3,\ R \text{ near } R_{c_2}. \tag{5.33}$$

Proof: We only need to prove Eq. (5.32). Under the assumptions of the lemma together with Eqs. (5.11), (5.19), and (5.20), by the discussion in case (2) at the beginning of this section, we know that $\{\beta_{j_2 011}(R), \beta_{j_2 012}(R)\}$ is the only simple pair of complex eigenvalues of $L_R|_{\tilde{H}_1}$ with $\mathrm{Re}(\beta_{j_2 011}(R_{c_2})) = \mathrm{Re}(\beta_{j_2 012}(R_{c_2})) = 0$. Since $\beta_{j_2 013}(R)$ (real), $\beta_{j_2 011}(R)$, and $\beta_{j_2 012}(R)$ are zeros of $f_{j_2 01}$, we know that

$$\beta_{j_2 013}(R) = -(\mathrm{Re}(\beta_{j_2 011}(R)) + \mathrm{Re}(\beta_{j_2 012}(R))) - (2\sigma+1)\gamma^2_{j_2 01}.$$

Hence, Eq. (5.32) is equivalent to

$$\beta_{j_2 013}(R) \begin{cases} > -(2\sigma+1)\gamma^2_{j_2 01} & \text{if } R < R_{c_2}, \\ = -(2\sigma+1)\gamma^2_{j_2 01} & \text{if } R = R_{c_2}, \\ < -(2\sigma+1)\gamma^2_{j_2 01} & \text{if } R > R_{c_2}, \end{cases} \tag{5.34}$$

which is true as shown in Fig. 4. This completes the proof. □

Lemma 5.6: For fixed α_1, $\alpha_2 > 0$, and $\sigma > 1$, $R_{c_1} \to \infty$ as $\mathrm{Ro} \to 0$. More precisely, $R_{c_1} = O(\mathrm{Ro}^{-4/3})$.

Proof: Since $b_1 = \pi^2/\sigma^2 \mathrm{Ro}^2$, by Eq. (5.27), $x_{b_1} = O(b_1^{1/3})$ as $\mathrm{Ro} \to 0$. Hence,

$$R_{c_1} = O(f_{b_1}(x_{b_1})) = O(b_1^{2/3}) = O(\mathrm{Ro}^{-4/3}).$$

□

VI. PROOF OF MAIN THEOREMS

A. Center manifold reduction

We are now in a position to reduce Eqs. (2.1)–(2.5) to the center manifold. For any $\psi=(U,T)\in H_1$, we have

$$\psi=\sum_{(j,k,l)\in\Lambda_1}^{\infty}\sum_{q=1}^{3}(x_{jklq}\psi_1^{\beta_{jklq}}+y_{jklq}\psi_2^{\beta_{jklq}})+\sum_{(j,k,0)\in\Lambda_2}(x_{jk0}\psi_1^{\beta_{jk0}}+y_{jk0}\psi_2^{\beta_{jk0}})+\sum_{l=1}^{\infty}\sum_{q=1}^{3}x_{00lq}\psi^{\beta_{00lq}}.$$

Under assumption (3.6), the first critical Rayleigh number is given by

$$R_{c_1}=\frac{\gamma_{j_101}^6}{\alpha_{j_10}^2}+\frac{\pi^2}{\sigma^2\mathrm{Ro}^2\alpha_{j_10}^2}. \tag{6.1}$$

In this case, the multiplicity of the first eigenvalue is 2 and the reduced Eqs. (2.1)–(2.5) are given by

$$\frac{\mathrm{d}x_{j_1011}}{\mathrm{d}t}=\beta_{j_1011}(R)x_{j_1011}+\frac{1}{\langle\psi_1^{\beta_{j_1011}},\Psi_1^{\beta_{j_1011}}\rangle_H}\langle G(\psi,\psi),\Psi_1^{\beta_{j_1011}}\rangle_H,$$

$$\frac{\mathrm{d}y_{j_1011}}{\mathrm{d}t}=\beta_{j_1011}(R)y_{j_1011}+\frac{1}{\langle\psi_2^{\beta_{j_1011}},\Psi_2^{\beta_{j_1011}}\rangle_H}\langle G(\psi,\psi),\Psi_2^{\beta_{j_1011}}\rangle_H. \tag{6.2}$$

Here for $\psi_1=(U_1,T_1)$, $\psi_2=(U_2,T_2)$, and $\psi_3=(U_3,T_3)$,

$$G(\psi_1,\psi_2)=-(P(U_1\cdot\nabla)U_2,(U_1\cdot\nabla)T_2)^t$$

and

$$\langle G(\psi_1,\psi_2),\psi_3\rangle_H=-\int_0^1\int_0^{2\pi/\alpha_2}\int_0^{2\pi/\alpha_1}[\langle(U_1\cdot\nabla)U_2,U_3\rangle_{\mathbb{R}^3}+(U_1\cdot\nabla)T_2T_3]\mathrm{d}x\mathrm{d}y\mathrm{d}z,$$

where P is the Leray projection to L^2 fields. Let the center manifold function be denoted by

$$\Phi=\sum_{\beta\neq\beta_{j_1011}}(\Phi_1^{\beta}(x_{j_1011},y_{j_1011})\psi_1^{\beta}+\Phi_2^{\beta}(x_{j_1011},y_{j_1011})\psi_2^{\beta}). \tag{6.3}$$

The direct calculation shows that

$$G(\psi_1^{\beta_{j_1011}},\psi_1^{\beta_{j_1011}})=-\left(0,\frac{A_1\pi^2}{2j_1\alpha_1}\sin 2j_1\alpha_1x,0,\frac{A_2\pi}{2}\sin 2\pi z\right)^t,$$

$$G(\psi_1^{\beta_{j_1011}},\psi_2^{\beta_{j_1011}})=-\left(\frac{\pi^2}{2j_1\alpha_1}\cos 2\pi z,\frac{A_1\pi^2}{2j_1\alpha_1}(\cos 2\pi z-\cos 2j_1\alpha_1x),0,0\right)^t,$$

$$G(\psi_2^{\beta_{j_1011}},\psi_1^{\beta_{j_1011}})=-\left(\frac{-\pi^2}{2j_1\alpha_1}\cos 2\pi z,\frac{-A_1\pi^2}{2j_1\alpha_1}(\cos 2j_1\alpha_1x+\cos 2\pi z),0,0\right)^t, \tag{6.4}$$

$$G(\psi_2^{\beta_{j_1011}},\psi_2^{\beta_{j_1011}})=-\left(0,\frac{-A_1\pi^2}{2j_1\alpha_1}\sin 2j_1\alpha_1x,0,\frac{A_2\pi}{2}\sin 2\pi z\right)^t.$$

$$G(\psi_1^{\beta_{j_1011}},\Psi_1^{\beta_{j_1011}}) = -\left(0,\frac{C_1\pi^2}{2j_1\alpha_1}\sin 2j_1\alpha_1 x,0,\frac{C_2\pi}{2}\sin 2\pi z\right)^t,$$

$$G(\psi_1^{\beta_{j_1011}},\Psi_2^{\beta_{j_1011}}) = -\left(\frac{\pi^2}{2j_1\alpha_1}\cos 2\pi z,\frac{C_1\pi^2}{2j_1\alpha_1}(\cos 2\pi z-\cos 2j_1\alpha_1 x),0,0\right)^t,$$

$$G(\psi_2^{\beta_{j_1011}},\Psi_1^{\beta_{j_1011}}) = -\left(\frac{-\pi^2}{2j_1\alpha_1}\cos 2\pi z,\frac{-C_1\pi^2}{2j_1\alpha_1}(\cos 2j_1\alpha_1 x+\cos 2\pi z),0,0\right)^t, \tag{6.5}$$

$$G(\psi_2^{\beta_{j_1011}},\Psi_2^{\beta_{j_1011}}) = -\left(0,\frac{-C_1\pi^2}{2j_1\alpha_1}\sin 2j_1\alpha_1 x,0,\frac{C_2\pi}{2}\sin 2\pi z\right)^t,$$

where $A_1=A_1(\beta_{j_1011})$, $A_2=A_2(\beta_{j_1011})$, $C_1=C_1(\beta_{j_1011})$, and $C_2=C_2(\beta_{j_1011})$.

Hereafter, we make the following convention:

$$o(2)=o(x_{j_1011}^2+y_{j_1011}^2)+O(|\beta_{j_1011}(R)|\cdot(x_{j_1011}^2+y_{j_1011}^2)),$$

$$o(3)=o((x_{j_1011}^2+y_{j_1011}^2)^{3/2})+O(|\beta_{j_1011}(R)|\cdot(x_{j_1011}^2+y_{j_1011}^2)^{3/2}),$$

$$o(4)=o((x_{j_1011}^2+y_{j_1011}^2)^2)+O(|\beta_{j_1011}(R)|\cdot(x_{j_1011}^2+y_{j_1011}^2)^2).$$

By Theorem 4.2 and Eqs. (6.4) and (6.5), we obtain

$$\Phi=\Phi_1^{\beta_{(2j_1)00}}\psi_1^{\beta_{(2j_1)00}}+\Phi_2^{\beta_{(2j_1)00}}\psi_2^{\beta_{(2j_1)00}}+\Phi_1^{\beta_{0021}}\psi_1^{\beta_{0021}}+o(2), \tag{6.6}$$

where

$$\Phi_1^{\beta_{(2j_1)00}}=\frac{A_1\pi^2}{\sigma\alpha_{(2j_1)0}^4}(x_{j_1011}^2-y_{j_1011}^2)+o(2),\qquad \psi_1^{\beta_{(2j_1)00}}=(0,-2j_1\alpha_1\sin 2j_1\alpha_1 x,0,0)^t,$$

$$\Phi_2^{\beta_{(2j_1)00}}=\frac{A_1\pi^2}{\sigma\alpha_{(2j_1)0}^4}(2x_{1011}y_{1011})+o(2),\qquad \psi_2^{\beta_{(2j_1)00}}=(0,2j_1\alpha_1\cos 2\alpha_1 x,0,0)^t,$$

$$\Phi_1^{\beta_{0021}}=\frac{-A_2}{8\pi}(x_{j_1011}^2+y_{j_1011}^2)+o(2),\qquad \psi_1^{\beta_{0021}}=(0,0,0,\sin 2\pi z)^t.$$

Note that for any $\psi_i\in H_1$ $(i=1,2,3)$,

$$\langle G(\psi_1,\psi_2),\psi_2\rangle_H=0, \tag{6.7}$$

$$\langle G(\psi_1,\psi_2),\psi_3\rangle_H=-\langle G(\psi_1,\psi_3),\psi_2\rangle_H, \tag{6.8}$$

and for any $\psi_i\in E_{jkl}$ $(i=1,2,3)$,

$$\langle G(\psi_1,\psi_2),\psi_3\rangle_H=0. \tag{6.9}$$

The direct calculation shows that

$$G(\tilde{\psi},\psi_i^{\beta_{j_1011}})=0\quad \text{for } \tilde{\psi}\in\{\psi_1^{\beta_{(2j_1)00}},\psi_2^{\beta_{(2j_1)00}},\psi_1^{\beta_{0021}}\},\ i=1,2. \tag{6.10}$$

Then, by $\psi=x_{j_1011}\psi_1^{\beta_{j_1011}}+y_{j_1011}\psi_2^{\beta_{j_1011}}+\Phi(x_{j_1011},y_{j_1011})$ and Eqs. (6.4) and (6.10), we derive that

$$\langle G(\psi,\psi),\Psi_1^{\beta_{j_1011}}\rangle_H=\langle G(\psi_1^{\beta_{j_1011}},\Phi),\Psi_1^{\beta_{j_1011}}\rangle_H x_{j_1011}+\langle G(\psi_2^{\beta_{j_1011}},\Phi),\Psi_1^{\beta_{j_1011}}\rangle_H y_{j_1011}+o(3)$$
$$=-\langle G(\psi_1^{\beta_{j_1011}},\Psi_1^{\beta_{j_1011}}),\Phi\rangle_H x_{j_1011}-\langle G(\psi_2^{\beta_{j_1011}},\Psi_1^{\beta_{j_1011}}),\Phi\rangle_H y_{j_1011}+o(3)$$
$$=-\frac{2A_1C_1\pi^6}{\sigma\alpha_1\alpha_2\sigma^4_{(2j_1)0}}(x^2_{j_1011}-y^2_{j_1011})x_{j_1011}-\frac{A_2C_2\pi^2}{8\alpha_1\alpha_2}(x^2_{j_1011}+y^2_{j_1011})x_{j_1011}+o(3),$$

$$-\frac{2A_1C_1\pi^6}{\sigma\alpha_1\alpha_2\alpha^4_{(2j_1)0}}(2x_{j_1011}y_{j_1011})y_{j_1011}=-\left(\frac{2A_1C_1\pi^6}{\sigma\alpha_1\alpha_2\alpha^4_{(2j_1)0}}+\frac{A_2C_2\pi^2}{8\alpha_1\alpha_2}\right)(x^2_{j_1011}+y^2_{j_1011})x_{j_1011}+o(3).$$

Similarly, we obtain

$$\langle G(\psi,\psi),\Psi_2^{\beta_{j_1011}}\rangle_H=-\left(\frac{2A_1C_1\pi^6}{\sigma\alpha_1\alpha_2\alpha^4_{(2j_1)0}}+\frac{A_2C_2\pi^2}{8\alpha_1\alpha_2}\right)(x^2_{j_1011}+y^2_{j_1011})y_{j_1011}+o(3).$$

Hence, the reduction equations are given by

$$\frac{\mathrm{d}x_{j_1011}}{\mathrm{d}t}=\beta_{j_1011}(R)x_{j_1011}+\delta(x^2_{j_1011}+y^2_{j_1011})x_{j_1011}+o(3),$$

$$\frac{\mathrm{d}y_{j_1011}}{\mathrm{d}t}=\beta_{j_1011}(R)y_{j_1011}+\delta(x^2_{j_1011}+y^2_{j_1011})y_{j_1011}+o(3), \tag{6.11}$$

where

$$\delta=-\left(\frac{2A_1C_1\pi^4}{\sigma\alpha^4_{(2j_1)0}}+\frac{A_2C_2}{8}\right)\Big/\left(\frac{\pi^2}{j_1^2\alpha_1^2}(1+A_1C_1)+1+A_2C_2\right)<0. \tag{6.12}$$

A standard energy estimate on Eqs. (6.11) together with the center manifold theory shows that, for $R\leq R_{c_1}$, $(U,T)=0$ is locally asymptotically stable for problems (2.1)–(2.5). Hence, by Theorem 4.1, the solutions to Eqs. (2.1)–(2.5) bifurcate from $(U,T,R)=(0,R_{c_1})$ to an attractor Σ_R. Moreover, by Eqs. (6.11) and (6.12) together with Theorem 5.10 in Ref. 18, we conclude that Σ_R is homeomorphic to S^1 in H.

B. Completion of the proof of Theorem 3.1

In this section, we prove that Σ_R consists of steady state solutions. It is clear that the first eigenvalue of $L_R|_{\tilde{H}_1}$ is simple for $R\approx R_{c_1}$. By the Kransnoselski bifurcation theorem (see among others Chow and Hale[4] and Nirenberg[21]), when R crosses R_{c_1}, the equations bifurcate from the basic solution to a steady state solution in $\tilde{H}$. Therefore, the attractor Σ_R contains at least one steady state solution. Second, it is easy to check that Eqs. (2.1)–(2.5) defined in H are translation invariant in the x direction. Hence, if $\psi_0(x,y,z)=(U(x,y,z),T(x,y,z))$ is a steady state solution, then $\psi_0(x+\rho,y,z)$ are steady state solutions as well. By the periodic condition in the x direction, the set

$$S_{\psi_0}=\{\psi_0(x+\rho,y,z)|\rho\in\mathbb{R}\}$$

is a cycle homeomorphic to S^1 in H. Therefore the steady state of Eqs. (2.1)–(2.5) generates a cycle of steady state solutions. Hence, the bifurcated attractor Σ_R consists of steady state solutions. The proof of Theorem 3.1 is complete.

C. Proof of Theorem 3.2

The proof follows directly from the classical Hopf bifurcation theorem and Lemma 5.5.

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[1] Cessi, P. and Ierley, G. R., "Symmetry-breaking multiple equilibria in quasi-geostrophic, wind-driven flows," J. Phys. Oceanogr. **25**, 1196–1202 (1995).
[2] Charney, J. and DeVore, J., "Multiple flow equilibria in the atmosphere and blocking," J. Atmos. Sci. **36**, 1205–1216 (1979).
[3] Chen, Z.-M., Ghil, M., Simonnet, E., and Wang, S., "Hopf bifurcation in quasi-geostrophic channel flow," SIAM J. Appl. Math. **64**, 343–368 (2003).
[4] Chow, S. N., and Hale, J. K., *Methods of bifurcation theory*, Grundlehren der Mathematischen Wissenschaften Vol. 251 (Springer-Verlag, New York, 1982).
[5] Dijkstra, H. A., *Nonlinear Physical Oceanography: A Dynamical Systems Approach to the Large-Scale Ocean Circulation and El Niño* (Kluwer, Dordrecht, 2000).
[6] Ghil, M. and Childress, S., *Topics in Geophysical Fluid Dynamics: Atmospheric Dynamics, Dynamo Theory, and Climate Dynamics* (Springer-Verlag, New York, 1987).
[7] Henry, D., *Geometric Theory of Semilinear Parabolic Equations*, Lecture Notes in Mathematics Vol. 840 (Springer-Verlag, Berlin, 1981).
[8] Ierley, G., and Sheremet, V. A., "Multiple solutions and advection-dominated flows in the wind-driven circulation. I: Slip," J. Mar. Res. **53**, 703–737 (1995).
[9] Jiang, S., Jin, F.-F., and Ghil, M., "Multiple equilibria, periodic, and aperiodic solutions in a wind-driven, double-gyre, shallow-water model," J. Phys. Oceanogr. **25**, 764–786 (1995).
[10] Jin, F. F., and Ghil, M., "Intraseasonal oscillations in the extratropics: Hopf bifurcation and topographic instabilities," J. Atmos. Sci. **47**, 3007–3022 (1990).
[11] Legras, B., and Ghil, M., "Persistent anomalies, blocking and variations in atmospheric predictability," J. Atmos. Sci. **42**, 433–471 (1985).
[12] Lions, J. L., Temam, R., and Wang, S., "New formulations of the primitive equations of the atmosphere and applications," Nonlinearity **5**, 237–288 (1992).
[13] Lions, J. L., Temam, R., and Wang, S., "On the equations of large-scale ocean," Nonlinearity **5**, 1007–1053 (1992).
[14] Lorenz, E. N., "Deterministic nonperiodic flow," J. Atmos. Sci. **20**, 130–141 (1963).
[15] Lorenz, E. N., "The mechanics of vacillation," J. Atmos. Sci. **20**, 448–464 (1963).
[16] Ma, T., and Wang, S., *Bifurcation Theory and Applications*, World Scientific Series on Nonlinear Science Vol. 53 (World Scientific, Singapore, 2005).
[17] Ma, T., and Wang, S., "Dynamic bifurcation of nonlinear evolution equations and applications," Chin. Ann. Math., Ser. B **26**, 185–206 (2005).
[18] Ma, T., and Wang, S., *Geometric Theory of Incompressible Flows with Applications to Fluid Dynamics*, Mathematical Surveys and Monographs Vol. 119 (American Mathematical Society, Providence, RI, 2005).
[19] Meacham, S., and Berloff, P., "Instability of a steady, barotropic, wind driven circulation," J. Mar. Res. **55**, 885–913 (1997).
[20] Meacham, S., and Berloff, P., "On the stability of the wind-driven circulation," J. Mar. Res. **56**, 937–993 (1998).
[21] Nirenberg, L., *Topics in Nonlinear Functional Analysis* (Courant Institute of Mathematical Sciences, New York University, New York, 1974), With a chapter by E. Zehnder, Notes by R. A. Artino, Lecture Notes, 1973–1974.
[22] Pedlosky, J., *Geophysical Fluid Dynamics*, 2nd ed. (Springer-Verlag, New York, 1987).
[23] Pedlosky, J., "Resonant topographic waves in barotropic and baroclinic flows," J. Atmos. Sci. **38**, 2626–2641 (1981).
[24] Speich, S., Dijkstra, H., and Ghil, M., "Successive bifurcations in a shallow-water model, applied to the wind-driven ocean circulation," Nonlinear Processes Geophys. **2**, 241–268 (1995).
[25] Stommel, H., "Thermohaline convection with two stable regimes of flow," Tellus **13**, 224–230 (1961).
[26] Temam, R., *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Applied Mathematical Sciences Vol. 68, 2nd ed. (Springer-Verlag, New York, 1997).
[27] Veronis, G., "An analysis of wind-driven ocean circulation with limited Fourier components," J. Atmos. Sci. **20**, 577–593 (1963).
[28] Veronis, G., "Wind-driven ocean circulation, Part II: Numerical solution of the nonlinear problem," Deep-Sea Res. **13**, 31–55 (1966).