The main objective of this article is to study the dynamics of the stratified rotating Boussinesq equations, which are a basic model in geophysical fluid dynamics. First, for the case where the Prandtl number is greater than 1, a complete stability and bifurcation analysis near the first critical Rayleigh number is carried out. Second, for the case where the Prandtl number is smaller than 1, the onset of the Hopf bifurcation near the first critical Rayleigh number is established, leading to the existence of nontrivial periodic solutions. The analysis is based on a newly developed bifurcation and stability theory for nonlinear dynamical systems (both finite and infinite dimensional) by two of the authors [T. Ma and S. Wang, Bifurcation Theory and Applications, World Scientific Series on Nonlinear Sciences Vol. 53 (World Scientific, Singapore, 2005)]. © 2007 American Institute of Physics. [DOI: 10.1063/1.2710350]

I. INTRODUCTION

The phenomena of the atmosphere and oceans are extremely rich in their organization and complexity, and a lot of them cannot be produced by experiments. These phenomena involve a broad range of temporal and spatial scales. As we know, both the atmospheric and oceanic flows are flows under the rotation of the earth. In fact, fast rotation and small aspect ratio are two main characteristics of the large-scale atmospheric and oceanic flows. The small aspect ratio characteristic leads to the primitive equations, and the fast rotation leads to the quasigeostrophic equations. These are fundamental equations in the study of atmospheric and oceanic flows; see Ghil and Childress,6 Lions et al.,12,13 and Pedlosky.22 Furthermore, convection occurs in many regimes of the atmospheric and oceanic flows.

Key problems in the study of climate dynamics and in geophysical fluid dynamics are the understanding and the prediction of the periodic, quasiperiodic, aperiodic, and fully turbulent characteristics of large-scale atmospheric and oceanic flows. Stability/bifurcation theory enables one to determine how different flow regimes appear and disappear as control parameters, such as the Reynolds number, vary. It, therefore, provides one with a powerful tool to explore the theoretical capability in the predictability problem. Most studies so far have only considered systems of ordinary differential equations (ODEs) that are obtained by projecting the partial differential equations (PDEs) onto a finite-dimensional solution space, either by finite differencing or by...
truncating a Fourier expansion (see Ghil and Childress, and further references therein). These were pioneered by Lorenz, Stommel, and Veronis among others, who explored the bifurcation structure of low-order models of atmospheric and oceanic flows. More recently, pseudo-arclength continuation methods have been applied to atmospheric (Legras and Ghil) and oceanic (Speich et al. and Dijkstra) models with increasing horizontal resolution. These numerical bifurcation studies have produced so far fairly reliable results for two classes of geophysical flows: (i) atmospheric flows in a periodic midlatitude channel, in the presence of bottom topography and a forcing jet, and (ii) oceanic flows in a rectangular midlatitude basin, subject to wind stress on its upper surface; see among others Charney and DeVore, Pedlosky, Legras and Ghil, and Jin and Ghil for saddle-node and Hopf bifurcations in the atmospheric channel, and Refs. for saddle-node, pitchfork, or Hopf bifurcations in the oceanic basin.

The main objective of this article is to conduct bifurcation and stability analysis for the original PDEs that govern geophysical flows. This approach should allow us to overcome some of the inherent limitations of the numerical bifurcation results that dominate the climate dynamics literature up to this point, and to capture the essential dynamics of the governing PDE systems.

The present article addresses the stability and transitions of basic flows for the stratified rotating Boussinesq equations. These equations are fundamental equations in the geophysical fluid dynamics; see among others Pedlosky. We obtain two main results in this article. The first is to conduct a rigorous and complete bifurcation and stability analysis near the first eigenvalue of the linearized problem. The second is the onset of the Hopf bifurcation, leading to the existence of periodic solutions of the model.

The detailed analysis is carried out in two steps. The first is a detailed study of the eigenvalue problem for the linearized problem around the basic state. In comparison to the classical Bénard convection problem, the linearized problem here is non-self-adjoint, leading to much more complicated spectrum, and more complicated dynamics. We derive, in particular, two critical Rayleigh numbers and . Here is the first critical Rayleigh number for the case where the Prandtl number is greater than 1, and is the first critical Rayleigh number for the case where the Prandtl number is less than 1. Moreover, leads to the onset of the steady state bifurcation, while leads to the onset of the Hopf bifurcation. Both parameters are explicitly given in terms of the physical parameters. The crucial issues here include (1) a complete understanding of the spectrum, (2) identification of the critical Rayleigh numbers, and most importantly (3) the verification of the principle of exchange of stabilities near these critical Rayleigh numbers.

The second step is to conduct a rigorous nonlinear analysis to derive the bifurcations at both critical Rayleigh numbers based on the classical Hopf bifurcation theory and on a newly developed dynamic bifurcation theory by two of the authors. This new dynamic bifurcation theory is centered at a new notion of bifurcation, called attractor bifurcation for dynamical systems, both finite dimensional and infinite dimensional, together with new strategies for the Lyapunov-Schmidt reduction and the center manifold reduction procedures. The bifurcation theory has been applied to various problems from science and engineering, including, in particular, the Kuramoto-Sivashinsky equation, the Cahn-Hillard equation, the Ginzburg-Landau equation, reaction-diffusion equations in biology and chemistry, the Bénard convection problem, and the Taylor problem; see Refs., and references therein.

We remark that the non-self-adjointness of the linearized problem gives rise to the onset of the Hopf bifurcation. We prove that the Hopf bifurcation appears at the Rayleigh number . As mentioned earlier, the understanding and prediction of the periodic, quasiperiodic, aperiodic, and fully turbulent characteristics of large-scale atmospheric and oceanic flows are key issues in the study of climate dynamics and in geophysical fluid dynamics. It is hoped that the study carried out in this article will provide some insights in these important issues.

Also, we would like to mention that rigorous proof of the existence of periodic solutions for a fluid system is normally a very difficult task from the mathematical point of view. For instance, with a highly involved analysis, proved the existence of a Hopf bifurcation in an idealized Fourier space.

The paper is organized as follows. Section II gives the basic setting of the problem. Section III
states the main results. The proofs of the main results occupy the remaining part of the paper: Sec. IV recapitulates the essentials of the attractor bifurcation theory, Sec. V is on the eigenanalysis, and Sec. VI is on the central manifold reduction and the completion of the proofs.

II. STRATIFIED ROTATING BOUSSINESQ EQUATIONS IN GEOPHYSICAL FLUID DYNAMICS

The stratified rotating Boussinesq equations are basic equations in the geophysical fluid dynamics, and their nondimensional form is given by

\[ \frac{\partial U}{\partial t} = \sigma(\Delta U - \nabla p) + \sigma R Te - \frac{1}{Ro} e \times (U \cdot \nabla) U, \]

\[ \frac{\partial T}{\partial t} = \Delta T + w - (U \cdot \nabla) T, \]

\[ \text{div } U = 0, \]

for \((x,y,z)\) in the nondimensional domain \(\Omega = \mathbb{R}^2 \times (0,1)\), where \(U = (u,v,w)\) are the velocity fields, \(e = (0,0,1)\) is the unit vector in the \(z\) direction, \(\sigma\) is the Prandtl number, \(R\) is the thermal Rayleigh number, \(Ro\) is the Rossby number, \(T\) is the temperature function, and \(p\) is the pressure function. We refer the interested readers to Pedlosky\textsuperscript{22} and Lions et al.\textsuperscript{13} for the derivation of this model and the related parameters. In particular, the term \((1/Ro)e \times U\) represents the Coriolis force, the \(w\) term in the temperature equation is derived using the stratification, and the definition of the Rayleigh number \(R\) is as follows:

\[ R = \frac{g \alpha \beta h^4}{\kappa v}. \]

We consider the periodic boundary condition in the \(x\) and \(y\) directions,

\[ (U,T)(x,y,z,t) = (U,T)(x+2j \pi/\alpha_1, y, z, t) = (U,T)(x, y + 2k \pi/\alpha_2, z, t), \]

for any \(j,k \in \mathbb{Z}\). At the top and bottom boundaries, we impose the free-free boundary conditions:

\[ (T,w) = 0, \quad \frac{\partial u}{\partial z} = 0, \quad \frac{\partial v}{\partial z} = 0, \quad z = 0, 1. \]

It is natural to put the constraint

\[ \int_{\Omega} u \, dx \, dy \, dz = \int_{\Omega} v \, dx \, dy \, dz = 0. \]

The initial value conditions are given by

\[ (U,T) = (\bar{U}, \bar{T}) \quad \text{at } t = 0. \]

Let

\[ H = \{(U,T) \in L^2(\Omega)^4 | \text{div } U = 0, w|_{z=0,1} = 0, (u,v) \text{ satisfies Eqs. (2.3) and (2.5)}\}, \]

\[ H_1 = \{(U,T) \in H^2(\Omega)^4 \cap H | (U,T) \text{ satisfies Eqs. (2.3) - (2.5)}\}, \]

\[ \mathcal{H} = \{(U,T) \in H | (u,v,w,T)(-x,-y,z) = (-u,-v,w,T)(x,y,z)\}. \]
\[ \tilde{H}_1 = H_1 \cap \tilde{H}. \]

Let \( L_R = -A - B_R : H_1 \to H (\tilde{H}_1 \to \tilde{H}) \) and \( G : H_1 \to H (\tilde{H}_1 \to \tilde{H}) \) be defined by

\[ A\psi = \left( -P \left[ \sigma \Delta U - \frac{1}{R_0} U \times U \right], -\Delta T \right), \]

\[ B_R \psi = (-P[\sigma R T], -w), \]

for any \( \psi = (U, T) \in H_1 (\tilde{H}_1) \), where

\[ G(\psi_1, \psi_2) = (-P[(U_1 \cdot \nabla)U_2], -(U_1 \cdot \nabla)T_2), \]

for any \( \psi_1 = (U_1, T_1) \), \( \psi_2 = (U_2, T_2) \in H_1 \). Here \( P \) is the Leray projection to \( L^2 \) fields; for a detailed account of the function spaces, see among many others Ref. 26.

**Remark 2.1:** Note that \( \tilde{H}_1 \) and \( \tilde{H} \) are invariant under the bilinear operator \( G \) in the sense that

\[ G(\psi_1, \psi_2) \in \tilde{H} \text{ for } \psi_1, \psi_2 \in \tilde{H}_1. \]

Hence, \( \tilde{H}_1 \) and \( \tilde{H} \) are invariant under the operator \( L_R + G \).

Then, the Boussinesq equations (2.1)–(2.5) can be written in the following operator form:

\[ \frac{d\psi}{dt} = L_R \psi + G(\psi), \quad \psi = (U, T). \tag{2.7} \]

### III. MAIN RESULTS

#### A. Definition of attractor bifurcation

To state the main theorems of this article, we proceed with the definition of attractor bifurcation, first introduced by Ma and Wang in Refs. 16 and 17.

Let \( H \) and \( H_1 \) be two Hilbert spaces, and \( H_1 \hookrightarrow H \) be a dense and compact inclusion. We consider the following nonlinear evolution equations:

\[ \frac{du}{dt} = L_\lambda u + G(u, \lambda), \]

\[ u(0) = u_0, \tag{3.1} \]

where \( u : [0, \infty) \to H \) is the unknown function, \( \lambda \in \mathbb{R} \) is the system parameter, and \( L_\lambda : H_1 \to H \) are parametrized linear completely continuous fields depending continuously on \( \lambda \in \mathbb{R}^1 \), which satisfy

\[ -L_\lambda = A + B_\lambda \quad \text{a sectorial operator}, \]

\[ A : H_1 \to H \quad \text{a linear homeomorphism}, \tag{3.2} \]

\[ B_\lambda : H_1 \to H \quad \text{parametrized linear compact operators}. \]

It is easy to see\(^7\) that \( L_\lambda \) generates an analytic semigroup \( \{e^{tL_\lambda}\}_{t \geq 0} \). Then, we can define fractional power operators \((-L_\lambda)^\mu\) for any \( 0 \leq \mu \leq 1 \) with domain \( H_\mu = D((-L_\lambda)^\mu) \) such that \( H_{\mu_1} \subset H_{\mu_2} \) if \( \mu_1 > \mu_2 \), and \( H_0 = H \).
Furthermore, we assume that the nonlinear terms \( G(\cdot, \lambda) : H_\mu \to H \) for some \( 1 > \mu \geq 0 \) are a family of parametrized \( C^r \) bounded operators \( (r \geq 1) \) continuously depending on the parameter \( \lambda \in \mathbb{R}^1 \), such that

\[
G(u, \lambda) = o(\|u\|_H), \quad \forall \lambda \in \mathbb{R}^1. \tag{3.3}
\]

In this paper, we are interested in the sectorial operator \(-L_\lambda = A + B_\lambda\) such that there exists an eigenvalue sequence \( \{\rho_k\} \subset \mathbb{C}^1 \) and an eigenvector sequence \( \{e_k, h_k\} \subset H_1 \) of \( A \):

\[
A e_k = \rho_k e_k, \quad \bar{e}_k = e_k + i h_k, \quad \text{Re} \rho_k \to \infty \quad (k \to \infty), \quad |\text{Im} \rho_k| \leq c, \tag{3.4}
\]

for some \( a, c > 0 \), such that \( \{e_k, h_k\} \) is a basis of \( H \). Also we assume that there is a constant 0 < \( \theta < 1 \) such that

\[
B_\lambda : H_\theta \to H \text{ bounded}, \quad \forall \lambda \in \mathbb{R}^1. \tag{3.5}
\]

Under conditions (3.4) and (3.5), the operator \(-L_\lambda = A + B_\lambda\) is a sectorial operator.

Let \( \{S_\lambda(t)\}_{t \geq 0} \) be an operator semigroup generated by Eq. (3.1). Then, the solution of Eq. (3.1) can be expressed as \( \psi(t, \psi_0) = S_\lambda(t)\psi_0 \), for any \( t \geq 0 \).

**Definition 3.1:** A set \( \Sigma \subset H \) is called an invariant set of Eqs. (3.1) if \( S(t)\Sigma = \Sigma \) for any \( t \geq 0 \). An invariant set \( \Sigma \subset H \) of Eqs. (3.1) is called an attractor if \( \Sigma \) is compact, and there exists a neighborhood \( W \subset H \) of \( \Sigma \) such that for any \( \psi_0 \in W \) we have

\[
\lim_{t \to \infty} \text{dist}_p(\psi(t, \psi_0), \Sigma) = 0.
\]

**Definition 3.2:**

1. We say that the solution of Eqs. (3.1) bifurcates from \((\psi, \lambda) = (0, \lambda_0) \) to an invariant set \( \Omega_\lambda \) if there exists a sequence of invariant sets \( \{\Omega_{\lambda_n}\} \) of Eqs. (3.1) such that \( 0 \in \Omega_{\lambda_n} \), \( \lim_{n \to \infty} \lambda_n = \lambda_0 \), and

\[
\lim_{n \to \infty} \max_{x \in \Omega_{\lambda_n}} |x| = 0.
\]

2. If the invariant sets \( \Omega_\lambda \) are attractors of Eqs. (3.1), then the bifurcation is called an attractor bifurcation.

**B. Main theorems**

In this article, we consider two cases:

\[
\sigma > 1 \quad \text{and} \quad R_{c_1} \text{ is obtained only at } (j, k, l) = (j_1, 0, 1), \tag{3.6}
\]

\[
\sigma < 1 \quad \text{and} \quad R_{c_2} \text{ is obtained only at } (j, k, l) = (j_2, 0, 1), \tag{3.7}
\]

for some \( j_1, j_2 \in \mathbb{N} \), where \( R_{c_1} \) and \( R_{c_2} \) are defined in Eqs. (5.18) and (5.22), respectively. In the above cases, \( R_{c_1} \) and \( R_{c_2} \) are given by the following formulas:

\[
R_{c_1} = \left( j_1^2 \alpha_1^2 + \frac{\pi^2}{j_1^2 \alpha_1^2} \right)^3 + \frac{\pi^2}{\sigma \cdot \text{Re} \cdot j_1^2 \alpha_1^2},
\]
Remark 3.1:

(1) Condition (3.6) guarantees that for $R=R_{c_1}$, the first eigenvalue of $L_{R|H_1}$ ($L_{R|\tilde{H}_1}$) is real and of multiplicity 2 (1); see Remark (5.1).

(2) Condition (3.7) guarantees that, for $R=R_{c_2}$, there exists only one simple pair of conjugate complex eigenvalues of $L_{R|H_1}$ crossing the imaginary axis; see Lemma (5.5)

(3) Condition (3.6) or (3.7) can be satisfied easily; see Lemmas (5.3) and (5.4).

**Theorem 3.1:** Assume Eq. (3.6). Then, the following assertions for problems (2.1)–(2.5) defined in $H$ hold true.

(1) If $R \leq R_{c_1}$, the steady state $(U, T) = 0$ is locally asymptotically stable.

(2) For $R > R_{c_1}$, the problem bifurcates from $(U, T) = (0, R_{c_1})$ to an attractor $\Sigma_R = S^1$, consisting of only steady state solutions (Fig. 1).

**Theorem 3.2:** Assume Eq. (3.7) and

$$Ro^2 < \frac{(1 - \sigma)^2 \pi^2}{\sigma^2 (1 + \sigma)(j^2 \alpha_1^2 + \pi^3)^3}.$$

The following statements are true.

(1) For problems (2.1)–(2.5) defined in $H$, the steady state $(U, T) = 0$ is locally asymptotically stable if $R < R_{c_2}$.

(2) For problems (2.1)–(2.5) defined in $\tilde{H}$, a Hopf bifurcation occurs generically when $R$ crosses $R_{c_2}$.

**IV. PRELIMINARIES**

**A. Attractor bifurcation theory**

Consider Eq. (3.1) satisfying Eqs. (3.2) and (3.3). We start with the principle of exchange of stabilities (PES). Let the eigenvalues (counting the multiplicity) of $L_{\lambda}$ be given by $\beta_1(\lambda), \beta_2(\lambda), \ldots$. Suppose that
It is known that \( \dim E_0 = m \).

**Theorem 4.1:** (Ma and Wang\textsuperscript{16,17}) Assume that the conditions (3.2)–(3.5), (4.1), and (4.2) hold true, and \( u=0 \) is locally asymptotically stable for Eq. (3.1) at \( \lambda = \lambda_0 \). Then, the following assertions hold true.

1. For \( \lambda > \lambda_0 \), Eq. (3.1) bifurcates from \((u, \lambda) = (0, \lambda_0)\) to attractors \( \Sigma_{\lambda} \), having the same homology as \( S^{m-1} \), with \( m-1 \leq \dim \Sigma_{\lambda} = m \), which is connected if \( m > 1 \).
2. For any \( u_\lambda \in \Sigma_{\lambda} \), \( u_\lambda \) can be expressed as
   \[
   u_\lambda = v_\lambda + o(\|v_\lambda\|_{H^1}), \quad v_\lambda \in E_0.
   \]
3. There is an open set \( U \subset H \) with \( 0 \in U \) such that the attractor \( \Sigma_{\lambda} \) bifurcated from \((0, \lambda_0)\) attracts \( U \setminus \Gamma \) in \( H \), where \( \Gamma \) is the stable manifold of \( u=0 \) with codimension \( m \).

**B. Center manifold theory**

A crucial ingredient for the proof of the main theorems using the above attractor bifurcation theorem is an approximation formula for center manifold functions; see Ref. 16.

Let \( H_1 \) and \( H \) be decomposed into
   \[
   H_1 = E_1^h \oplus E_2^h, \quad H = \tilde{E}_1^h \oplus \tilde{E}_2^h,
   \]
for \( \lambda \) near \( \lambda_0 \in \mathbb{R}^1 \), where \( E_1^h, E_2^h \) are invariant subspaces of \( L_\lambda \), such that \( \dim E_1^h < \infty, \tilde{E}_1^h = E_1^h, \tilde{E}_2^h = \) closure of \( E_2^h \) in \( H \). In addition, \( L_\lambda \) can be decomposed into \( L_\lambda = L_1^h \oplus L_2^h \) such that for any \( \lambda \) near \( \lambda_0 \),
   \[
   L_1^h = L_\lambda|E_1^h : E_1^h \rightarrow \tilde{E}_1^h,
   \]
   \[
   L_2^h = L_\lambda|E_2^h : E_2^h \rightarrow \tilde{E}_2^h,
   \]
where all eigenvalues of \( L_1^h \) possess negative real parts, and the eigenvalues of \( L_2^h \) possess non-negative real parts at \( \lambda = \lambda_0 \). Furthermore, with \( \mu < 1 \) given by Eq. (3.3), let
   \[
   E_2^h(\mu) = \) closure of \( E_2^h \) in \( H_\mu \).

By the classical center manifold theorem (see among others Refs. 7 and 26), there exists a neighborhood of \( \lambda_0 \) given by \( |\lambda - \lambda_0| < \delta \) for some \( \delta > 0 \), a neighborhood \( B_\delta \subset E_2^h \) of \( x=0 \), and a \( C^1 \) center manifold function \( \Phi(\cdot, \lambda): B_\delta \rightarrow E_2^h(\theta) \), called the center manifold function, depending continuously on \( \lambda \). Then, to investigate the dynamic bifurcation of Eq. (3.1) it suffices to consider the finite-dimensional system as follows:
\[
\frac{dx}{dt} = L^\lambda x + g_1(x, \Phi_\lambda(x), \lambda), \quad x \in B_\lambda \subset E^\lambda_1.
\]  
(4.5)

Hence, an approximation formula for the center manifold function \( \Phi_\lambda \) is crucial for the bifurcation and stability study.

Let the nonlinear operator \( G \) be in the following form:

\[
G(u, \lambda) = G_n(u, \lambda) + o(\|u\|^n),
\]
(4.6)
for some integer \( n \geq 2 \). Here \( G_n: H_1 \times \cdots \times H_1 \rightarrow H \) is an \( n \)-multilinear operator, and \( G_n(u, \lambda) = G_n(u, \ldots, u, \lambda) \).

**Theorem 4.2:** (Ma and Wang\(^{16}\)) Under the conditions (4.3), (4.4), and (4.6), the center manifold function \( \Phi(x, \lambda) \) can be expressed as

\[
\Phi(x, \lambda) = (-L^\lambda)^{-1} P_2 G_n(x, \lambda) + o(\|x\|^n) + O(\text{Re} \beta \|x\|^n),
\]
(4.7)
where \( L^\lambda \) is as in Eq. (4.4), \( P_2: H \rightarrow E_2 \) the canonical projection, \( x \in E^\lambda_1 \), and \( \beta = (\beta_1(\lambda), \ldots, \beta_m(\lambda)) \) the eigenvectors of \( L^\lambda_1 \).

**V. EIGENVALUE PROBLEM**

The eigenvalue problem of the linearized problem of Eqs. (2.1)–(2.4) is given by

\[
\sigma(\Delta U - \nabla p) + \sigma RT e - \frac{1}{Ro} e \times U = \beta U,
\]

\[
\Delta T + w = \beta T,
\]
(5.1)

\[
\text{div} \ U = 0,
\]

supplemented with Eqs. (2.3) and (2.4). For \( \psi = (U, T) \) satisfying Eqs. (2.3) and (2.4), we expand the field \( \psi \) in Fourier series,

\[
\psi(x, y, z) = \sum_{j,k=\pm \infty} \psi_{jk}(z)e^{(j\alpha_1 \varepsilon + k\alpha_2 y)}.
\]
(5.2)

Plugging Eq. (5.2) into Eq. (5.1), we obtain the following system of ordinary differential equations:

\[
\sigma(D_{jk} u_{jk} - ij \alpha_1 p_{jk}) + \frac{1}{Ro} v_{jk} = \beta u_{jk},
\]

\[
\sigma(D_{jk} v_{jk} - ik \alpha_2 p_{jk}) - \frac{1}{Ro} u_{jk} = \beta v_{jk},
\]

\[
D_{jk} w_{jk} - p'_{jk} + RT_{jk} = \sigma^{-1} \beta w_{jk},
\]

\[
D_{jk} T_{jk} + w_{jk} = \beta T_{jk},
\]

\[
ij \alpha_1 u_{jk} + ik \alpha_2 v_{jk} + w'_{jk} = 0,
\]
(5.3)
\[ u_j' \vert_{\zeta=0.1} = u_j' \vert_{\zeta=0.1} = w_k' \vert_{\zeta=0.1} = T_j \vert_{\zeta=0.1} = 0, \]

for \( j, k \in \mathbb{Z} \), where \( \zeta' = d/d\zeta \), \( D_{jk} = d^2/d\zeta^2 - \alpha_j^2 \), and \( \alpha_j^2 = j^2 + k^2 \alpha_j^2 \). If \( w_{jk} \neq 0 \), Eq. (5.3) can be reduced to a single equation for \( w_{jk}(z) \):

\[
(D_{jk} - \beta)(\sigma D_{jk} - \beta)^2 D_{jk} + \frac{1}{R_0^2} (D_{jk} - \beta)(D_{jk} + \alpha_j^2) + \sigma R \alpha_j^2 (\sigma D_{jk} - \beta) \right) w_{jk} = 0, \tag{5.4}
\]

\[
w_{jk} = w_{jk}' = w_{jk}'' = w_{jk}''' = 0 \quad \text{at} \quad z = 0, 1, \tag{5.5}
\]

for \( j, k \in \mathbb{Z} \). Thanks to Eq. (5.5), \( w_{jk} \) can be expanded in a Fourier sine series,

\[
w_{jk}(z) = \sum_{l=1}^{\infty} w_{jk} \sin l\pi z, \tag{5.6}
\]

for \((j,k) \in \mathbb{Z} \times \mathbb{Z} \). Substituting Eq. (5.6) into Eq. (5.4), we see that the eigenvalues \( \beta \) of the problem (5.1) satisfy the cubic equations

\[
\beta^3 + (2\sigma + 1) \gamma_{jkl}^2 \beta^2 + \left[ (\sigma^2 + 2\sigma) \gamma_{jkl}^4 + \frac{l^2 \pi^2}{R_0^2 \gamma_{jkl}^2} - \sigma R \alpha_j^2 \right] \beta + \sigma^2 \gamma_{jkl}^2 - \sigma^2 R \alpha_j^2 + \frac{l^2 \pi^2}{R_0^2} = 0, \tag{5.7}
\]

for \( j, k \in \mathbb{Z} \) and \( l \in \mathbb{N} \), where \( \gamma_{jkl}^2 = \alpha_j^2 + l^2 \pi^2 \). In the following discussions, we let

\[
g_{jkl}(\beta) = (\beta + \gamma_{jkl}^2)[(\beta + \sigma \gamma_{jkl}^2)^2 + l^2 \pi^2 R \alpha_j^2 \gamma_{jkl}^2],
\]

\[
h_{jkl}(\beta) = \sigma R \alpha_j^2 \gamma_{jkl}^2 (\beta + \sigma \gamma_{jkl}^2), \tag{5.8}
\]

\[
f_{jkl}(\beta) = g_{jkl}(\beta) - h_{jkl}(\beta),
\]

and \( \beta_{jkl1}(R), \beta_{jkl2}(R), \) and \( \beta_{jkl3}(R) \) be the zeros of \( f_{jkl}(R) \) with

\[
\Re(\beta_{jkl1}) \geq \Re(\beta_{jkl2}) \geq \Re(\beta_{jkl3}).
\]

A. Eigenvectors

In the following discussions, we consider the following index sets:

\[
\Lambda_1 = \{(j,k,l) \in \mathbb{Z}^2 \times \mathbb{N} | j \geq 0, (j,k) \neq (0,0) \},
\]

\[
\Lambda_2 = \{(j,k,l) \in \mathbb{Z}^2 \times \{0\} | j \geq 0, (j,k) \neq (0,0) \},
\]

\[
\Lambda_3 = \{(j,k,l) \in \{0\} \times \mathbb{N} \},
\]

\[
\Lambda = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3.
\]

1. For \((j,k,0) \in \Lambda_2 \), we define

\[
\psi_{10}^j = (k\alpha_2 \sin(j\alpha_1 x + k\alpha_2 y), -j\alpha_1 \sin(j\alpha_1 x + k\alpha_2 y), 0, 0)',
\]

\[
\psi_{20}^j = (-k\alpha_2 \cos(j\alpha_1 x + k\alpha_2 y), j\alpha_1 \cos(j\alpha_1 x + k\alpha_2 y), 0, 0)',
\]
\[ E_{jk0} = \text{span}\{\psi_{jk0}^1, \psi_{jk0}^2\}, \]
\[ \beta_{\lambda_j} = \bigcup_{i(k,0)} = \beta_{jk0}, \]
where \( \beta_{jk0} = -\sigma \gamma_{jk0} = -\sigma j^2 \alpha_j^2 = -\sigma (j^2 \alpha_j^2 + 2k^2 \alpha_j^2) \). It is not hard to see that \( L_{R}(\psi_{jk0}^1) = \beta_{jk0} \psi_{jk0}^1 \)
and \( L_{R}(\psi_{jk0}^2) = \beta_{jk0} \psi_{jk0}^2 \).

2. For \((0,0,1) \in \Lambda_3\), we define
\[ \psi_{001}^1 = (0,0,0, \sin l \pi z)', \quad \psi_{001}^2 = (\cos l \pi z, 0,0,0)', \]
\[ \psi_{001}^3 = (0,\cos l \pi z, 0,0)', \quad E_{001} = \text{span}\{\psi_{001}^1, \psi_{001}^2, \psi_{001}^3\}, \]
\[ \beta_{\lambda_3} = \bigcup_{i=1}^\infty \left( \psi_{001}^1 \right), \quad \beta_{\lambda_3} = \bigcup_{l=1}^\infty \left( \psi_{001}^3 \right), \]
where \( \beta_{001} = -\gamma_{001} = -l^2 \pi^2, \quad \beta_{002} = -\gamma_{002} - (1/Ro)', \quad \beta_{003} = -\gamma_{003} + (1/Ro)' \). It is easy to check that
\[ L_{R}(\psi_{001}^1) = \beta_{001} \psi_{001}^1, \]
\[ L_{R}(\psi_{001}^2) = -\sigma \gamma_{002} \psi_{002}^2 - \frac{1}{Ro} \psi_{003}^1, \]
\[ L_{R}(\psi_{001}^3) = \frac{1}{Ro} \psi_{002}^1 - \sigma \gamma_{003} \psi_{003}^3. \]

3. For \((j,k,l) \in \Lambda_1\), we define
\[ \phi_{jkl}^1 = \left( -\frac{j \alpha_j \pi}{\alpha_{jk}^2} \sin(j \alpha_j x + k \alpha_j y) \cos l \pi z, \right. \]
\[ \left. -\cos(j \alpha_j x + k \alpha_j y) \sin l \pi z, 0 \right)', \]
\[ \phi_{jkl}^2 = \left( \frac{k \alpha_j \pi}{\alpha_{jk}^2} \sin(j \alpha_j x + k \alpha_j y) \cos l \pi z, \right. \]
\[ \left. -\frac{j \alpha_j \pi}{\alpha_{jk}^2} \sin(j \alpha_j x + k \alpha_j y) \cos l \pi z, 0 \right)', \]
\[ \phi_{jkl}^3 = (0,0,0, \cos(j \alpha_j x + k \alpha_j y) \sin l \pi z)', \]
\[ \phi_{jkl}^4 = \left( \frac{j \alpha_j \pi}{\alpha_{jk}^2} \cos(j \alpha_j x + k \alpha_j y) \cos l \pi z, \right. \]
\[ \left. \frac{k \alpha_j \pi}{\alpha_{jk}^2} \cos(j \alpha_j x + k \alpha_j y) \cos l \pi z, \right)', \]
\[ \sin(j \alpha_j x + k \alpha_j y) \sin l \pi z, 0 \right)', \]
\[ \phi_{jkl}^5 = \left( -\frac{k \alpha_j \pi}{\alpha_{jk}^2} \cos(j \alpha_j x + k \alpha_j y) \cos l \pi z, \right. \]
\[ \left. -\frac{j \alpha_j \pi}{\alpha_{jk}^2} \cos(j \alpha_j x + k \alpha_j y) \cos l \pi z, 0 \right)', \]
\[ \phi_{jkl}^6 = (0,0,0, \sin(j \alpha_j x + k \alpha_j y) \sin l \pi z)', \]
\[ E_{jkl}^1 = \text{span}\{\phi_{jkl}^1, \phi_{jkl}^2, \phi_{jkl}^3\}, \quad E_{jkl}^2 = \text{span}\{\phi_{jkl}^4, \phi_{jkl}^5, \phi_{jkl}^6\}, \]
The dual vector for

\[ E_{jkl} = E_{jkl}^1 \oplus E_{jkl}^2, \]

where

\[ \beta_{\Lambda_1} = \bigcup_{(j,k,l) \in \Lambda_1} \bigcup_{q=1}^3 \{ \beta_{jklq} \}. \]

It is easy to check that \( E_{jkl}^1 \) and \( E_{jkl}^2 \) are invariant subspaces of the linear operator \( L_R \), i.e., \( L_R(E_{jkl}^1) \subseteq E_{jkl}^1 \) and \( L_R(E_{jkl}^2) \subseteq E_{jkl}^2 \). The characteristic polynomial of \( L_R|_{E_{jkl}^1} \) ( \( L_R|_{E_{jkl}^2} \)) is given by \( f_{jkl} \) as defined in Eq. (5.8). Since \( E_{jkl}^1(E_{jkl}^1) \) is of dimension 3, the (generalized) eigenvectors of \( L_R|_{E_{jkl}^1} \), \( \bigcup_{q=1}^3 \{ \psi_{jklq}^{\beta_{jklq}} \} \), form a basis of \( E_{jkl}^1(E_{jkl}^1) \), i.e., span \( \bigcup_{q=1}^3 \{ \psi_{jklq}^{\beta_{jklq}} \} = E_{jkl}^1 \) (span \( \bigcup_{q=1}^3 \{ \psi_{jklq}^{\beta_{jklq}} \} = E_{jkl}^2 \)). If \( \beta_{jklq} \) is a real zero of \( f_{jkl} \), the eigenvector corresponding to \( \beta_{jklq} \) in \( E_{jkl}^1(E_{jkl}^1) \) is given by

\[ \psi_{jklq}^{\beta_{jklq}} = \phi_{jklq}^1 + A_1(\beta_{jklq}) \phi_{jklq}^2 + A_2(\beta_{jklq}) \phi_{jklq}^3. \]

(5.9)

where

\[ A_1(\beta) = \frac{-1}{\text{Re}(\beta + \sigma \gamma_{jkl})}, \quad A_2(\beta) = \frac{1}{\beta + \gamma_{jkl}^2}. \]

If \( \beta_{jklq}, \beta_{jklq}^{\ast} \) are zeros of \( f_{jkl} \), the (generalized) eigenvectors corresponding to \( \beta_{jklq} \) and \( \beta_{jklq}^{\ast} \) in \( E_{jkl}^1(E_{jkl}^2) \) are given by

\[ \psi_{jklq}^{\beta_{jklq}} = \phi_{jklq}^1 + R_1(\beta_{jklq}) \phi_{jklq}^2 + R_2(\beta_{jklq}) \phi_{jklq}^3, \]

\[ \psi_{jklq}^{\beta_{jklq}^{\ast}} = I_1(\beta_{jklq}) \phi_{jklq}^2 + I_2(\beta_{jklq}) \phi_{jklq}^3. \]

(5.11)

where

\[ R_1(\beta) = \text{Re}(A_1(\beta)), \quad R_2(\beta) = \text{Re}(A_2(\beta)), \]

\[ I_1(\beta) = \text{Im}(A_1(\beta)), \quad I_2(\beta) = \text{Im}(A_2(\beta)). \]

The dual vector corresponding to \( \psi_{jklq}^{\beta_{jklq}} \) (\( \psi_{jklq}^{\beta_{jklq}^{\ast}} \)) is given by

\[ \Psi_{jklq}^{\beta_{jklq}} = \phi_{jklq}^1 + C_1(\beta_{jklq}) \phi_{jklq}^2 + C_2(\beta_{jklq}) \phi_{jklq}^3, \]

\[ \Psi_{jklq}^{\beta_{jklq}^{\ast}} = \phi_{jklq}^1 + C_1(\beta_{jklq}) \phi_{jklq}^2 + C_2(\beta_{jklq}) \phi_{jklq}^3. \]

(5.13)

where

\[ C_1(\beta) = \frac{1}{\text{Re}(\beta + \sigma \gamma_{jkl})}, \quad C_2(\beta) = -\frac{\sigma R}{\beta + \gamma_{jkl}^2}. \]

The dual vector \( \Psi_{jklq}^{\beta_{jklq}} \) (\( \Psi_{jklq}^{\beta_{jklq}^{\ast}} \)) satisfies

\[ (\psi_{jklq}^{\beta_{jklq}}^*, \psi_{jklq}^{\beta_{jklq}})_H = 0, \quad (\psi_{jklq}^{\beta_{jklq}^{\ast}}^*, \psi_{jklq}^{\beta_{jklq}^{\ast}})_H = 0. \]

(5.15)

for \( q \neq q^* \).

We note that \( E_{j,k,l} \) is orthogonal to \( E_{j,k,l}^{\ast} \) for \( (j_1,k_1,l_1) \neq (j_2,k_2,l_2) \) and \( E_{j,k,l}^1 \) is orthogonal to \( E_{j,k,l}^2 \) for \( (j,k,l) \in \Lambda_1 \). Hence, the dual vector \( \Psi_{jklq}^{\beta_{jklq}} \) (\( \Psi_{jklq}^{\beta_{jklq}^{\ast}} \)) satisfies

\[ (\psi_{jklq}^{\beta_{jklq}}^*, \psi_{jklq}^{\beta_{jklq}})_H = 0, \quad (\psi_{jklq}^{\beta_{jklq}^{\ast}}^*, \psi_{jklq}^{\beta_{jklq}^{\ast}})_H = 0. \]
\begin{align*}
\langle \psi, \Psi^B_{jkl}(\beta) \rangle_H &= 0 \quad \text{for} \quad \psi \in (\cup_{(j',k',l')}^+ \{ E_{j',k'}^{(1)} \}) \cup E_{jkl}^2, \\
((\psi, \Psi^B_{jkl}(\beta))_H &= 0 \quad \text{for} \quad \psi \in (\cup_{(j',k',l')}^+ \{ E_{j',k'}^{(1)} \}) \cup E_{jkl}^1). \tag{5.16}
\end{align*}

In view of the Fourier expansion, we see that \( \cup_{(j,k,l) \in \Lambda} E_{jkl} \) is a basis of \( H_1 \) and \( (\cup_{(j,k,l) \in \Lambda} E_{jkl}^1) \cup (\cup_{(j,k,l) \in \Lambda} \{ \psi^{B_0} \}) \cup (\cup_{(j,k,l) \in \Lambda} \{ \psi^{B_1} \}) \) is a basis of \( \tilde{H}_1 \). Hence, by the discussion above, we have the following conclusions.

(a) The set \( \beta_{\Lambda_1} = \beta_{\Lambda_1} \cup \beta_{\Lambda_2} \cup \beta_{\Lambda_3} \) consists of all eigenvalues of \( L_{1|H_1} \), and the (generalized) eigenvectors of \( L_{1|H_1} \) form a basis of \( H_1 \).

(b) The set \( \beta_{\tilde{\Lambda}_1} = \beta_{\Lambda_1} \cup \beta_{\Lambda_2} \cup \beta_{\Lambda_3} \) consists of all eigenvalues of \( L_{1|\tilde{H}_1} \), and the (generalized) eigenvectors of \( L_{1|\tilde{H}_1} \) form a basis of \( \tilde{H}_1 \).

(c) \( \text{Re}(\beta) < 0 \) for each \( \beta \in \beta_{\Lambda_2} \cup \beta_{\Lambda_3} \).

\textbf{Lemma 5.1:} If \( R \) is small, then \( \text{Re}(\beta_{\tilde{\Lambda}_0}(R)) < 0 \) for each \( \beta_{\tilde{\Lambda}_0} \in \beta_{\Lambda_1} \).

\textbf{Proof:} Plugging \( \beta = \gamma_{jkl}^* \beta^* \) into \( f_{jkl} \), we get \( f_{jkl}(\beta) = \gamma_{jkl}^* \tilde{f}_{jkl}(\beta^*) \), where

\[
\tilde{f}_{jkl}(\beta^*) = (\beta^* + 1)(\beta^* + \sigma)^2 + \frac{l^2 \pi^2}{\gamma_{jkl}} (\beta^* + \sigma) - \sigma R \frac{\alpha_{jkl}}{\gamma_{jkl}^*} (\beta^* + \sigma).
\]

Hence, we only need to show that the real part of each zero of \( \tilde{f}_{jkl} \) is strictly negative when \( R \) is small. We observe that \( \tilde{f}_{jkl}(\beta^*) > 0 \) for all \( \beta^* > 0 \) provided \( R < 1 + \sigma^{-3} \). Therefore, if all zeros of \( \tilde{f}_{jkl} \) are real numbers, we are done.

For the case where only one of the zeros of \( \tilde{f}_{jkl} \) is real, this real zero, \( \beta_1^* \), is a perturbation of \(-1\). There exists an \( \epsilon \) (depending on \( \sigma \) only) such that \( \beta_1^* < 0 \) provided \( R < \epsilon \). This makes the real part of the other two zeros of \( \tilde{f}_{jkl} \) strictly negative and the proof is complete. \( \square \)

\section{B. Characterization of critical Rayleigh numbers}

Based on the above discussion, we know that only the eigenvalues in \( \beta_{\Lambda_1} \) depend on the Rayleigh number \( R \). Hence, to study the principle of exchange of stabilities for problem (5.1), it suffices to focus on the problem on the set \( \beta_{\Lambda_1} \). We proceed with the following two cases.

\textbf{Case 1:} \( \beta = 0 \) is a zero of \( f_{jkl} \) if and only if the constant term of the polynomial \( f_{jkl} \) is 0. In this case, we have

\[
R = \frac{\gamma_{jkl}^*}{\alpha_{jkl}^*} + \frac{l^2 \pi^2}{\sigma \gamma_{jkl}^*} \geq \frac{(\alpha_{jkl}^* + \pi)^3}{\alpha_{jkl}^*} + \frac{\pi^2}{\sigma \gamma_{jkl}^*}.
\tag{5.17}
\]

Hence, the critical Rayleigh number \( R_{c1} \) is given by

\[
R_{c1} = \min_{(j,k,l) \in \Lambda_1} \left\{ \frac{\gamma_{jkl}^*}{\alpha_{jkl}^*} + \frac{l^2 \pi^2}{\sigma \gamma_{jkl}^*} \right\} = \frac{\gamma_{jkl}^*}{\alpha_{jkl}^*} + \frac{\pi^2}{\sigma \gamma_{jkl}^*}.
\tag{5.18}
\]

for some \( (j_1,k_1,l_1) \in \Lambda_1 \).

\textbf{Case 2:} A careful analysis on Eq. (5.7) shows that \( \beta = a i \ (a \neq 0) \), a purely imaginary number, is a zero of \( f_{jkl} \) if and only if the following two equations hold true:

\[
(\sigma^2 + 2\sigma) \frac{\gamma_{jkl}^*}{\gamma_{jkl}^*} + \frac{l^2 \pi^2}{\sigma \gamma_{jkl}^*} - \sigma R \frac{\alpha_{jkl}^*}{\gamma_{jkl}^*} > 0,
\]

\[
(\sigma^2 + 2\sigma) \frac{\gamma_{jkl}^*}{\gamma_{jkl}^*} + \frac{l^2 \pi^2}{\sigma \gamma_{jkl}^*} - \sigma R \frac{\alpha_{jkl}^*}{\gamma_{jkl}^*} < 0.
\]
(2σ + 1)γ^2_{jkl} \left[ (σ^2 + 2σ)γ^2_{jkl} + \frac{l^2 \pi^2}{Ro^2 γ^2_{jkl}} - αR \frac{α^2_{jkl}}{2} γ^2_{jkl} \right] = α^2 γ^2_{jkl} - α^2 Rα^2_{jkl} + \frac{l^2 \pi^2}{Ro^2}.

In this case, we have

\[ R = \frac{2(σ + 1)γ^6_{jkl}}{α^2_{jkl}} + \frac{2l^2 \pi^2}{(σ + 1)Ro^2 α^2_{jkl}}, \quad (5.19) \]

\[ R < \frac{(σ + 2)γ^6_{jkl}}{α^2_{jkl}} + \frac{l^2 \pi^2}{σRo^2 α^2_{jkl}}. \quad (5.20) \]

Plugging Eq. (5.20) into Eq. (5.19), we derive an upper bound for Ro^2:

\[ Ro^2 < \frac{(1 - α)^2 l^2 \pi^2}{α^2(1 + σ)γ^6_{jkl}}, \quad (5.21) \]

which could only hold true when σ<1.

As in Case 1, the minimum of the right hand side of Eq. (5.19) is always obtained at l=1. Hence, the critical Rayleigh number R_c is given by

\[ R_c = \min_{(j,k,l) = 1} \left\{ \frac{2(σ + 1)γ^6_{jkl}}{α^2_{jkl}} + \frac{2l^2 \pi^2}{(σ + 1)Ro^2 α^2_{jkl}} \right\} = \frac{2(σ + 1)γ^6_{jkl}}{(σ + 1)Ro^2 α^2_{jkl}}, \quad (5.22) \]

for some (j,k,l) ∈ A_1. In the case of σ<1, Eq. (5.21) with l=1 implies R_c is smaller than R_c^1. Hence, for problems (2.1)–(2.5), R_c^1 is the first critical Rayleigh number if σ>1 and R_c^2 is the first critical Rayleigh number if σ<1. Therefore, the principle of exchange of stabilities is given by Lemmas 5.2 and 5.5.

**Lemma 5.2:** For fixed σ>1 and Ro>0, suppose that (α^2_{j,k,l},l)=(α^2_{j,k,l},1) minimizes the right hand side of Eq. (5.17), then

\[ β_{j,k,l}(l) = \begin{cases} <0 & \text{if } R < R_c, \\ 0 & \text{if } R = R_c, \\ >0 & \text{if } R > R_c, \end{cases} \quad q = 1,2,3, \quad R \text{ near } R_c. \quad (5.23) \]

**Proof:** By the above discussion, we only need to show that the first eigenvalue crosses the imaginary axis. We note that f_{j,k,l}(β)=0 is equivalent to g_{j,k,l}(β)=h_{j,k,l}(β), i.e.,

\[ (β + γ^2_{j,k,l})[(β + σγ^2_{j,k,l})^2 + l^2 \pi^2 Ro^2 γ^2_{j,k,l}] = σRα^2_{j,k,l} γ^2_{j,k,l} (β + σγ^2_{j,k,l}). \quad (5.25) \]

We see that both g_{j,k,l} and h_{j,k,l} are strictly increasing for β>−γ^2_{j,k,l} (since σ>1). Let Γ_1 be the graph of η=g_{j,k,l}(β) and Γ_2 be the graph of η=h_{j,k,l}(β), as shown in Fig. 2. When R=R_c, point S_0, the intersecting point of Γ_1 and Γ_2 corresponding to β_{j,k,l}(R) [i.e., the β coordinate of S_0 is β_{j,k,l}(R)], is on the η axis. When R increases (decreases), S_0 becomes S_1 (S_2). This proves Eq. (5.23) and the proof is complete. □

**Remark 5.1:**

(1) In the proof of Lemma 5.2, as shown by Eq. (5.25) and Fig. 2, we see that, for R=R_c^1, the first eigenvalue β_{j,k,l} is a simple zero of f_{j,k,l}(β). We have seen in Sec. V A that there are eigenvectors \( ψ_{j,k,l}^1, ψ_{j,k,l}^2 \) corresponding to \( β_{j,k,l} \) and \( β_{j,k,l}^2 \) respectively. Therefore, the
multiplicity of the first eigenvalue of \( L_{H_1} (L_{H_1}) \) is \( m_{H_1} = 2m \) \((m_{H_1} = m)\), where \( m \) is the number of \((j,k,1)'s \) satisfying \( \alpha_2^2 = \alpha_2^2 \). Hence, condition (3.6) guarantees that, for \( R \approx R_{c_2} \), the first eigenvalue of \( L_{H_1} (L_{H_1}) \) is real and of multiplicity 2 (1).

(2) For the classical Bénard problem without rotation, the second term on the right hand side of Eq. (5.17), hence the second term on the right hand side of Eq. (5.18), is not presented. Therefore, the first critical Rayleigh number of the classical Bénard problem depends only on the aspect of ratio, while the first critical Rayleigh number of the rotating problem depends on the aspect of ratio, the Prandtl number, and the Rossby number. It is clear that the first critical Rayleigh number of fast rotating flows is remarkably larger than the first critical Rayleigh number of the classical Bénard problem. This indicates that the rotating flows are much more stable than the nonrotating flows.

(3) \( R_{c_1} \) is the first critical Rayleigh number if the Prandtl number is greater than 1. For the case where the Prandtl number is smaller than 1, \( R_{c_2} \) is the first critical Rayleigh number and, in general, there are a few critical values between \( R_{c_2} \) and \( R_{c_1} \).

For \( x > 0, \ b > 0 \), we define

\[
    f_b(x) = \frac{(x + \pi^2)^3 + b}{x}. \tag{5.26}
\]

Let \( x = \alpha_{jk}^2 \), then the right hand side of Eq. (5.18) could be expressed as \( f_{b_1}(x) \), where \( b_1 = \pi^2/\sigma^2 \text{Ro}^2 \), and the second line of Eq. (5.22) could be expressed as \( 2(\sigma + 1)f_{b_2}(x) \), where \( b_2 = \pi^2/(\sigma + 1)^2 \text{Ro}^2 \). Consider

\[
    f'_b(x) = \frac{(2x - \pi^2)(x + \pi^2)^2 - b}{x^2}. \tag{5.27}
\]

As shown in Fig. 3, it is easy to see that

(a) for \( x \in (0, \infty) \), \( f_b(x) \) has only one critical number \( x_b \);
(b) \( f'_b(x) < 0 \) if \( x < x_b \);
(c) \( f'_b(x) > 0 \) if \( x > x_b \);
(d) \( f_b(x_b) \) is the global minimum of \( f_b(x) \); and
(e) \( x_b \) is strictly increasing in \( b \), hence, \( x_{b_1} > x_{b_2} > \pi^2/2 \).

In Lemmas 5.3 and 5.4, we consider the following different conditions:

\[
    x_{b_1} \leq \alpha_1^2 < \alpha_2^2, \tag{5.28}
\]
Lemma 5.3:

1. Condition (3.6) holds true under the assumption (5.28).
2. Generically, condition (3.6) holds true under the assumption (5.29).

Proof:

1. Under the assumption (5.28), by (c), we conclude that $R_{c_1}$ is only obtained at $(j,k,l) = (1,0,1)$, i.e., $j_1 = 1$.

2. Under the assumption (5.29), there exists $j^* \geq 2$ such that $j^* \alpha_1^2 < 2x_{b_1} < (j^* + 1)^2 \alpha_1^2$. We note that

\[
(j^* + 1)^2 \alpha_1^2 = \begin{cases} 
< 2j^* \alpha_1^2 < 2x_{b_1} < \alpha_1^2 & \text{if } j^* \geq 3, \\
= 9 \alpha_1^2 < 9x_{b_1} < 2x_{b_1} < \alpha_1^2 & \text{if } j^* = 2.
\end{cases}
\]

Hence, by (b) and (c), we conclude that

\[
R_{c_1} = \min \{ f_{b_1} (j^* \alpha_1^2), f_{b_1} (j^* + 1)^2 \alpha_1^2) \},
\]

i.e., $j_1 = j^*$ or $j_1 = j^* + 1$. Note that, by (b) and (c), generically $f_{b_1} (j^* \alpha_1^2) \neq f_{b_1} (j^* + 1)^2 \alpha_1^2)$. The proof is complete.

Lemma 5.4:

1. Condition (3.7) holds true under the assumption (5.30).
2. Generically, condition (3.7) holds true under the assumption (5.31).

Proof: Consider

\[
\alpha_1^2 \leq \frac{1}{5} x_{b_1} < 2x_{b_1} < \alpha_2^2, \tag{5.29}
\]

\[
x_{b_2} \leq \alpha_1^2 < \alpha_2^2, \tag{5.30}
\]

\[
\alpha_1^2 \leq \frac{1}{5} x_{b_2} < 2x_{b_2} < \alpha_2^2. \tag{5.31}
\]
Hence, Eq. \(5.11\) is equivalent to

\[ R_{c_2} = \min_{(j,k,l) \in \Lambda_1} \{ 2(\sigma + 1) f_{c_2} (\alpha_{jk}^2) \} . \]

The rest part of the proof is the same as the proof of Lemma 5.3.

**Lemma 5.5:** Assume Eq. (3.7), \( R = R_{c_2} \) and \( R \) satisfies Eq. (5.21) for \( \{ j,k,l \} = \{ j_2,0,1 \} \), i.e., \( \text{Ro}^2 < (1-\sigma)^2/(\sigma^2(1+\sigma) \gamma_j^0_01) \), then \( \{ \beta_{j_2,01}(R), \beta_{j_2,012}(R) \} \) (\( \beta_{j_2,011}(R) = \bar{\beta}_{j_2,012}(R) \)) is the only simple pair of complex eigenvalues of the problem (5.1) in space \( \bar{H}_1 \) satisfying

\[
\text{Re} (\beta_{j_2,011}(R)) = \begin{cases} < 0 & \text{if } R < R_{c_2}, \\ \text{Re} & \text{if } R = R_{c_2}, \\ > 0 & \text{if } R > R_{c_2}, \end{cases} \tag{5.32}
\]

or

\[
\text{Re} (\beta_{j_2,012}(R)) < 0 \quad \text{for } (\alpha_{j_2,0,1}) \neq (\alpha_{j_2,0,1}), \quad q = 1,2,3, \; R \text{ near } R_{c_2}. \tag{5.33}
\]

**Proof:** We only need to prove Eq. (5.32). Under the assumptions of the lemma together with Eqs. (5.11), (5.19), and (5.20), by the discussion in case (2) at the beginning of this section, we know that \( \{ \beta_{j_2,011}(R), \beta_{j_2,012}(R) \} \) is the only simple pair of complex eigenvalues of \( L_{(j_2)} \bar{H}_1 \) with \( \text{Re}(\beta_{j_2,011}(R_{c_2})) = \text{Re}(\beta_{j_2,012}(R_{c_2})) = 0 \). Since \( \beta_{j_2,011}(R) \) (real), \( \beta_{j_2,011}(R) \), and \( \beta_{j_2,012}(R) \) are zeros of \( f_{j_2,01} \), we know that

\[
\beta_{j_2,013}(R) = - (\text{Re}(\beta_{j_2,011}(R)) + \text{Re}(\beta_{j_2,012}(R))) - (2\sigma + 1) \gamma_j^{01}. \]

Hence, Eq. (5.32) is equivalent to

\[
\beta_{j_2,013}(R) = \begin{cases} > (2\sigma + 1) \gamma_j^{01} & \text{if } R < R_{c_2}, \\ = (2\sigma + 1) \gamma_j^{01} & \text{if } R = R_{c_2}, \\ < (2\sigma + 1) \gamma_j^{01} & \text{if } R > R_{c_2}, \end{cases} \tag{5.34}
\]

which is true as shown in Fig. 4. This completes the proof.

**Lemma 5.6:** For fixed \( \alpha_1, \alpha_2 > 0 \), and \( \sigma > 1 \), \( R_{c_1} \to \infty \) as \( R \to 0 \). More precisely, \( R_{c_1} = O(R\text{Ro}^{-4/3}) \).

**Proof:** Since \( b_1 = \pi^2 / \sigma^2 \text{Ro}^2 \), by Eq. (5.27), \( x_{b_1} = O(b_1^{1/3}) \) as \( R \to 0 \). Hence,

\[
R_{c_1} = O(f_{b_1}(x_{b_1})) = O(b_1^{2/3}) = O(R\text{Ro}^{-4/3}) .
\]

\( \square \)
VI. PROOF OF MAIN THEOREMS

A. Center manifold reduction

We are now in a position to reduce Eqs. (2.1)--(2.5) to the center manifold. For any \( \psi = (U, T) \in H_1 \), we have

\[
\psi = \sum_{(j, k, l) = l_1}^{\infty} \sum_{q=1}^{3} (x_j \delta_{ij} y_{k \delta_{il}} + y_j \delta_{ij} y_{k \delta_{il}}) + \sum_{(j, k, l) = l_2}^{\infty} (x_j \delta_{ij} y_{k \delta_{il}} + y_j \delta_{ij} y_{k \delta_{il}}) + \sum_{j=1}^{\infty} \sum_{k=1}^{3} x_{k00j} \delta_{ij} y_{k \delta_{il}}.
\]

Under assumption (3.6), the first critical Rayleigh number is given by

\[
R_c = \frac{\gamma_j^0}{T_0} + \frac{\pi^2}{2 \rho \alpha_j^2}.
\]  

In this case, the multiplicity of the first eigenvalue is 2 and the reduced Eqs. (2.1)--(2.5) are given by

\[
\frac{dx_j}{dt} = \beta_j (R)x_j + \frac{1}{\gamma_j^0} \langle G(\psi, \psi), \psi_j \rangle_H
\]

\[
\frac{dy_j}{dt} = \beta_j (R)y_j + \frac{1}{\gamma_j^0} \langle G(\psi, \psi), \psi_j \rangle_H.
\]

Here for \( \psi_1 = (U_1, T_1) \), \( \psi_2 = (U_2, T_2) \), and \( \psi_3 = (U_3, T_3) \),

\[
G(\psi_1, \psi_2) = - (P(U_1 \cdot \nabla)U_2, (U_1 \cdot \nabla)T_2)^t
\]

and

\[
\langle G(\psi_1, \psi_2), \psi_3 \rangle_H = - \int_0^1 \int_0^{2\pi m_2} \int_0^{2\pi m_1} [(U_1 \cdot \nabla)U_2, U_3, (U_1 \cdot \nabla)T_2T_3] dx dy dz,
\]

where \( P \) is the Leray projection to \( L^2 \) fields. Let the center manifold function be denoted by

\[
\Phi = \sum_{\rho \neq \beta_j (0)} (\Phi_1^\rho(x_{j_1011}y_{j_1011}) \psi_1^\rho + \Phi_2^\rho(x_{j_1011}y_{j_1011}) \psi_2^\rho).
\]

The direct calculation shows that

\[
G(\psi_1^\rho, \psi_2^\rho) = - \left( 0, A_1 \pi^2, \frac{A_1 \pi^2}{2j_1 \alpha_1}, 0, \frac{A_2 \pi}{2} \sin 2 \pi z \right)^t,
\]

\[
G(\psi_2^\rho, \psi_2^\rho) = - \left( \frac{\pi^2}{2j_1 \alpha_1}, \cos 2 \pi z, \frac{A_1 \pi^2}{2j_1 \alpha_1} (\cos 2 \pi z - \cos 2 j_1 \alpha_1 x), 0, 0 \right)^t,
\]

\[
G(\psi_2^\rho, \psi_2^\rho) = - \left( \frac{\pi^2}{2j_1 \alpha_1}, \cos 2 \pi z, - \frac{A_1 \pi^2}{2j_1 \alpha_1} (\cos 2 j_1 \alpha_1 x + \cos 2 \pi z), 0, 0 \right)^t,
\]

\[
G(\psi_2^\rho, \psi_2^\rho) = - \left( 0, -\frac{A_1 \pi^2}{2j_1 \alpha_1} \sin 2 j_1 \alpha_1 x, 0, \frac{A_2 \pi}{2} \sin 2 \pi z \right)^t.
\]
By Theorem 4.2 and Eqs. (6.4) and (6.5), we obtain

\[
\Phi = \Phi_1^{(2j,00),i} \psi_1^{(2j,00),i} + \Phi_2^{(2j,00),i} \psi_2^{(2j,00),i} + \Phi_1^{(0021),i} \psi_1^{(0021),i} + o(2),
\]

where

\[
\Phi_1^{(2j,00),i} = \frac{A_1 \pi^2}{\sigma a^{(2j),0}} (x_{j,011}^2 - y_{j,011}^2) + o(2), \quad \psi_1^{(2j,00),i} = (0, -2j_1 \alpha_1 \sin 2j_1 \alpha_1 x, 0, 0)',
\]

\[
\Phi_2^{(2j,00),i} = \frac{A_1 \pi^2}{\sigma a^{(2j),0}} (2x_{0111}y_{0111}) + o(2), \quad \psi_2^{(2j,00),i} = (0, 2j_1 \alpha_1 \cos 2j_1 \alpha_1 x, 0, 0)',
\]

\[
\Phi_1^{(0021),i} = \frac{-A_2}{8 \pi} (x_{j,011} + y_{j,011}^2) + o(2), \quad \psi_1^{(0021),i} = (0, 0, 0, \sin 2 \pi \tau)'.
\]

Note that for any \( \psi_i \in H_1 \) (i=1,2,3),

\[
\langle G(\psi_1, \psi_2), \psi_3 \rangle_H = 0,
\]

(6.7)

and for any \( \psi_i \in E_{j \beta} \) (i=1,2,3),

\[
\langle G(\psi_1, \psi_2), \psi_3 \rangle_H = -\langle G(\psi_1, \psi_3), \psi_2 \rangle_H,
\]

(6.8)

The direct calculation shows that

\[
G(\tilde{\psi}, \psi_i^{(001)}) = 0 \quad \text{for} \quad \tilde{\psi} \in \{ \psi_1^{(2j,00),i}, \psi_2^{(2j,00),i}, \psi_1^{(0021),i} \}, \quad i = 1, 2.
\]

(6.10)

Then, by \( \psi = x_{j,011} \psi_1^{(001)} + y_{j,011} \psi_2^{(001)} + \Phi(x_{j,011}, y_{j,011}) \) and Eqs. (6.4) and (6.10), we derive that
\[ (G(\psi, \psi), \Psi_1^{(0,1)})_{H} = (G(\psi_1^{(0,1)}, \Phi), \Psi_1^{(0,1)})_{H} x_{j,011} + (G(\psi_2^{(0,1)}, \Phi), \Psi_1^{(0,1)})_{H} y_{j,011} + o(3) \]
\[ = - (G(\psi_1^{(0,1)}, \Phi), \Psi_1^{(0,1)})_{H} x_{j,011} - (G(\psi_1^{(0,1)}, \Phi), \Psi_1^{(0,1)})_{H} y_{j,011} + o(3) \]
\[ = - \frac{2A_1C_1}{\sigma_1 \sigma_2 \alpha_2^4(2j_{011})} \left( x_{j,011}^2 - y_{j,011}^2 \right) x_{j,011} - \frac{A_2C_2}{8 \sigma_1 \sigma_2} \left( x_{j,011}^2 + y_{j,011}^2 \right) x_{j,011} + o(3), \]
\[ - \frac{2A_1C_1}{\sigma_1 \sigma_2 \alpha_2^4(2j_{011})} \left( 2x_{j,011}y_{j,011} \right) y_{j,011} = - \left( \frac{2A_1C_1}{\sigma_1 \sigma_2 \alpha_2^4(2j_{011})} + \frac{A_2C_2}{8 \sigma_1 \sigma_2} \right) \left( x_{j,011}^2 + y_{j,011}^2 \right) y_{j,011} + o(3). \]

Similarly, we obtain
\[ (G(\psi, \psi), \Psi_2^{(0,1)})_{H} = - \left( \frac{2A_1C_1}{\sigma_1 \sigma_2 \alpha_2^4(2j_{011})} + \frac{A_2C_2}{8 \sigma_1 \sigma_2} \right) \left( x_{j,011}^2 + y_{j,011}^2 \right) y_{j,011} + o(3). \]

Hence, the reduction equations are given by
\[ \frac{d x_{j,011}}{d t} = \beta_{j,011}(R)x_{j,011} + \delta x_{j,011}^2 + y_{j,011}^2 x_{j,011} + o(3), \]
\[ \frac{d y_{j,011}}{d t} = \beta_{j,011}(R)y_{j,011} + \delta x_{j,011}^2 + y_{j,011}^2 y_{j,011} + o(3), \] (6.11)
where
\[ \delta = - \frac{2A_1C_1}{\sigma_1 \sigma_2 \alpha_2^4(2j_{011})} \left( x_{j,011}^2 + y_{j,011}^2 \right) x_{j,011} + \frac{A_2C_2}{8 \sigma_1 \sigma_2} \left( x_{j,011}^2 + y_{j,011}^2 \right) y_{j,011} + o(3). \] (6.12)

A standard energy estimate on Eqs. (6.11) together with the center manifold theory shows that, for \( R \leq R_{c_1} \), \((U, T)=0\) is locally asymptotically stable for problems (2.1)–(2.5). Hence, by Theorem 4.1, the solutions to Eqs. (2.1)–(2.5) bifurcate from \((U, T)=(0, R_{c_1})\) to an attractor \( \Sigma_R \). Moreover, by Eqs. (6.11) and (6.12) together with Theorem 5.10 in Ref. 18, we conclude that \( \Sigma_R \) is homeomorphic to \( S^1 \) in \( H \).

**B. Completion of the proof of Theorem 3.1**

In this section, we prove that \( \Sigma_R \) consists of steady state solutions. It is clear that the first eigenvalue of \( L_H^{\beta_{j,011}} \) is simple for \( R=\tilde{R}_{c_1} \). By the Kransnoselski bifurcation theorem (see among others Chow and Hale [4] and Nirenberg [21]), when \( R \) crosses \( \tilde{R}_{c_1} \), the equations bifurcate from the basic solution to a steady state solution in \( \tilde{H} \). Therefore, the attractor \( \Sigma_R \) contains at least one steady state solution. Second, it is easy to check that Eqs. (2.1)–(2.5) defined in \( H \) are translation invariant in the \( \chi \) direction. Hence, if \( \psi_0(x, y, z) = (U(x, y, z), T(x, y, z)) \) is a steady state solution, then \( \psi_0(x+\rho, y, z) \) are steady state solutions as well. By the periodic condition in the \( \chi \) direction, the set
\[ S_{\psi_0} = \{ \psi_0(x+\rho, y, z) | \rho \in \mathbb{R} \} \]
is a cycle homeomorphic to \( S^1 \) in \( H \). Therefore the steady state of Eqs. (2.1)–(2.5) generates a cycle of steady state solutions. Hence, the bifurcated attractor \( \Sigma_R \) consists of steady state solutions. The proof of Theorem 3.1 is complete.

**C. Proof of Theorem 3.2**

The proof follows directly from the classical Hopf bifurcation theorem and Lemma 5.5.
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