Superfluidity of helium-3

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Abstract

This article presents a phenomenological dynamic phase transition theory – modeling and analysis – for liquid helium-3. We derived two new models, for liquid helium-3 with or without applied field, by introducing three wave functions and using a unified dynamical Ginzburg–Landau model. The analysis of these new models leads to predictions of existence of (1) a metastable region, (2) a new phase C in a narrow region, and (3) switch points of transition types on the coexistence curves near two triple points. It is hoped that these predictions will be useful for designing better physical experiments and lead to better understanding of the physical mechanism of superfluidity.

1. Introduction

Superfluidity is a phase of matter in which "unusual" effects are observed when liquids, typically of helium-4 or helium-3, overcome friction by surface interaction at a stage, known as the "lambda point" for helium-4, at which the liquid's viscosity becomes zero. Experiments have indicated that helium atoms have two stable isotopes $^4\text{He}$ and $^3\text{He}$. $^3\text{He}$ contains two electrons, two protons and one neutron. Hence it has a fractional spin and obeys the Fermi–Dirac statistics. The liquid $^3\text{He}$ has two types of superfluid phases: phase A and phase B. In particular, if we apply a magnetic field on the liquid $^3\text{He}$, then there will be a third superfluid phase, called the A$_1$ phase.

The main objectives of this article are (1) to establish some dynamical Ginzburg–Landau models for $^3\text{He}$ with or without applied magnetic fields, and (2) to study superfluid dynamic transitions and their physical significance.

Consider the case without applied magnetic field. The modeling is based on two main ingredients as follows. First, instead of using a single wave function $\psi$, we use three wave functions to represent the Anderson–Brinkman–Morel (ABM) state and the Balian–Werthamer (BM) state. More precisely, we introduce three complex valued functions $\psi_0$, $\psi_1$, $\psi_2$ to characterize the superfluidity of $^3\text{He}$, with $\psi_0$ for the state $|\uparrow\uparrow\rangle$, $\psi_1$ for the state $|\downarrow\downarrow\rangle$, and $\psi_2$ for the state $|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle$. Then we are able to formulate a Ginzburg–Landau (GL) energy in terms of these three wave functions and the density function $\rho_0$ of the normal fluid state.

Second, we use a unified time-dependent Ginzburg–Landau model for equilibrium phase transitions, developed recently by the authors of [4,5], to derive a general time-dependent GL model for $^3\text{He}$.

In a nutshell, the model is obtained by a careful examination of the classical phase transition diagrams and by using both mathematical and physical insights offered by a recently developed dynamical transition theory as briefly recalled in the Appendix. The model of the case with applied magnetic field can be derived in the same fashion.
Fig. 1. The coexistence curve of $^3$He in the case without an applied magnetic field.

Fig. 2. Derived theoretical PT-phase diagram of $^3$He: The region $H = H_1 \cup H_2$ is the metastable domain, where the solid state and the superfluid state appear randomly depending on fluctuations. The curve $bcd$ is the first critical curve where phase transition between normal fluid and superfluid states occurs.

With the models at our disposal, we can study the dynamic phase transitions of liquid $^3$He, and derive some physical predictions.

To be precise, we first recall the classical phase transition diagrams of $^3$He, as shown in Fig. 1; see among many others Ginzburg [1], Reichl [7] and Onuki [6]. From this diagram, we see that there are two coexistence curves, with one separating the solid state and the phase A and phase B superfluid states, and the other separating the superfluid states and the normal liquid state. The transition crossing the coexistence curve between the solid and superfluid states is first order (Type-II in the sense of dynamic classification scheme given in the Appendix), and the transition crossing the coexistence curve between the superfluid and normal liquid states is second order (Type-I in the sense of dynamic classification scheme). In addition, the second transition between phase A and phase B superfluid states is first order (Type-II).

We would like to mention that the Ginzburg–Landau theory with only one wave function cannot describe this phase transition diagram, and this is one of the mains reasons that we need a new Ginzburg–Landau model as discussed above to study the phase transition dynamics for $^3$He.

The models established are analyzed using a recently developed dynamical transition theory, leading to some interesting physical predictions. Here we address briefly the new results derived. For the case without the applied magnetic field, the main results obtained are synthesized in a theoretical PT-phase diagram of $^3$He given by Fig. 2.

One prediction from our results is the existence of a metastable region $H = H_1 \cup H_2$, in which the solid state and the superfluid states $A$ and $B$ appear randomly depending on fluctuations. In particular, in $H_1$, the phase $B$ superfluid state and the solid may appear, and in $H_2$ the phase $A$ superfluid state and the solid state may appear.

Another prediction is the possible existence of phase $C$ superfluid state, which is characterized by the wave function $\psi_2$, representing $| \uparrow \downarrow \rangle + | \downarrow \uparrow \rangle$. However, phase $C$ region is very narrow, which may be the reason why it is hard to be observed in experiments.

Also, the results predict that near the two triple points $b$ and $c$, there is a possibility of the existence of two switch points, where the transition on the corresponding coexistence curve switches types at each switch point. The existence of such switch points depends on the physical parameters.
One important new ingredient for the analysis is a dynamic transition theory developed recently by the authors [2,3]. With this theory, we derive a new dynamic phase transition classification scheme, which classifies phase transitions into three categories: Type-I, Type-II and Type-III, corresponding respectively to the continuous, the jump and mixed transitions in the dynamic transition theory.

The case with applied field can be addressed in a similar fashion; see Section 5.

This article is organized as follows. Section 2 introduces a new dynamic model for liquid helium-3 without applied magnetic field. The phase transition dynamics is given in Section 3, and the types of transitions in different regimes are given in Section 4. Section 5 deals with the case with applied magnetic field, and Section 6 gives a summary on the physical predictions of the new models.

2. Dynamic model for liquid $^3$He with zero applied magnetic field

The superfluidity of liquid $^3$He was found in 1971 by D.M. Lee, D.D. Osheroff, and R.C. Richardson, and its transition temperature is $T \sim 10^{-3}$ K under $p = 1$ atm ($10^5$ Pa). There are two superfluid phases A and B if there is no applied magnetic field. Fig. 1 provides the phase diagram of the liquid $^3$He in the PT-plane.

Since the atoms $^3$He are fermions, to form the superfluid phase they must be paired to become bosons. When no magnetic field is applied, there are two pairing states: the Anderson–Brinkman–Morel (ABM) state and the Balian–Werthamer (BW) state, given respectively by

$$\sqrt{2}\tilde{\phi} = (a + ib)|↑↑⟩ + (a - ib)|↓↓⟩,$$

$$\sqrt{2}\tilde{\phi} = (a + ib)|↑↑⟩ + (a - ib)|↓↓⟩ + c|↑↓⟩ + |↓↑⟩].$$

The ABM state corresponds to the superfluid phase A, and the BW state to the phase B. Also, we call state $|↑↓⟩ + |↓↑⟩$ as phase C, which appears in the theory developed in this article.

We introduce three complex valued functions $\psi_0, \psi_1, \psi_2$ to characterize the superfluidity of $^3$He, in which $\psi_0$ represents the state $|\uparrow\uparrow⟩$, $\psi_1$ the state $|\downarrow\downarrow⟩$, and $\psi_2$ the state $|↑↓⟩ + |↓↑⟩$.

Let $\rho_n$ be the normal fluid density, $\rho_a, \rho_b$ and $\rho_c$ represent the densities of superfluid phases A, B and C respectively. Then we have

$$\rho_n = \tau_0|\psi_0|^2 + \tau_1|\psi_1|^2 \quad (\tau_0 > 0, \tau_1 > 0),$$

$$\rho_a = \tau_2|\psi_0|^2 + \tau_3|\psi_1|^2 + \tau_4|\psi_2|^2 \quad (\tau_2, \tau_3 > 0, \tau_4 > 0),$$

$$\rho_c = \tau_5|\psi_2|^2 \quad (\tau_5 > 0).$$

The total density of $^3$He is given by

$$\rho = \begin{cases} 
\rho_n & \text{in the normal state,} \\
\rho_n + \rho_a & \text{in the phase A state,} \\
\rho_n + \rho_b & \text{in the phase B state,} \\
\rho_n + \rho_c & \text{in the phase C state.}
\end{cases}$$

Physically, the states $\psi_0, \psi_1$ and $\psi_2$ are independent, and consequently there are no coupling terms $|\nabla(\psi_i + \psi_j)|^2$ and $|\psi_i + \psi_j|^2$ in the free energy density. Since in the case without applied magnetic field $\psi_0$ and $\psi_1$ are equal, their coefficients in the free energy should be the same. Thus, we have the following Ginzburg–Landau free energy for $^3$He with $H = 0$:

$$G(\psi_0, \psi_1, \psi_2, \rho_n) = \frac{1}{2} \int_{\Omega} \left[ k_1 \frac{h^2}{m} |\nabla \psi_0|^2 + \alpha_1 |\psi_0|^2 + \alpha_2 |\psi_n|^2 + \frac{\alpha_3}{2} |\psi_0|^4 + \frac{k_1 h^2}{m} |\nabla \psi_1|^2 + \alpha_1 |\psi_1|^2 + \alpha_2 |\psi_n|^2 + \frac{\alpha_3}{2} |\psi_1|^4 + \frac{k_2 h^2}{m} |\nabla \psi_2|^2 + \beta_1 |\psi_2|^2 + \frac{\beta_2}{4} |\psi_2|^4 + \beta_3 |\psi_0|^2 |\psi_2|^2 + \beta_4 |\psi_0|^2 |\psi_2|^2 + \beta_5 |\psi_n|^2 |\psi_2|^2 + 2 \mu_2 \rho_n^4 + \frac{\mu_3}{2} \rho_n^3 - p \left( \rho_n + \frac{\mu_0}{2} \rho_n^2 \right) \right] dx,$$  \hspace{1cm} (1)

where the coefficients depend on $T$ and $p$, and for $1 \leq i \leq 3, j = 2, 3, 4$,

$$k_1 > 0, \quad \beta_1 > 0, \quad \alpha_2 > 0, \quad \alpha_3 > 0, \quad \mu_3 > 0, \quad \mu_2 < 0.$$  \hspace{1cm} (2)

For $\alpha_1, \beta_1$ and $\mu_1$, there are regions $A_i, B_i, C_i$ ($i = 1, 2$) in the PT-plane $\mathbb{R}_+^2$ such that $\Lambda_1 + \Lambda_2 = \tilde{B}_1 + \tilde{B}_2 = \tilde{C}_1 + \tilde{C}_2 = \mathbb{R}_+^2$, and

$$\alpha_1 = \alpha_1(T, p) \begin{cases} 
> 0 & \text{if } (T, p) \in A_1, \\
< 0 & \text{if } (T, p) \in A_2.
\end{cases}$$  \hspace{1cm} (3)
\[ \beta_1 = \beta_1(T, p) \begin{cases} > 0 & \text{if } (T, p) \in B_1, \\ < 0 & \text{if } (T, p) \in B_2, \end{cases} \] (4)

\[ \mu_1 = \mu_1(T, p) \begin{cases} > 0 & \text{if } (T, p) \in C_1, \\ < 0 & \text{if } (T, p) \in C_2. \end{cases} \] (5)

It is known that for $^3\text{He}$, $\mu_1 = \mu_1(T, p)$ is not monotone on $T$. In fact, at $T_m = 0.318 \, \text{K}$, $p_m = 29.31 \times 10^5 \, \text{Pa}$, we have

\[ \mu_1(T_m, p_m) = 0, \quad \frac{\partial \mu_1(T_m, p_m)}{\partial T} = 0, \] (6)

where $m = (T_m, p_m)$ is as shown in Fig. 1. Near the point $m$ the famous Pomeranchuk effect takes place, i.e., when pressure increases, the liquid $^3\text{He}$ will absorb heat to undergo a transition to solid state.

By the normalized model \((A.6)\), we infer from \((1)\) the following time-dependent GL equations for the superfluidity of liquid $^3\text{He}$:

\[ \begin{align*}
\frac{\partial \psi_0}{\partial t} &= \frac{k_1h^2}{m} \Delta \psi_0 - \alpha_1 \psi_0 - \alpha_2 \rho_1 \psi_0 - \beta_3 |\psi_2|^2 \psi_0 - \alpha_3 |\psi_0|^2 \psi_0, \\
\frac{\partial \psi_1}{\partial t} &= \frac{k_1h^2}{m} \Delta \psi_1 - \alpha_1 \psi_1 - \alpha_2 \rho_1 \psi_1 - \beta_3 |\psi_2|^2 \psi_1 - \alpha_3 |\psi_1|^2 \psi_1, \\
\frac{\partial \psi_2}{\partial t} &= \frac{k_1h^2}{m} \Delta \psi_2 - \beta_1 \psi_2 - \beta_3 |\psi_0|^2 \psi_2 - \beta_3 |\psi_1|^2 \psi_2 - \beta_4 \rho_2 \psi_2 - \beta_2 |\psi_2|^2 \psi_2, \\
\frac{\partial \rho_n}{\partial t} &= \frac{k_3 \Delta \rho_n - (\mu_1 - \mu_0 \rho_0) \rho_n - \mu_2 \rho_n^2 - \mu_3 \rho_n^3 - \frac{\alpha_2}{2} |\psi_0|^2 - \frac{\alpha_2}{2} |\psi_1|^2 - \frac{\beta_4}{2} |\psi_2|^2 - p. \end{align*} \] (7)

The nondimensional form of \((7)\) can be written as

\[ \begin{align*}
\frac{\partial \psi_0}{\partial t} &= \Delta \psi_0 + \lambda_1 \psi_0 - a_1 \rho_0 \psi_0 - a_2 |\psi_2|^2 \psi_0 - a_3 |\psi_0|^2 \psi_0, \\
\frac{\partial \psi_1}{\partial t} &= \Delta \psi_1 + \lambda_1 \psi_1 - a_1 \rho_0 \psi_1 - a_2 |\psi_2|^2 \psi_1 - a_3 |\psi_1|^2 \psi_1, \\
\frac{\partial \psi_2}{\partial t} &= \kappa_1 \Delta \psi_2 + \lambda_2 \psi_2 - b_1 \rho_0 \psi_2 - b_2 |\psi_0|^2 \psi_2 - b_3 |\psi_1|^2 \psi_2 - b_3 |\psi_2|^2 \psi_2, \\
\frac{\partial \rho_n}{\partial t} &= \kappa_2 \Delta \rho_n + \lambda_3 \rho_n - c_1 |\psi_0|^2 - c_1 |\psi_1|^2 - c_2 |\psi_2|^2 + c_3 \rho_n^2 - c_4 \rho_n^3, \end{align*} \] (8)

where

\[ \begin{align*}
\lambda_1 &= -\frac{m^2}{h^2 k_1} (\alpha_1 + \alpha_2 \rho_0^0), \\
\lambda_2 &= -\frac{m^2}{h^2 k_1} (\beta_1 + b_1 \rho_0^0), \\
\lambda_3 &= -\frac{m^2}{h^2 k_1} (\mu_1 - \mu_0 \rho_0^0 \mu_2 - 3(\rho_0^0)^2 \mu_3). \end{align*} \] (9)

and $\rho_0^0$ is determined by the state state solution of the form $(\psi_0, \psi_1, \psi_2, \rho_n) = (0, 0, 0, \rho_0^0)$ of \((7)\). By \((2)\) the coefficients in \((8)\) satisfy

\[ a_i > 0, \quad b_i > 0, \quad c_j > 0, \quad \forall 1 \leq i \leq 3, \, 1 \leq j \leq 4. \]

The boundary conditions are the Neumann conditions

\[ \frac{\partial (\psi_0, \psi_1, \psi_2, \rho_n)}{\partial n} = 0 \quad \text{on} \quad \partial \Omega. \] (10)

When $p$ is a constant, the problem \((8)\) with \((10)\) can be reduced to the following system of ordinary differential equations:

\[ \begin{align*}
\frac{d \rho_1}{d t} &= \lambda_1 \rho_1 - a_1 \rho_0 \rho_1 - a_2 \rho_2 \rho_1 - a_3 \rho_1^2, \\
\frac{d \rho_2}{d t} &= \lambda_2 \rho_2 - b_1 \rho_0 \rho_2 - b_2 \rho_1 \rho_2 - b_3 \rho_2^2, \\
\frac{d \rho_n}{d t} &= \lambda_3 \rho_n - c_1 \rho_1 - c_2 \rho_2 + c_3 \rho_n^2 - c_4 \rho_n^3, \end{align*} \] (11)

where $\rho_1 = |\psi_0|^2 + |\psi_1|^2$ and $\rho_2 = |\psi_2|^2$. 
3. Critical parameter curves and PT-phase diagram

3.1. Critical parameter curves

Critical parameter curves in the PT-plane are given by
\[
l_i = \{(T, p) \in \mathbb{R}_+^2 | \lambda_i(T, p) = 0\}, \quad i = 1, 2, 3,
\]
where \(\lambda_i = \lambda_i(T, p)\) are defined by (9).

It is clear that the critical parameter curves \(l_i\) are associated with the PT-phase diagram of \(^3\)He. As in the last section if we can determine the critical parameter curves \(l_i\) (1 \(\leq i \leq 3\)), then we obtain the PT-phase diagram.

Phenomenologically, according to the experimental PT-phase diagram (Fig. 1), the parameter curves \(l_i\) \((i = 1, 2, 3)\) in the PT-plane should be as schematically illustrated in Fig. 3(a)–(c). The combination of the diagrams (a)–(c) in Fig. 3 gives Fig. 4, in which the real line \(\overline{bm}\) stands for the coexistence curve of the solid and liquid phases, and \(\overline{bcd}\) for the coexistence curve of superfluid and normal liquid phases.

Now we rigorously examine phase transitions in different regimes determined by the equations and the critical parameter curves.
3.2. States in the metastable region

We consider the dynamical properties of transitions for (11) in the metastable region. It is clear that at point \( b = (T_b, p_b) \),
\[
\lambda_1(T_b, p_b) = \lambda_3(T_b, p_b) = 0, \quad \lambda_2(T_b, p_b) < 0,
\]
and the metastable region \( H_1 \) near the triple point \( b \) is defined by
\[
H_1 = \{(T, p) \in \mathbb{R}^2 \mid (\lambda_1, \lambda_2, \lambda_3)(T, p) = (+, -, +)\}.
\]

To study the structure of flows of (11) for \( (T, p) \in H \) it is necessary to consider Eqs. (11) at the point \( b = (T_b, p_b) \), and (12) which are given by
\[
\frac{d\rho_1}{dt} = -a_1 \rho_n \rho_1 - a_3 \rho_1^2,
\]
\[
\frac{d\rho_n}{dt} = -c_1 \rho_1 + c_3 \rho_n^2 - c_4 \rho_n^3.
\]

We know that
\[
c_3 > 0 \quad \text{for} \quad (T, p) \subset H_1.
\]

Eqs. (13) have the following two steady state solutions:
\[
Z_1 = (\rho_1, \rho_n) = (0, c_3/a_4),
\]
\[
Z_2 = (\rho_1, \rho_n) = \left(\frac{a_1}{a_3} \alpha, -\alpha\right),
\]
\[
\alpha = \frac{c_3}{2c_4} \left(\sqrt{1 + \frac{4c_1a_1c_4}{a_3c_3^2}} - 1\right).
\]

By direct computation, we can prove that the eigenvalues of the Jacobian matrices of (13) at \( Z_1 \) and \( Z_2 \) are negative. Hence, \( Z_1 \) and \( Z_2 \) are stable equilibrium points of (13). Physically, \( Z_1 \) stands for solid state, and \( Z_2 \) for superfluid state. The topological structure of (13) is schematically illustrated by Fig. 5(a), the two regions \( R_1 \) and \( R_2 \) divided by curve \( AO \) in Fig. 5(b) are the basins of attraction of \( Z_1 \) and \( Z_2 \) respectively.

We note that in \( H_1 \), \( \lambda_1 \) and \( \lambda_3 \) are small, i.e.,
\[
0 < \lambda_1(T, p), \quad \lambda_2(T, p) \ll 1, \quad \text{for} \quad (T, p) \in H,
\]
and (13) can be considered as a perturbed system of (11).

Thus, for \( (T, p) \in H_1 \) the system (11) has four steady state solutions \( \tilde{Z}_i = \tilde{Z}(T, p) \) \( (1 \leq i \leq 4) \) such that
\[
\lim_{(T, p) \to (T_c, p_c)} (\tilde{Z}_1, \tilde{Z}_2, \tilde{Z}_3, \tilde{Z}_4)(T, p) = (Z_1, Z_2, 0, 0),
\]
and \( \tilde{Z}_1 \) and \( \tilde{Z}_2 \) are stable, representing solid state and liquid He-3 state respectively, \( \tilde{Z}_3 \) and \( \tilde{Z}_4 \) are two saddle points. The topological structure of (13) for \( (T, p) \in H_1 \) is schematically shown in Fig. 5(c), and the basins of attraction of \( \tilde{Z}_1 \) and \( \tilde{Z}_2 \) are \( R_1 \) and \( R_2 \) as illustrated by Fig. 5(d).

3.3. First phase transition

On the coexistence curve \( \tilde{bc} \),
\[
\lambda_1 = 0, \quad \lambda_2 < 0, \quad \lambda_3 < 0.
\]
Hence the first phase transition crossing \( \tilde{bc} \) is between normal fluid state and phase A superfluid state.

On the coexistence curve \( cd \),
\[
\lambda_1 < 0, \quad \lambda_2 = 0, \quad \lambda_3 < 0.
\]
In this case, the first phase transition crossing this coexistence curve is between the normal fluid state and the phase C superfluid state.
3.4. Second phase transitions

When \((T, p)\) crosses the curve segment \(bcd\), (11) will undergo a second transition. We need to consider two cases.

**Second transition crossing \(f'c\)**

If \((T, p)\) passes through this curve segment \(bc\), then the first transition solution is given by

\[
(\rho_1, \rho_2, \rho_n) = (\rho_1^*, 0, \rho_n^*),
\]

\[
\rho_1^* = \rho_1^*(T, p) > 0,
\]

\[
\rho_n^* = \rho_n^*(T, p) < 0.
\]

Take a transformation

\[
\rho'_1 = \rho_1 - \rho_1^*, \quad \rho'_2 = \rho_2, \quad \rho'_n = \rho_n - \rho_n^*.
\]

Then, system (11) is in the following form (for simplicity, we drop the primes):

\[
\begin{align*}
\frac{d\rho_1}{dt} &= \tilde{\lambda}_1 \rho_1 - a_1 \rho_1^* \rho_n - a_2 \rho_1^* \rho_2 - a_2 \rho_1 \rho_2 - a_3 \rho_1^3, \\
\frac{d\rho_2}{dt} &= \tilde{\lambda}_2 \rho_2 - b_1 \rho_1 \rho_2 - b_2 \rho_1^2 \rho_2 - b_3 \rho_2^2, \\
\frac{d\rho_n}{dt} &= \tilde{\lambda}_3 \rho_n - c_1 \rho_1 - c_2 \rho_2 + (c_3 - 3c_4 \rho_n^*) \rho_n^2 - c_4 \rho_n^3,
\end{align*}
\]
where

\[
\tilde{\lambda}_1 = \lambda_1 + a_1|\rho_n^*| - 2a_3\rho_n^*, \\
\tilde{\lambda}_2 = \lambda_2 + b_1|\rho_n^*| - b_2\rho_n^*, \\
\tilde{\lambda}_3 = \lambda_3 - 2c_3|\rho_n^*| - 3c_4\rho_n^{2*}.
\]

(16)

The linear operator in (15) reads

\[
L = \begin{pmatrix}
\tilde{\lambda}_1 & -a_2\rho_n^* & -a_1\rho_n^* \\
0 & \tilde{\lambda}_2 & 0 \\
-c_1 & -c_2 & \tilde{\lambda}_3
\end{pmatrix}.
\]

The three eigenvalues of \(L\) are

\[
\beta_1 = \tilde{\lambda}_2, \quad \beta_{\pm} = \frac{1}{2} \left[ \tilde{\lambda}_1 + \tilde{\lambda}_3 \pm \sqrt{(\tilde{\lambda}_3 - \tilde{\lambda}_1)^2 + 4a_1c_1\rho_n^*} \right].
\]

(17)

It is known that the transition solution \((\rho_1^*, 0, \rho_n^*)\) is stable near \(\hat{bc}\). Therefore the eigenvalues of \(L\) satisfy

\[
\beta_1(T, p) < 0, \quad \beta_{\pm}(T, p) < 0 \quad \text{for} \ (T, p) \ \text{near} \ \hat{bc}.
\]

However, near \(\hat{fc}\) there is a curve segment \(\hat{f}c\) such that

\[
\hat{f}c = \{(T, p) \in \mathbb{R}_+^2 | \beta_1(T, p) = 0 \}.
\]

Thus, system (15) has a transition on \(\hat{f}c\), which is called the second transition of (11), and \(\hat{f}c\) is the coexistence curve of phases A and B; see Fig. 2.

Second transition crossing \(\hat{ch}'\)

If \((T, p)\) passes through this curve segment \(\hat{bc}\), then the first transition solution is given by

\[
(\rho_1, \rho_2, \rho_n) = (0, \eta_2, \eta_n), \\
\eta_2 = \eta_2(T, p) > 0, \\
\eta_n = \eta_n(T, p) < 0.
\]

Take a transformation

\[
\rho'_1 = \rho, \quad \rho'_2 = \rho_2 - \eta_2, \quad \rho'_n = \rho_n - \eta_n.
\]

Then system (11) is in the following form (for simplicity, we drop the primes):

\[
\begin{align*}
\frac{d\rho_1}{dt} & = \tilde{\lambda}_1\rho_1 - a_1\rho_1\rho_n - a_2\rho_1\rho_2 - a_3\rho_1^2, \\
\frac{d\rho_2}{dt} & = \tilde{\lambda}_2\rho_2 - b_1\eta_2\rho_n - b_2\eta_2\rho_1 - b_1\rho_2\rho_n - b_2\rho_1\rho_2 - b_3\rho_2^2, \\
\frac{d\rho_n}{dt} & = \tilde{\lambda}_3\rho_n - c_1\rho_1 - c_2\rho_2 + (c_3 - 3c_4\eta_n)\rho_n^2 - c_4\rho_n^3,
\end{align*}
\]

(18)

where

\[
\tilde{\lambda}_1 = \lambda_1 + a_1|\eta_n| - a_2\eta_2, \\
\tilde{\lambda}_2 = \lambda_2 + b_1|\eta_n| - 2b_2\eta_2, \\
\tilde{\lambda}_3 = \lambda_3 - 2c_3|\eta_n| - 3c_4\eta_n^2.
\]

(19)

The linear operator in (18) reads

\[
L = \begin{pmatrix}
\tilde{\lambda}_1 & 0 & -0 \\
-b_2\eta_2 & \tilde{\lambda}_2 & -b_1\eta_2 \\
-c_1 & -c_2 & \tilde{\lambda}_3
\end{pmatrix}.
\]

The three eigenvalues of \(L\) are

\[
\beta_1 = \tilde{\lambda}_1, \quad \beta_{\pm} = \frac{1}{2} \left[ \tilde{\lambda}_1 + \tilde{\lambda}_3 \pm \sqrt{(\tilde{\lambda}_3 - \tilde{\lambda}_1)^2 + 4b_1c_2\eta_2} \right].
\]

(20)
It is known that the transition solution \((\rho^*_1, 0, \rho^*_n)\) is stable near \(\hat{bc}\). Therefore the eigenvalues of \(L\) satisfy
\[
\beta_1(T, p) < 0, \quad \beta_{\pm}(T, p) < 0 \quad \text{for} \quad (T, p) \text{ near } \hat{cd}.
\]
However, near \(\hat{hc}\) there is a curve segment \(\hat{hc}\) such that
\[
\hat{hc} = \{(T, p) \in \mathbb{R}_+^2 | \beta_1(T, p) = \hat{\sim} \lambda_1 = 0\}.
\]
Thus, system (18) has a transition on \(\hat{hc}\), which is called the second transition of (11), and \(\hat{hc}\) is the coexistence curve of phases C and B; see Fig. 2.

In summary, with the above analysis and the dynamic transition theory, we arrive at the following transition theorem:

**Theorem 1.** Define a few regions in the PT-plane (see Fig. 2) by
\[
\begin{align*}
E_1 &= \{(T, p) \in \mathbb{R}_+^2 | (\lambda_1, \lambda_2, \lambda_3)(T, p) = (-, -, +)\}, \\
E_2 &= \{(T, p) \in \mathbb{R}_+^2 | (\lambda_1, \lambda_2, \lambda_3)(T, p) = (-, -, -)\}, \\
H_1 &= \{(T, p) \in \mathbb{R}_+^2 | (\lambda_1, \tilde{\lambda}_2, \lambda_3)(T, p) = (+, -)\}, \\
H_2 &= \{(T, p) \in \mathbb{R}_+^2 | (\lambda_1, \tilde{\lambda}_2, \lambda_3)(T, p) = (+, +, +)\}, \\
Region f'bc &= \{(T, p) \in \mathbb{R}_+^2 | (\lambda_1, \lambda_2, \lambda_3)(T, p) = (+, -, -)\}, \\
Region cdh' &= \{(T, p) \in \mathbb{R}_+^2 | (\lambda_1, \tilde{\lambda}_2, \lambda_3)(T, p) = (-, -, +)\}.
\end{align*}
\]
and let the Region Og'f'ch' be the complement of the sum of the above regions. Then the following conclusions hold true:

1. If \((T, p) \in E_1\), the phase of \(^3\)He is in solid state.
2. If \((T, p) \in E_2\), the phase is in normal liquid state.
3. If \((T, p) \in Region f'bc\), the phase is in phase A superfluid state.
4. If \((T, p) \in Region cdh'\), the phase is in phase C superfluid state.
5. If \((T, p) \in Region Og'f'ch'\), the phase is in phase B superfluid state.
6. If \((T, p) \in H_1\), there are two regions \(R_1\) and \(R_2\) in the state space \((\rho_1, \rho_2, \rho_n)\) such that, under a fluctuation which is described by the initial value \((x_0, y_0, z_0)\) in (11): If \((x_0, y_0, z_0) \in R_1\) then the phase is in solid state, and if \((x_0, y_0, z_0) \in R_2\) then it is in phase A superfluid state.
7. If \((T, p) \in H_2\), there are two regions \(K_1\) and \(K_2\) in the state space \((\rho_1, \rho_2, \rho_n)\) such that if \((x_0, y_0, z_0) \in K_1\) then the phase is in solid state, and if \((x_0, y_0, z_0) \in K_2\) then it is in phase B superfluid state.

4. **Classification of superfluid transitions**

In this section, we classify the superfluid transitions of (11) crossing various coexistence curves. First we consider the transitions crossing curve segments \(cd\) and \(bc\) in Fig. 2. Obviously, we have
\[
\begin{align*}
\hat{cd} &= \{(T, p) \in \mathbb{R}_+^2 | (\lambda_1, \lambda_2, \lambda_3)(T, p) = (-, 0, -)\}, \\
\hat{bc} &= \{(T, p) \in \mathbb{R}_+^2 | (\lambda_1, \lambda_2, \lambda_3)(T, p) = (0, 0, -)\}.
\end{align*}
\]
Let
\[
\begin{align*}
A_1 &= a_1c_1 - a_3|\lambda_3|, \quad A_2 = a_1c_2 - a_2|\lambda_3|, \\
B_1 &= b_1c_2 - b_3|\lambda_3|, \quad B_2 = b_1c_1 - b_2|\lambda_3|.
\end{align*}
\]
In fact, it is obvious that \(A_2 = B_2\).

**Theorem 2.** For system (11) we have the following assertions:

1. As \((T_0, p_0) \in \hat{cd}\), the transition of (11) at \((T_0, p_0)\) is between the phase B and normal liquid. Furthermore if \(B_1 \leq 0\), then it is Type- I, and if \(B_1 > 0\), then it is Type- II.
2. As \((T_0, p_0) \in \hat{bc}\), the transition is between the phase A and normal liquid. Moreover, if \(A_1 \leq 0\), then it is Type- I, and if \(A_1 > 0\), then it is Type- II.

**Theorem 2** provides conditions for the first transition of (11). The following theorem gives sufficient conditions for the second transition near \(\hat{fc}\) in Fig. 2.

Obviously, the curve \(\hat{fc}\) is given by
\[
\hat{fc} = \{(T, p) \in \mathbb{R}_+^2 | (\lambda_1, \lambda_2, \lambda_3)(T, p) = (+, 0, -)\}.
\]
To set up the second transition theorem, we need to assume the following conditions. Let \(\varepsilon > 0\) be small. Suppose that
\[
B_2 > -\varepsilon, \quad b_1a_3 - b_2a_1 > -\varepsilon \quad \text{for} \quad (T, p) \in \hat{fc},
\]
(21)
and the gap between \( \hat{bc} \) and \( \hat{fc} \) is small, i.e.,

\[
|T_2 - T_3| = O(\varepsilon) \quad \forall (T_2, p) \in \hat{bc}, \ (T_3, p) \in \hat{fc}.
\]  

(22)

We also assume that

\[
(a_1, c_1) = O(\varepsilon), \quad (b_1, c_2, c_3, a_3, b_3) = O(1),
\]

(23)

\[
a_3 B_1 - b_2 A_2 > 0 \quad \text{in } \hat{fc} \text{ with } A_1 \leq 0.
\]

(24)

**Theorem 3.** Under conditions (21) and (22), there exists a curve segment \( \hat{f}'c \) near \( \hat{fc} \) as shown in Fig. 2 such that (11) has the second transition from the first transition solution \((\rho_1^*, 0, \rho_2^*), \) i.e., (15) has a transition from \((\rho_1, \rho_2, \rho_n) = 0 \) in \( \hat{f}'c, \) and the transition solutions \((\tilde{\rho}_1, \tilde{\rho}_2, \tilde{\rho}_n) \) satisfy that \( \tilde{\rho}_2 > 0. \) In addition, if (23) and (24) hold true, then this transition is Type-II.

Physical experiments display that the superfluid transition of liquid \( ^3\text{He} \) between the normal liquid and superfluid phase \( B \) is continuous. Hence, it is necessary to give the conditions of Type-I transition of (11) at the intersecting point \( C \) of two curves \( \lambda_1 = 0 \) and \( \lambda_2 = 0. \)

**Theorem 4.** Let \((T_0, p_0)\) be the point \( C \) that \( \lambda_1(T_0, p_0) = 0 \) and \( \lambda_2(T_0, p_0) = 0. \) Then the transition of (11) at \((T_0, p_0)\) is Type-I if and only if one of the following two conditions hold true:

(i) \( A_1 \leq 0, \ B_1 \leq 0, \ A_2 = B_2 < 0, \)

(ii) \( A_1 \leq 0, \ B_1 = 0, \ A_2 = B_2 \geq 0 \) and \( A_1 B_1 > A_2 B_2. \)

In particular, if the transition is Type-I, then for \( \lambda_1 > 0, \lambda_2 > 0 \) near \((p_0, T_0), \) there are four types of topological structures of the transitions on center manifold, which are classified as follows:

1. The transition is of the structure as shown in Fig. 6(a), if

   \[
   \lambda_1 |B_1| + \lambda_2 A_2 > 0 \quad \text{and} \quad \lambda_2 |A_1| + \lambda_1 B_2 > 0.
   \]

2. The transition is of the structure as shown in Fig. 6(b), if

   \[
   \lambda_1 |B_1| + \lambda_2 A_2 < 0 \quad \text{and} \quad \lambda_2 |A_1| + \lambda_1 B_2 < 0.
   \]

3. The transition is of the structure as shown in Fig. 6(c), if

   \[
   \lambda_1 |B_1| + \lambda_2 A_2 < 0 \quad \text{and} \quad \lambda_2 |A_1| + \lambda_1 B_2 > 0.
   \]
(4) The transition has the structure as shown in Fig. 6(d), if
\[
\lambda_1 |B_1| + \lambda_2 A_2 > 0 \quad \text{and} \quad \lambda_2 |A_1| + \lambda_1 B_2 < 0.
\]

Before the proof of these theorems, we give the following remark.

**Remark 1.** Physically, the transition between normal liquid and superfluid for $^3$He is generally Type-I, the transition between superfluid phases $A$ and $B$ is Type-II, and the region of phase $A$ is narrow. Therefore, under conditions (21)–(24) and
\[
B_1 < 0 \quad \text{in } \mathbb{C},
\]
\[
B_1 \leq 0, \quad A_1 < 0 \quad \text{near point } C = \tilde{T}_1 \cap \tilde{T}_2 = (T_0, p_0),
\]
the above theorems (**Theorems 2–4**) provide a precise mathematical proof for superfluid transitions of liquid $^3$He with no applied magnetic field.

By condition (23) we see that
\[
a_1 \cdot c_3 = O(e^2), \quad a_3 = O(1).
\]

Assertion (1) of **Theorem 2** implies that only in a very small range of $(T, p)$ near the change point of superfluid and solid, the transition between normal liquid and superfluid is II-type, and this range is
\[
0 > \lambda_3(T, p) > -\frac{a_1 c_1}{a_3} = -O(e^2).
\]

Moreover, the superfluid density of phase $A$ near the solid phase is in the quantitative order $e^3$, i.e.,
\[
\rho_1 = \frac{a_1^2 c_1}{a_2^2 c_2} = O(e^3).
\]

Hence, difference between the Type-I and the Type-II phase transitions in experiments is very small.

Note that **Theorem 3** is also valid if condition (21) is replaced by
\[
\frac{\partial \lambda_2}{\partial T} \gg 1 \quad \text{for } (T, p) \text{ near } \hat{c}.
\]

**Proof of Theorem 2.** As $(T_0, p_0) \in \hat{b}c, \lambda_2(T_0, p_0) < 0$ and the space of $(\rho_1, 0, \rho_n)$ is invariant for (11). Therefore, the transition equations of (11) at $(T_0, p_0)$ are referred to the following form
\[
\begin{align*}
\frac{d\rho_1}{dt} &= \lambda_1 \rho_1 - a_1 \rho_n \rho_1 - a_3 \rho_1^2, \\
\frac{d\rho_2}{dt} &= \lambda_2 \rho_2 - c_1 \rho_1 + c_3 \rho_n - c_4 \rho_3, \\
\frac{d\rho_n}{dt} &= \lambda_3 \rho_n - c_1 \rho_1 + c_3 \rho_n^2 - c_4 \rho_3.
\end{align*}
\]

The second order approximation of the center manifold function $\rho_n$ of (25) satisfies the equation
\[
\lambda_3 \rho_n + c_3 \rho_n^2 = c_1 \rho_1.
\]

Its solution is
\[
\rho_n = \frac{c_1 \rho_1}{|\lambda_3|} + \frac{c_1^2 c_3}{|\lambda_3|^3} \rho_1^2 + o(\rho_1^2).
\]

Putting $\rho_n$ in the first equation of (25) we get the reduced equation of (11) on center manifold as follows
\[
\frac{d\rho_1}{dt} = \lambda_1 \rho_1 + \frac{1}{|\lambda_3|} A_1 \rho_1^2 - \frac{a_1 c_1 c_3}{|\lambda_3|^3} \rho_1^3 + o(\rho_1^3).
\]

Assertion (2) follows from (26).

Likewise, if $(T_0, p_0) \in \tilde{c}d, \lambda_1(T_0, p_0) < 0$ and the space of $(0, \rho_2, \rho_n)$ is invariant for (11), therefore in the same fashion as above we can prove Assertion (1). The proof is complete. ■

**Proof of Theorem 3.** We proceed with the following two cases.

**Case 1:** $A_1 \leq 0$ in $bc$. In this case, by **Theorem 2**, the transition of (11) in $bc$ is Type-I, and the transition solution $(\rho_1^*, 0, \rho_n^*)$ satisfies that
\[
\rho_n^* = \frac{c_1}{|\lambda_3|} \rho_1^*.
\]
The equations describing the second transition are given by (15) and the eigenvalues in (16) and (17) are rewritten as

\[ \beta_1 = \lambda_2 + b_1 |\rho_n^*| - b_2 \rho_1^* = \lambda_2 + \frac{1}{|\lambda_3|} B_2 \rho_1^*, \tag{27} \]

\[ \beta_+ = \frac{1}{2} [\tilde{\lambda}_1 + \tilde{\lambda}_3 + \sqrt{(\lambda_1 - \tilde{\lambda}_3)^2 + 4a_1c_1 \rho_1^*}] = (\text{by (22)} \text{ and } \rho_1^*, \tilde{\lambda}_1 \text{ being small}) \]

\[ = -\lambda_1 + o(|\lambda_1|), \tag{28} \]

\[ \beta_1 < \beta_+. \tag{29} \]

In addition, we know that

\[ \lambda_1 = 0, \quad \lambda_2 < 0 \quad \text{on } \tilde{c}, \]

\[ \lambda_2 (T - \delta, p) > 0 \quad \text{for } (T, p) \in \tilde{c} \text{ and } \delta > 0. \]

Hence, by assumptions (21) and (22), from (27) to (29) we can infer that there exists a curve segment \( \tilde{c}c \) near \( \tilde{c}c' \) such that for \( (T_2, p) \in \tilde{c}c \) and \( (T_0, p) \in \Gamma_3' \) we have

\[ \beta_1 (T, p) \begin{cases} < 0 & \text{if } T_0 < T \leq T_2, \\ = 0 & \text{if } T = T_0, \\ > 0 & \text{if } T < T_0, \end{cases} \tag{30} \]

and \( \beta_- < \beta_+ = -\lambda_1 + o(|\lambda_1|) < 0 \). Hence, by Theorem 6, the system (15) has a transition on \( \Gamma_3' \).

To determine the transition type, we consider the center manifold function of (15), which satisfies

\[ \begin{pmatrix} \tilde{\lambda}_1 \\ a_1 \rho_1^* \\ -c_1 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \rho_n \end{pmatrix} = \begin{pmatrix} a_2 \rho_1^* \rho_2 \\ c_2 \rho_2 \\ \rho_n \end{pmatrix} + o(\rho_2). \tag{31} \]

The solution of (31) is

\[ \rho_1 = \frac{a_2 \tilde{\lambda}_3 + c_1 a_1}{\lambda_3 \lambda_1 - c_1 a_1 \rho_1^*} \rho_1^* \rho_2 + o(\rho_2), \]

\[ \rho_n = \frac{\tilde{\lambda}_1 c_2 + c_1 a_2 \rho_1^*}{\lambda_3 \lambda_1 - c_1 a_1 \rho_1^*} \rho_2 + o(\rho_2). \]

From (16), (22) and (23) we can obtain

\[ \rho_1 \simeq \frac{A_2 \rho_2}{a_3 \lambda_1}, \quad \rho_n \simeq -\frac{c_2}{|\lambda_3|} \rho_2. \tag{32} \]

Inserting the center manifold function (32) into the second equation of (15), we get the reduced equation as

\[ \frac{d\rho_2}{dt} = \tilde{\lambda}_2 \rho_2 + \frac{1}{|\lambda_3|} \left( B_1 - \frac{b_2 A_2}{a_3} \right) \rho_2^2. \]

By (24), the transition of (15) is Type-II.

Case 2. \( A_1 > 0 \) in \( \tilde{c}c \). In this case, the transition of (11) in \( \tilde{c}c \) is Type-II, and the transition solution in \( \tilde{c}c \) is

\[ \rho_1^* = \frac{a_1}{a_3} |\rho_n^*|, \quad \rho_n^* = -\frac{c_3}{2c_4} \left( \sqrt{1 + \frac{4c_4 c_1 a_1}{a_3 c_3^2}} - 1 \right). \]

The eigenvalue \( \beta_1 \) in (17) reads

\[ \beta_1 = \lambda_2 + \frac{1}{a_3} (b_1 a_3 - b_2 a_1) |\rho_n^*|. \]

By (21) and (22) it implies that (30) holds. Hence (11) has a second transition in \( \Gamma_3' \) for \( A_1 > 0 \) in \( \tilde{c}c \).

Under condition (23), we have

\[ \rho_n^* = -\frac{a_1 c_1}{a_3 c_3}, \quad \tilde{\lambda}_1 \simeq -a_3 \rho_1^*, \quad \tilde{\lambda}_3 \simeq \lambda_3 - \frac{2a_1 c_1}{a_3}. \]
By $A_1 > 0$ we get that $|\lambda_3| < 0(e^2)$. Thus, the solutions of (31) can be rewritten as

\[
\rho_1 \sim \frac{a_1 c_1}{\lambda_1 \lambda_3 - c_1 a_1 \rho_1} \rho_1^* \rho_2 \sim \frac{c_2}{c_1} \rho_2,
\]

\[
\rho_2 \sim -\frac{a_2 c_2}{a_1 c_1} \rho_2.
\]

Putting $\rho_1$ and $\rho_2$ into the second equation of (15), we obtain the reduced equation on the center manifold as

\[
d\rho_2 \over dt \sim \tilde{\lambda}_2 \rho_2 + \left( \frac{b_1 c_2 a_3}{a_1 c_1} - \frac{c_2 b_2}{c_1} - b_3 \right) \rho_2^2.
\]

Due to (23) we see that

\[
\frac{b_1 c_2 a_3}{a_1 c_1} - \frac{c_2 b_2}{c_1} - b_3 > 0.
\]

Therefore, this transition of (15) is Type-II.

It is clear that the second transition solutions $(\tilde{\rho}_1, \tilde{\rho}_2, \tilde{\rho}_3)$ satisfy that $\tilde{\rho}_2 > 0$. Thus, the theorem is proved. 

**Proof of Theorem 4.** At point $C = (T_0, p_0)$, $\lambda_1(T_0, p_0) = 0$, $\lambda_2(T_0, p_0) = 0$. Hence, the center manifold function of (11) at $(T_0, p_0)$ reads

\[
\rho_n = -\frac{c_1}{|\lambda_3|} \rho_1 - \frac{c_2}{|\lambda_3|} \rho_2 + \frac{c_3}{|\lambda_3|^2} (c_1 \rho_1 + c_2 \rho_2)^2.
\]

Putting $\rho_n$ in the first two equations of (11) we get the reduced equations on the center manifold as

\[
\frac{d \rho_1}{dt} = \lambda_1 \rho_1 + \frac{1}{|\lambda_3|} (A_1 \rho_1^2 + A_2 \rho_1 \rho_2) - \frac{c_3 a_1}{|\lambda_3|^3} (c_1 \rho_1 + c_2 \rho_2)^2 \rho_1,
\]

\[
\frac{d \rho_2}{dt} = \lambda_2 \rho_2 + \frac{1}{|\lambda_3|} (B_1 \rho_2^2 + B_2 \rho_1 \rho_2) - \frac{c_3 b_1}{|\lambda_3|^3} (c_1 \rho_1 + c_2 \rho_2)^2 \rho_2.
\]

To verify the Type-I transition, by the attractor bifurcation theorem [2], it suffices to consider the following equations:

\[
\frac{d \rho_1}{dt} = A_1 \rho_1^2 + A_2 \rho_1 \rho_2 - \frac{c_3 a_1}{|\lambda_3|^2} (c_1 \rho_1 + c_2 \rho_2)^2 \rho_1,
\]

\[
\frac{d \rho_2}{dt} = B_1 \rho_2^2 + B_2 \rho_1 \rho_2 - \frac{c_3 b_1}{|\lambda_3|^2} (c_1 \rho_1 + c_2 \rho_2)^2 \rho_2.
\]

Since (8) have variational structure, the flows of (34) are of gradient type. Therefore, $(\rho_1, \rho_2) = 0$ has no elliptic region for (34). Hence, in the same fashion as used in Section 6.3 in [2], one can prove that the region

\[S = \{(\rho_1, \rho_2) \in \mathbb{R}^2 | \rho_1 > 0, \rho_2 > 0\}\]

is a stable parabolic region. Namely, $(\rho_1, \rho_2) = 0$ is an asymptotically stable singular point of (34) if and only if one of the two conditions (i) and (ii) holds true. Thus, we only need to prove Assertions (1)–(4).

For Type-I transition at point $C = (T_0, p_0)$, by condition (i) and (ii), $A_1 < 0$ and $B_1 < 0$. Hence, as $\lambda_1 > 0$, $\lambda_2 > 0$ there are bifurcated solutions of (33) in the $\rho_1$-axis and $\rho_2$-axis as

\[z_1 = (\rho_1^*, 0) = \left( \frac{|\lambda_3|}{|A_1|} \lambda_1, 0 \right), \quad z_2 = (0, \rho_2^*) = \left( 0, \frac{|\lambda_3|}{|B_1|} \lambda_2 \right).
\]

The Jacobian matrices of (33) at $z_1$ and $z_2$ are given by

\[
J(z_1) = \begin{pmatrix}
-\lambda_1 \\
0 \\
\frac{A_2}{|B_1|} - \lambda_2 \\
\frac{B_2}{|B_1|} - \lambda_1
\end{pmatrix},
\]

\[
J(z_2) = \begin{pmatrix}
\lambda_1 \\
0 \\
\frac{B_1}{|B_1|} - \lambda_2 \\
\frac{A_1}{|A_1|} - \lambda_1
\end{pmatrix}.
\]

Since (33) has at most three bifurcated singular points in region 5, there are only four types of Type-I transitions, as shown in Fig. 6(a)–(d), and each type is completely determined by the signs of the eigenvalues of $J(z_1)$ and $J(z_2)$. Thus, from the eigenvalues $\beta(z_1) = \frac{1}{|A_1|} (\lambda_1 |A_1| + \lambda_2 B_2)$ of $J(z_1)$ and $\beta(z_2) = \frac{1}{|B_1|} (\lambda_1 |B_1| + \lambda_2 A_2)$ of $J(z_2)$ we can readily derive Assertions (1)–(4). The proof is complete. 

\[\square\]
5. Liquid $^3$He with nonzero applied field

When liquid $^3$He is placed in a magnetic field $H$, the superfluid transition is different from that with $H = 0$. Experiments show that as a magnetic field is applied, a new superfluid phase $A_1$ appears, and the region of phase $A$ can be extended to the bottom at $p = 0$. The $PT$-phase diagram is schematically illustrated by Fig. 7.

As a magnetic field $H$ is applied, a new pairing state appears, in which the spin of pairing atoms is parallel to the magnetic field. This new state corresponds to the phase $A_1$, and is expressed by

$$\sqrt{2}\langle \Phi \rangle = 2a|\uparrow \uparrow\rangle.$$ 

We introduce the complex valued functions $\psi_0$ to represent the state $|\uparrow \uparrow\rangle$, $\psi_1$ the state $|\downarrow \downarrow\rangle$, and $\psi_2$ the state $|\uparrow \downarrow \rangle + |\downarrow \uparrow\rangle$. Let $\rho_0, \rho_1, \rho_2$ stand for the densities of superfluid phases $A_1, A, B$ respectively. Then we have

$$\rho_0 = |\psi_0|^2,$$

$$\rho_1 = \tau_0|\psi_0|^2 + \tau_1|\psi_1|^2 \quad (\tau_0 > 0, \tau_1 > 0),$$

$$\rho_2 = \tau_2|\psi_0|^2 + \tau_3|\psi_1|^2 + \tau_4|\psi_2|^2 \quad (\tau_2, \tau_3 \geq 0, \tau_4 > 0),$$

and the total density of liquid $^3$He in a magnetic field is given by

$$\rho = \begin{cases} 
\rho_0 + \rho_1 & \text{at state } A_1, \\
\rho_0 + \rho_2 & \text{at state } A, \\
\rho_0 + \rho_2 & \text{at state } B.
\end{cases}$$

Thus, similarly to (1), for liquid $^3$He with $H \neq 0$ we give the Ginzburg–Landau free energy in the following form. For simplicity we take the nondimensional form:

$$G(\psi_0, \psi_1, \psi_2, \rho_n) = \frac{1}{2} \int_{\Omega} \left[ \kappa_0|\nabla \psi_0|^2 - \lambda_0|\psi_0|^2 + \alpha_0\rho_n|\psi_0|^2 + \alpha_1|\psi_1|^2|\psi_1|^2 + \alpha_2|\psi_0|^2|\psi_2|^2 + \frac{\alpha_3}{2}|\psi_1|^4 \\
+ \kappa_1|\nabla \psi_1|^2 - \lambda_1|\psi_1|^2 + \alpha_1\rho_n|\psi_1|^2 + \alpha_2|\psi_1|^2|\psi_2|^2 + \frac{\alpha_4}{2}|\psi_1|^4 \\
+ \kappa_2|\nabla \psi_2|^2 - \lambda_2|\psi_2|^2 + \alpha_1\rho_n|\psi_2|^2 + \frac{\alpha_5}{2}|\psi_2|^4 + \kappa_3|\nabla \rho_n|^2 - \lambda_3|\rho_n|^2 - \frac{\alpha_6}{3}\rho_n^3 - \frac{\alpha_7}{4}\rho_n^4 \right] dx. \quad (35)$$

The equations describing liquid $^3$He with $H \neq 0$ read

$$\frac{\partial \psi_0}{\partial t} = \kappa_0 \Delta \psi_0 + \lambda_0 \psi_0 - \alpha_0 \rho_n \psi_0 - \alpha_1|\psi_1|^2\psi_0 - \alpha_2|\psi_2|^2\psi_0 - \alpha_3|\psi_0|^2\psi_0,$$

$$\frac{\partial \psi_1}{\partial t} = \kappa_1 \Delta \psi_1 + \lambda_1 \psi_1 - \alpha_1 \rho_n \psi_1 - \alpha_1|\psi_0|^2\psi_1 - \alpha_2|\psi_2|^2\psi_1 - \alpha_3|\psi_1|^2\psi_1,$$

$$\frac{\partial \psi_2}{\partial t} = \kappa_2 \Delta \psi_2 + \lambda_2 \psi_2 - \alpha_1 \rho_n \psi_2 - \alpha_2|\psi_0|^2\psi_2 - \alpha_2|\psi_1|^2\psi_2 - \alpha_3|\psi_2|^2\psi_2,$$

$$\frac{\partial \rho_n}{\partial t} = \kappa_3 \Delta \rho_n + \lambda_3 \rho_n - \frac{\alpha_0}{2}|\psi_0|^2 - \frac{\alpha_1}{2}|\psi_1|^2 - \frac{\alpha_2}{2}|\psi_2|^2 + \frac{\alpha_3}{2}\rho_n^2 - \frac{\alpha_4}{3}\rho_n^3,$$

$$\frac{\partial}{\partial n}(\psi_0, \psi_1, \psi_2, \rho_n) = 0 \quad \text{on } \partial \Omega,$$

where the coefficients satisfy that for any $0 \leq i \leq 3$ and $1 \leq j \leq 3$,

$$\alpha_i > 0, \quad a_j > 0, \quad b_1, b_3, c_2, c_3 > 0.$$
Fig. 8. Curve $\gamma_0$ is $\lambda_0 = 0$, $\gamma_1$ is $\lambda_1 = 0$, $\gamma_2 = 0$, and $\gamma_3$ is $\lambda_3 = 0$.

Eq. (36) should be the same as (8) for $H = 0$. Therefore we assume that when $H = 0$,

$$\kappa_0 = \kappa_1, \quad \lambda_0 = \lambda_1, \quad a_0 = a_1, \quad a_1 = 0, \quad a_2 = a_2, \quad a_3 = a_3. \quad (37)$$

Based on the physical facts, we also assume that

$$\lambda_0 = \lambda_1(T, p), \quad \tilde{\lambda}(T, p, H) = 0 \quad \text{if} \quad H \neq 0,$$

$$\tilde{\lambda}(T, p, H) \to 0 \quad \text{if} \quad H \to 0. \quad (38)$$

When the magnetic field $H$ and the pressure $p$ are homogeneous on $\Omega$, problem (36) can be reduced to

$$\frac{d\rho_0}{dt} = \lambda_0 \rho_0 - \alpha_0 \rho_n \rho_0 - \alpha_1 \rho_1 \rho_0 - \alpha_2 \rho_2 \rho_0 - \alpha_3 \rho_0^2, \quad (39)$$

$$\frac{d\rho_1}{dt} = \lambda_1 \rho_1 - a_1 \rho_n \rho_1 - \alpha_1 \rho_0 \rho_1 - \alpha_2 \rho_2 \rho_1 - a_3 \rho_1^2,$$

$$\frac{d\rho_2}{dt} = \lambda_2 \rho_2 - b_1 \rho_n \rho_2 - \alpha_2 \rho_0 \rho_2 - a_2 \rho_2 \rho_1 - b_3 \rho_2^2,$$

$$\frac{d\rho_n}{dt} = \lambda_3 \rho_n - \frac{\alpha_0}{2} \rho_0 - \frac{a_1}{2} \rho_1 - \frac{b_1}{2} \rho_2 + c_2 \rho_2^2 - c_3 \rho_n^2,$$

where $\rho_i = |\psi_i|^2$ ($i = 0, 1, 2$), and $\lambda_i$ ($i = 1, 2, 3$) are as in (11).

Let $\lambda_1, \lambda_2, \text{and} \lambda_3$ be that as shown in Fig. 3(a)–(c) respectively. Then, due to (38) the curves $\lambda_j(T, p) = 0 (0 \leq j \leq 3)$ in the $PT$-plane are schematically illustrated by Fig. 8.

Let the applied magnetic field $H \neq 0$ such that

$$0 < \tilde{\lambda}(T, p, H) < \epsilon, \quad \text{for} \quad \epsilon > 0 \text{ small}, \quad (40)$$

where $\tilde{\lambda}$ is as in (38). Then, under conditions (21), (22), (37), (38) and (40), in the same fashion as in Theorems 2 and 3, we can prove the following transition theorem for (39).

**Theorem 5.** Assume conditions (21), (22), (37), (38) and (40), then for $H \neq 0$ there exist two curve segments $\gamma' h'$ near $\lambda_1 = 0$ and $e' m'$ near $\lambda_2 = 0$ in the $PT$-plane as shown in Fig. 9 such that the following assertions hold true:
(1) System (39) has a transition in curve segment $\lambda_0 = 0$ with $\lambda_3 < 0$ (i.e., the curve segment $\hat{g}_n$ in Fig. 9), which is Type-I for $\alpha_0^2 - 2|\lambda_3|\alpha_3 \leq 0$, and is Type-II for $\alpha_0^2 - 2|\lambda_3|\alpha_3 > 0$. The transition solution is given by $(\rho_0^*, 0, 0, \rho_n^*)$ with $\rho_0^* > 0$, $\rho_n^* < 0$.

(2) The system has a second transition from $(\rho_0^*, 0, 0, \rho_n^*)$ in the curve segment $\hat{f}^n$ (i.e., $\lambda_1 - \alpha_1\rho_n^* - \alpha_1\rho_0^* = 0$), and the transition solution is as $(\rho_0^*, \rho_1^*, 0, \rho_n^*)$ with $\rho_0^* > 0$, $\rho_1^* > 0$ and $\rho_n^* < 0$.

(3) The system has a third transition from $(\rho_0^*, \rho_1^*, \rho_2^*, \rho_n^*)$ in the curve segment $\hat{e}^m$ (i.e., $\lambda_2 - b_1\rho_0^* - a_2\rho_0^* - a_2\rho_1^* = 0$), and the transition solution is $(\rho_0^*, \rho_2^*, \rho_0^*, \rho_n^*)$ with $\rho_0^* > 0$ (0 $\leq i \leq 2$) and $\rho_n^* < 0$.

**Remark 2.** The first transition of (39) in curve segment $\hat{g}_n$ corresponds to the phase transition of $^3$He in a magnetic field between the normal liquid and superfluid phase $A_1$, and the second transition in $\hat{f}^n$ corresponds to the phase transition between superfluid phases $A_1$ and $A$, and the third transition in $\hat{e}^m$ corresponds to the phase transition between superfluid phases $A$ and $B$; see Fig. 9.

**Remark 3.** The transition theorems, Theorems 1–3 and 5, provide a theoretical foundation to explain the PT-phase diagrams of superfluidity, meanwhile they support these models of liquid He which are based on the phenomenology.

### 6. Physical remarks

By carefully examining the classical phase transition diagrams and with both mathematical and physical insights offered by the dynamical transition theory, we derived two new models for superfluidity of $^3$He with or without applied field. A crucial component of these two models is the introduction of three wave functions to represent Anderson–Brinkman–Morel (ABM) and the Balian–Werthamer (BM) states.

Then we have obtained a theoretical PT-phase diagram of $^3$He as shown in Fig. 2, based on the models and the dynamic phase transition analysis. A few main characteristics of the results are as follows.

First, the analysis shows the existence of a metastable region $H = H_1 \cup H_2$, in which the solid state and the superfluid states $A$ and $B$ appear randomly depending on fluctuations. In particular, in $H_1$, the phase $B$ superfluid state and the solid may appear, and in $H_2$ the phase $A$ superfluid state and the solid state may appear.

Second, theoretics analysis suggests the existence of phase $C$ superfluid state, which is characterized by the wave function $\psi_2$, representing $| \uparrow \uparrow \rangle + | \downarrow \downarrow \rangle$. However, phase $C$ region is very narrow, which may be the reason for it being hard to be observed in experiments.

Third, the curve $\hat{b}c\hat{d}$ is the first critical curve where phase transition between normal fluid and superfluid states occurs. The curve $\hat{f}c$ is the coexistence curve between phases $A$ and $B$. The curve $\hat{b}c$ is the coexistence curve between normal fluid state and the phase $A$ superfluid state, the curve $\hat{c}d$ is the coexistence curve between normal fluid state and the phase $C$ superfluid state, the curve $\hat{c}d$ is the coexistence curve between the phases $B$ and $C$ superfluid states.

Fourth, Theorems 2–4 imply that near the two triple points $b$ and $c$, there is a possibility of the existence of two switch points, where the transition on the corresponding coexistence curve switches types at each switch point. The existence of such switch points depends on the physical parameters.

In comparison to classical results as shown in Fig. 1, our results lead to the predictions of the existence of (1) an metastable region $H$, (2) a new phase $C$ in a narrow region, and (3) switch points. It is hoped that these predictions will be useful for designing better physical experiments and lead to better understanding of the physical mechanism of superfluidity.

### Appendix. General principles of phase transition dynamics

In this Appendix, we introduce a new phase dynamic transition classification scheme to classify phase transitions into three categories: Type-I, Type-II and Type-III, corresponding to mathematically continuous, jump, and mixed transitions, respectively.

#### A.1. Dynamic transition theory

In sciences, nonlinear dissipative systems are generally governed by differential equations, which can be expressed in the following abstract form. Let $X$ and $X_1$ be two Banach spaces, and $X_1 \subset X$ a compact and dense inclusion. Hereafter we always consider the following nonlinear evolution equations

$$
\frac{du}{dt} = L_u u + G(u, \lambda), \quad u(0) = \varphi. \tag{A.1}
$$

where $u : [0, \infty) \rightarrow X$ is the unknown function, and $\lambda \in \mathbb{R}^1$ is the system parameter.

Assume that $L_u : X_1 \rightarrow X$ is a parameterized linear completely continuous field depending continuously on $\lambda \in \mathbb{R}^1$, which satisfies

$$
L_u = -A + B_u \quad \text{a sectorial operator},
$$

$$
A : X_1 \rightarrow X \quad \text{a linear homeomorphism},
$$

$$
B_u : X_1 \rightarrow X \quad \text{a linear compact operator}. \tag{A.2}
$$
The phase transitions for

\[ G(u, \lambda) = o(||u||_{X_\sigma}) \quad \forall \lambda \in \mathbb{R}^1. \]  

(A.3)

Hereafter we always assume the conditions (A.2) and (A.3), which represent that the system (A.1) has a dissipative structure.

A state of the system (A.1) at \( \lambda \) is usually referred to as a compact invariant set \( \Sigma_\lambda \). In many applications, \( \Sigma_\lambda \) is a singular point or a periodic orbit. A state \( \Sigma_\lambda \) of (A.1) is stable if \( \Sigma_\lambda \) is an attractor; otherwise \( \Sigma_\lambda \) is called unstable.

**Definition 1.** We say that the system (A.1) has a phase transition from a state \( \Sigma_\lambda \) at \( \lambda = \lambda_0 \) if \( \Sigma_\lambda \) is stable on \( \lambda < \lambda_0 \) (or on \( \lambda > \lambda_0 \)) and is unstable on \( \lambda > \lambda_0 \) (or on \( \lambda < \lambda_0 \)). The critical parameter \( \lambda_0 \) is called a critical point. In other words, the phase transition corresponds to an exchange of stable states.

Obviously, the attractor bifurcation of (A.1) is a type of transition. However, bifurcation and transition are two different, but related concepts.

Let \( \{\beta_j(\lambda) \in \mathbb{C} \mid j \in \mathbb{N}\} \) be the eigenvalues (counting multiplicity) of \( L_\lambda \), and assume that

\[
\begin{align*}
\text{Re} \beta_i(\lambda) &= \begin{cases} < 0 & \text{if } \lambda < \lambda_0, \\
= 0 & \text{if } \lambda = \lambda_0, \\
> 0 & \text{if } \lambda > \lambda_0,
\end{cases} & \forall 1 \leq i \leq m, \\
\text{Re} \beta_j(\lambda_0) &< 0 & \forall j \geq m + 1.
\end{align*}
\]

(A.4)

(A.5)

The following theorem is a basic principle of transitions from equilibrium states, which provides sufficient conditions and a basic classification for transitions of nonlinear dissipative systems. This theorem is a direct consequence of the center manifold theorems and the stable manifold theorems; we omit the proof.

**Theorem 6.** Let the conditions (A.4) and (A.5) hold true. Then, the system (A.1) must have a transition from \( (u, \lambda) = (0, \lambda_0) \) and there is a neighborhood \( U \subset X \) of \( u = 0 \) such that the transition is one of the following three types:

1. **Continuous Transition:** there exists an open and dense set \( \tilde{U}_\lambda \subset U \) such that for any \( \varphi \in \tilde{U}_\lambda \), the solution \( u_\lambda(t, \varphi) \) of (A.1) satisfies

\[
\lim_{\lambda \rightarrow \lambda_0} \limsup_{t \rightarrow \infty} ||u_\lambda(t, \varphi)||_X = 0.
\]

2. **Jump Transition:** for any \( \lambda_0 < \lambda < \lambda_0 + \varepsilon \) with some \( \varepsilon > 0 \), there is an open and dense set \( U_\lambda \subset U \) such that for any \( \varphi \in U_\lambda \),

\[
\limsup_{t \rightarrow \infty} ||u_\lambda(t, \varphi)||_X \geq \delta > 0,
\]

for some \( \delta > 0 \) independent of \( \lambda \).

3. **Mixed Transition:** for any \( \lambda_0 < \lambda < \lambda_0 + \varepsilon \) with some \( \varepsilon > 0 \), \( U \) can be decomposed into two open sets \( U_\lambda^1 \) and \( U_\lambda^2 \) (\( U_\lambda^1 \) not necessarily connected): \( \tilde{U} = \tilde{U}_\lambda^1 + \tilde{U}_\lambda^2, U_\lambda^1 \cap U_\lambda^2 = \emptyset \), such that

\[
\begin{align*}
\lim_{\lambda \rightarrow \lambda_0} \limsup_{t \rightarrow \infty} ||u(t, \varphi)||_X &= 0 & \forall \varphi \in U_\lambda^1, \\
\limsup_{t \rightarrow \infty} ||u(t, \varphi)||_X \geq \delta > 0 & \forall \varphi \in U_\lambda^2.
\end{align*}
\]

With this theorem at our disposal, we are in a position to give a new dynamic classification scheme for dynamic phase transitions.

**Definition 2.** The phase transitions for (A.1) at \( \lambda = \lambda_0 \) are classified using their dynamic properties: continuous, jump, and mixed as given in Theorem 6, which are called Type-I, Type-II and Type-III respectively.

An important aspect of the transition theory is to determine which of the three types of transitions given by Theorem 6 occurs in a specific problem. A corresponding dynamic transition theory has been developed recently by the authors for this purpose; see [3]. We refer interested readers to these references for details of the theory.

**A.2. New Ginzburg–Landau models for equilibrium phase transitions**

In this section, we recall a new time-dependent Ginzburg–Landau theory for modeling equilibrium phase transitions in statistical physics.
Consider a thermal system with a control parameter $\lambda$. By the mathematical characterization of gradient systems and the Le Châtelier principle, for a system with thermodynamic potential $\mathcal{H}(u, \lambda)$, the governing equations are essentially determined by the functional $\mathcal{H}(u, \lambda)$. When the order parameters $(u_1, \ldots, u_m)$ are nonconserved variables, i.e., the integers

$$\int_\Omega u_i(x, t) \, dx = a_i(t) \neq \text{constant}$$

then the time-dependent equations are given by

$$\frac{\partial u_i}{\partial t} = -\beta_i \frac{\delta}{\delta u_i} \mathcal{H}(u, \lambda) + \Phi_i(u, \nabla u, \lambda),$$

(A.6)

for $1 \leq i \leq m$, where $\beta_i > 0$ and $\Phi_i$ satisfy

$$\int_\Omega \sum_i \Phi_i \frac{\delta}{\delta u_i} \mathcal{H}(u, \lambda) \, dx = 0.$$  

(A.7)

The condition (A.7) is required by the Le Châtelier principle. In the concrete problem, the terms $\Phi_i$ can be determined by physical laws and (A.7). We remark here that following the Le Châtelier principle, one should have an inequality constraint. However physical systems often obey most simplified rules, as many existing models for specific problems are consistent with the equality constraint here. This remark applies to the constraint (A.13) as well.

When the order parameters are the number density and the system has no material exchange with the external, then $u_j$ ($1 \leq j \leq m$) are conserved, i.e.,

$$\int_\Omega u_j(x, t) \, dx = \text{constant}. $$  

(A.8)

This conservation law requires a continuity equation

$$\frac{\partial u_j}{\partial t} = -\nabla \cdot J_j(u, \lambda),$$

(A.9)

where $J_j(u, \lambda)$ is the flux of component $u_j$, satisfying

$$J_j = -k_j \nabla \left( \mu_j - \sum_{i \neq j} \mu_i \right),$$

(A.10)

where $\mu_i$ is the chemical potential of component $u_i$,

$$\mu_j - \sum_{i \neq j} \mu_i = \frac{\delta}{\delta u_j} \mathcal{H}(u, \lambda) - \phi_j(u, \nabla u, \lambda),$$

(A.11)

and $\phi_j(u, \lambda)$ is a function depending on the other components $u_i$ ($i \neq j$). Thus, from (A.9)–(A.11) we obtain the dynamical equations as follows

$$\frac{\partial u_j}{\partial t} = \beta_j \Delta \left[ \frac{\delta}{\delta u_j} \mathcal{H}(u, \lambda) - \phi_j(u, \nabla u, \lambda) \right],$$

(A.12)

for $1 \leq j \leq m$, where $\beta_j > 0$ are constants, and $\phi_j$ satisfy

$$\int_\Omega \sum_j \Delta \phi_j \cdot \frac{\delta}{\delta u_j} \mathcal{H}(u, \lambda) \, dx = 0.$$  

(A.13)

When $m = 1$, i.e., the system is a binary system, consisting of two components $A$ and $B$, then the term $\phi_i = 0$. The above model covers the classical Cahn–Hilliard model. It is worth mentioning that for multi-component systems, these $\phi_i$ play an important rule in deriving good time-dependent models.

If the order parameters $(u_1, \ldots, u_k)$ are coupled to the conserved variables $(u_{k+1}, \ldots, u_m)$, then the dynamical equations are

$$\frac{\partial u_i}{\partial t} = -\beta_i \frac{\delta}{\delta u_i} \mathcal{H}(u, \lambda) + \Phi_i(u, \nabla u, \lambda),$$

$$\frac{\partial u_j}{\partial t} = \beta_j \Delta \left[ \frac{\delta}{\delta u_j} \mathcal{H}(u, \lambda) - \phi_j(u, \nabla u, \lambda) \right],$$

(A.14)

for $1 \leq i \leq k$ and $k + 1 \leq j \leq m$. Here $\Phi_i$ and $\phi_i$ satisfy (A.7) and (A.13), respectively.

The model (A.14) we derive here gives a general form of the governing equations to thermodynamic phase transitions, and will play crucial role in studying the dynamics of equilibrium phase transitions in statistical physics.
References