GRAVITATIONAL FIELD EQUATIONS AND
THEORY OF DARK MATTER AND DARK ENERGY

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Abstract. The main objective of this article is to derive new gravitational field equations and to establish a unified theory for dark energy and dark matter. The gravitational field equations with a scalar potential $\varphi$ function are derived using the Einstein-Hilbert functional, and the scalar potential $\varphi$ is a natural outcome of the divergence-free constraint of the variational elements. Gravitation is now described by the Riemannian metric $g_{\mu\nu}$, the scalar potential $\varphi$ and their interactions, unified by the new field equations. From quantum field theoretic point of view, the vector field $\Phi_{\mu} = D_{\mu}\varphi$, the gradient of the scalar function $\varphi$, is a spin-1 massless bosonic particle field. The field equations induce a natural duality between the graviton (spin-2 massless bosonic particle) and this spin-1 massless bosonic particle. Both particles can be considered as gravitational force carriers, and as they are massless, the induced forces are long-range forces. The (nonlinear) interaction between these bosonic particle fields leads to a unified theory for dark energy and dark matter. Also, associated with the scalar potential $\varphi$ is the scalar potential energy density $\frac{c^4}{8\pi G}\Phi = \frac{c^4}{8\pi G}g^{\mu\nu}D_{\mu}D_{\nu}\varphi$, which represents a new type of energy caused by the non-uniform distribution of matter in the universe. The negative part of this potential energy density produces attraction, and the positive part produces repelling force. This potential energy density is conserved with mean zero: $\int_M \Phi dM = 0$. The sum of this potential energy density $\frac{c^4}{8\pi G}\Phi$ and the coupling energy between the energy-momentum tensor $T_{\mu\nu}$ and the scalar potential field $\varphi$ gives rise to a unified theory for dark matter and dark energy: The negative part of this sum represents the dark matter, which produces attraction, and the positive part represents the dark energy, which drives the acceleration of expanding galaxies. In addition, the scalar curvature of space-time obeys $R = \frac{8\pi G}{c^4}T + \Phi$. Furthermore, the proposed field equations resolve a few difficulties encountered by the classical Einstein field equations.

1. Introduction and summary. Two great mysteries concerning our universe are dark energy and dark matter, which are introduced to explain, respectively, the


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acceleration of the expanding galaxies and more matter than can be accounted for in our visible stars [27, 24, 12, 32, 28, 10]. The two leading models for dark energy are a cosmological constant and quintessence; see among many others [4, 5, 6, 12, 23, 31]. Most studies of dark matter are oriented toward to direct detection of likely candidates of dark matter such as weakly interactive massive particles (WIMP) [2] and axions. Despite if many attempts, the mysteries of dark matter and dark energy remain.

The main objective of this article is to derive a new set of gravitational field equations, and to establish a unified theory for dark matter and dark energy. We proceed as follows.

First, the natural starting point for this study is to fundamentally examine the Einstein field equations:

\[ R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{8\pi G}{c^4}T_{\mu\nu}, \]  

(1.1)

where \( g_{\mu\nu} \) is the Riemannian metric of the space-time, \( R_{\mu\nu} \) is the Ricci curvature tensor, \( R \) is the scalar curvature, \( c \) is the speed of light in vacuum, and \( T_{\mu\nu} \) is the energy-momentum tensor of matter; see among many others [14, 1].

Einstein equations are derived based on the following three basic principles:

- the principle of equivalence,
- the principle of general relativity, and
- the Lagrangian least action principle.

The first two principles amount to saying that the space-time is a 4-dimensional Riemannian manifold \((M, g_{\mu\nu})\), where the metric \( \{g_{\mu\nu}\} \) represents gravitational potential, and the third principle determines that the Riemannian metric \( \{g_{\mu\nu}\} \) is an extremum point of the Lagrangian action. There is no doubt that the most natural Lagrangian is the Einstein-Hilbert functional:

\[ L_{EH}(g_{\mu\nu}) = \int_M \left( R + \frac{8\pi G}{c^4}S \right) \sqrt{-g}dx, \]  

(1.2)

whose Euler-Lagrangian equations, \( \delta L_{EH} = 0 \), are exactly Einstein equations (1.1). Here \( S \) is the stress tensor the normal matter.

Second, there are a number of difficulties for the Einstein field equations. One such difficulty is that the Einstein field equations failed to explain the dark matter and dark energy, and the equations are inconsistent with the accelerating expansion of the galaxies. In spite of many attempts to modify the Einstein gravitational field equation to derive a consistent theory for the dark energy, the mystery of dark energy and dark matter remains.

Another issue is related to the existence of solutions. We shall show that there is no solution for the Einstein field equations for the spherically symmetric case with cosmic microwave background (CMB). One needs clearly to resolve this inconsistency caused by the non-existence of solutions.

In addition, from the Einstein equations (1.1), it is clear that

\[ R = \frac{8\pi G}{c^4}T, \]  

(1.3)

where \( T = g^{\mu\nu}T_{\mu\nu} \) is the energy-momentum density. A direct consequence of this formula is that the discontinuities of \( T \) give rise to the same discontinuities of the curvature and the discontinuities of space-time. This is certainly an inconsistency which needs to be resolved.
Furthermore, it has been observed that the universe is highly non-homogeneous as indicated by e.g. the “Great Walls”, filaments and voids. However, the Einstein equations do not appear to offer a good explanation of this inhomogeneity. With these issues, a fundamental level of examination of the Einstein equations are inevitable.

Third, the Bianchi identity implies that the Einstein tensor is conserved:

\[ D_\mu (R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R) = 0. \] (1.4)

Consequently, the energy-momentum tensor of matter \( T_{\mu \nu} \) is conserved as well. However, the presence of dark energy and dark matter suggests that the conservation of energy-momentum of matter should also include the dark matter and dark energy sectors. In other words, due to the presence of dark matter and dark energy, it is natural for us to propose the following postulate:

Postulate 1.1: The energy-momentum tensor \( T_{\mu \nu} \) of normal matter need not be conserved:

\[ D_\mu (T_{\mu \nu}) \neq 0. \] (1.5)

Fourth, it is our believe that the Einstein great vision of gravity is still valid. Namely, and modification of the Einstein equations should still obey 1) the principle of equivalence, 2) the Lagrangian least action principle. In addition, the most natural Lagrangian action functional is still the Einstein-Hilbert functional (1.2). Hence Postulate 1.1 and the energy-momentum conservation of the Einstein tensor leads us to derive the Euler-Lagrangian equations from the Einstein-Hilbert functional under energy-momentum conservation constraint:

\[ (\delta L_{EH}(g_{\mu \nu}), X) = 0 \quad \forall \ X_{\mu \nu} \text{ with } X_{\mu \nu} = X_{\nu \mu} \text{ and } D_\mu X_{\mu \nu} = 0. \] (1.6)

By an orthogonal decomposition theorem, Theorem 3.1, there is a vector field \( \Phi_\mu = D_\mu \varphi \) on the space-time manifold \( M \) such that

\[ R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = -\frac{8\pi G}{c^4} T_{\mu \nu} - D_\mu \Phi_\nu, \quad \Phi_\nu = D_\nu \varphi. \] (1.7)

Here \( \varphi : M \to \mathbb{R} \) is a scalar field. The corresponding energy-momentum conservation of matter is then given by

\[ D_\mu \left( D_\mu \Phi_\nu + \frac{8\pi G}{c^4} T_{\mu \nu} \right) = 0. \] (1.8)

It is remarkable that a generalized version of this constraint Lagrangian action principle is valid for all four interactions—both unified and decoupled. Hence we present it as a principle, called Principle of Interaction Dynamics (PID) in [20].

An alternative way to derive the new field equations (1.7) is to observe that with this postulate, by the orthogonal decomposition theorem, Theorem 3.1, again, there is a vector field \( \Phi_\mu \) on the space-time manifold \( M \) such that

\[ T_{\mu \nu} = \tilde{T}_{\mu \nu} - \frac{c^4}{8\pi G} D_\mu \Phi_\nu, \quad D_\mu \tilde{T}_{\mu \nu} = 0. \]

Hence the conserved Einstein tensor \( R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R \) must be balanced by the conserved energy-momentum \( \tilde{T}_{\mu \nu} \), leading to the gravitational field equations (1.7).

Fifth, an important remark is that the added term \( D_\mu \Phi_\nu \) cannot be derived by adding any scalar field in the Einstein-Hilbert functional. In fact, in order to derive this term, a natural term to add in the action functional should be the energy
density \( g^{\mu \nu} D_\mu D_\nu \phi \) and the resulting equations will then contain more terms, caused by the variation of metric \( g_{\mu \nu} \) in the covariant derivatives \( D_\mu \).

Hence the new gravitational field equations (1.7) are completely different from any classical extensions of the Einstein equations. Among many interesting extensions are the \( f(R) \) gravity theories, scalar/tensor field gravity theories and Elie Cartan gravity theories with torsion [3, 30, 11, 7, 4, 8, 25] and the references therein.

**Sixth**, from quantum field theoretic point of view, the vector field \( \Phi_\mu \) on the right-hand side of the field equations (1.7) is a spin-1 massless bosonic particle field. The field equations induce a natural duality between the spin-2 massless bosonic particle, called graviton, and this spin-1 massless bosonic particle. Both particles can be considered as gravitational force carriers, and as they are massless, the induced forces are long-range forces. The (nonlinear) interaction between these bosonic particle fields leads to a unified theory for dark energy and dark matter. Although, we do not exclude the existence of exotic particles in the universe, the main constituent of dark matter and dark energy is the new massless spin-1 particle field.

This duality was first introduced by the authors in [20, 21] for all four interactions. In these two papers, the authors introduce two basic principles: the principle of interaction dynamics (PID) and the principle of representation invariance (PRI). Intuitively, PID takes the variation of the action functional under energy-momentum conservation constraint. PRI requires that physical laws be independent of representations of the gauge groups. With these principles, the authors derive a unified field model for four interaction. One important outcome of this unified field model is a natural duality between the interacting fields \((g, A, W^{\pm}, Z, S^k)\), corresponding to graviton, photon, intermediate vector bosons \( W^{\pm} \) and \( Z \) and gluons, and the adjoint bosonic fields \((\Phi_\mu, \phi^0, \phi_w^a, \phi_s^k)\). This duality predicts two Higgs particles of similar mass with one due to weak interaction and the other due to strong interaction.

Also, PID and PRI can be applied directly to individual interactions. Of course, the main starting point of this program of research is initiated in this article. For weak and strong interactions, among other things, we derive 1) three levels of strong interaction potentials for quark, nucleon/hadron, and atom respectively, and 2) a weak interaction potential. These potential/force formulas offer a clear mechanism for both quark confinement and asymptotic freedom—a longstanding problem in particle physics. Also, we intend to offer our view on such questions as why our universe is as it is, by introducing energy levels for leptons and quarks as well as for hadrons, and by exploring essential characteristics of the potential/force formulas.

**Seventh**, gravitation is now described by the Riemmanian metric \( g_{\mu \nu} \), the bosonic vector field \( \Phi_\mu \) and their interactions, unified by the new gravitational field equations (1.7). In fact, with the work in this article and in [20, 21], we show that bosonic particles with even spins give rise mainly to attractive forces, and bosonic particles with odd spins give rise mainly to repulsive forces. Hence we argue that the spin-2 graviton associated with the Riemannian metric \( g_{\mu \nu} \) contributes to the attractive gravitation as described all classical theories by Galileo, Newton and Einstein. Repulsive nature of the spin-1 bosonic particle field explains dark energy and the (nonlinear) interactions of spin-2 graviton and the spin-1 massless bosonic field leads to a unified theory for dark energy and dark matter; see also the gravitational force formula (1.15) in the static and spherically symmetric case below.
EIGHTH, the energy-momentum density $T = g^\mu\nu T_{\mu\nu}$ and the scalar potential energy density $\frac{\delta}{8\pi G} \Phi = \frac{\delta}{8\pi G} g^{\mu\nu} D_\mu \Phi_\nu$ satisfy

$$R = \frac{8\pi G}{c^4} T + \Phi,$$

(1.9)

$$\int_M \Phi \sqrt{-g} dx = 0.$$

(1.10)

The scalar potential energy density $\frac{\delta}{8\pi G} \Phi$ has a number of important physical properties.

One important property is that if the matter is homogeneously distributed in the universe, then the new particle field $\Phi_\mu = \nabla \varphi$ vanishes, and consequently the scalar potential energy density is identically zero: $\Phi \equiv 0$. The presence of dark matter and dark energy can be considered as caused by the non-uniform distribution of matter in the universe. This scalar potential energy density varies as the galaxies move and matter of the universe redistributes. Like gravity, it affects every part of the universe as a field. In other words, the universe with uniformly distributed matter leads to identically zero scalar potential energy, and is unstable. It is this instability that leads to the existence of the dark matter and dark energy, and consequently the high non-homogeneity of the universe.

Also, the scalar curvature of space-time obeys (1.9). Consequently, when there is no normal matter present (with $T = 0$), the curvature $R$ of space-time is balanced by $R = \Phi$. Therefore, there is no real vacuum in the universe.

NINTH, to further explain the dark matter and dark energy phenomena, we consider a central matter field with total mass $M$ and radius $r_0$ and with spherical symmetry. With spherical coordinates, the corresponding Riemannian metric must be of the following form:

$$ds^2 = -e^u \c^2 dt^2 + e^v dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$

(1.11)

where $u = u(r)$ and $v = v(r)$ are functions of the radial distance. With the new gravitational field equations, the force exerted on an object with mass $m$ is given by

$$F = mMG \left[ -\frac{1}{r^2} - \frac{1}{\delta} \left( 2 + \frac{\delta}{r} \right) \varphi' + \frac{Rr}{\delta} \right], \quad R = \Phi \quad \text{for } r > r_0.$$

(1.12)

where $\delta = 2GM/c^2$, $R$ is the scalar curvature, and $\varphi$ is the scalar potential. The first term is the classical Newton gravitation, the second term is the coupling interaction between matter and the scalar potential $\varphi$, and the third term is the interaction generated by the scalar potential energy density $\frac{\delta}{8\pi G} \Phi$ as indicated in (1.9) ($R = \Phi$ for $r > r_0$). In this formula, the negative and positive values of each term represent respectively the attracting and repelling forces. It is then clear that the combined effect of the second and third terms in the above formula represent the dark matter, dark energy and their interactions with normal matter.

Also, importantly, this formula is a direct representation of the Einstein’s equivalence principle. Namely, the curvature of space-time induces interaction forces between matter.

In addition, one can derive a more detailed version of the above formula:

$$F = mMG \left[ -\frac{1}{r^2} + \left( 2 + \frac{\delta}{r} \right) \varphi r^2 + \frac{Rr}{\delta} + \frac{1}{\delta} \left( 2 + \frac{\delta}{r} \right) r^2 \int r^{-2} Rdr \right],$$

(1.13)
where \( \varepsilon > 0 \). The conservation law (1.10) of \( \Phi \) suggests that \( R \) behaviors as \( r^{-2} \) for \( r \) sufficiently large. Consequently the second term in the right hand side of (1.13) must dominate and be positive, indicating the existence of dark energy.

In fact, the above formula can be further simplified to derive the following approximate formula:

\[
F = mMG \left[ -\frac{1}{r^2} - \frac{k_0}{r} + k_1 r \right],
\]

(1.14)

\[
k_0 = 4 \times 10^{-18} \text{km}^{-1}, \quad k_1 = 10^{-57} \text{km}^{-3}.
\]

(1.15)

Again, in (1.14), the first term represents the Newton gravitation, the attracting second term stands for dark matter and the repelling third term is the dark energy.

Tenth, the new gravitational field equations (1.7) are mathematically consistent. In fact, as indicated in [14], there are 10 equations with only 6 independent unknowns in the classical Einstein equations. With the introduction of the vector field \( \Phi \), the gravitational field equations are a more consistent system with 10 equations and with 10 unknowns. For example, with the spherically symmetric case with cosmic microwave background (CMB). Namely, the existence of solutions in this case can be proved.

Also, our theory suggests that the curvature \( R \) is always balanced by \( \Phi \) in the entire space-time by (1.9), and the space-time is no longer flat. Namely the entire space-time is also curved and is filled with dark energy and dark matter. In particular, the discontinuities of \( R \) induced by the discontinuities of the energy-momentum density \( T \), dictated by the Einstein field equations, are no longer present thanks to the balance of \( \Phi \).

Eleventh, the mathematical part of this article is devoted to a rigorous derivation of the new gravitational field equations. We observe that (1.6) does not imply \( \delta L_{EH}(g_{\mu\nu}) = 0 \), the classical Einstein equations. Instead, it shows that \( \delta L_{EH}(g_{\mu\nu}) \) is orthogonal to all symmetric divergence-free tensor fields \( X \). Hence we need to decompose general tensor fields on Riemannian manifolds into divergence-free and gradient parts. For this purpose, an orthogonal decomposition theorem is derived in Theorem 3.1. In particular, given an \((r,s)\) tensor field \( u \in L^2(T^r_s M) \), we have

\[
u = \nabla \psi + v, \quad \text{div } v = 0, \quad \psi \in H^1(T^r_s - 1).
\]

(1.16)

The gradient part is acting on an \((r,s-1)\) tensor field \( \psi \).

When restricting to a \((0,2)\) symmetric tensor field \( u \), the gradient part in the above decomposition is given by \( \nabla \psi_\mu \) where \( \psi_\mu \) is a \((0,1)\) tensor. Then using symmetry, we show in Theorem 3.2 that this \((0,1)\) tensor \( \psi \) can be uniquely determined, up to addition to constants, by the gradient of a scalar field \( \varphi \):

\[
\psi = \nabla \varphi, \quad \varphi \in H^2(M),
\]

and consequently we obtain the following decomposition for general symmetric \((0,2)\) tensor fields:

\[
u_{\mu\nu} = v_{\mu\nu} + D_\mu D_\nu \varphi, \quad D^\mu v_{\mu\nu} = 0, \quad \varphi \in H^2(M).
\]

(1.17)

We remark here that the orthogonal decompositions (1.16) and (1.17) are reminiscent of the orthogonal decomposition of vectors fields into gradient and divergence parts, which are crucial for studying incompressible fluid flows; see among many others [18, 19].

This article is divided into two parts. The physically inclined readers can go directly to the physics part after reading this Introduction.
Part 1. Mathematics

2. Preliminaries.

2.1. Sobolev spaces of tensor fields. Let \((M, g_{ij})\) be an \(n\)-dimensional Riemannian manifold with metric \((g_{ij})\), and \(E = T^r_s M\) be an \((r, s)\)-tensor bundle on \(M\). A mapping \(u : M \to E\) is called a section of the tensor-bundle \(E\) or a tensor field. In a local coordinate system \(x\), a tensor field \(u\) can be expressed component-wise as follows:

\[
u = \left\{ u_{i_1 \cdots i_s}^{j_1 \cdots j_r}(x) \mid 1 \leq i_1, \cdots, i_s, j_1, \cdots, j_r \leq n \right\},
\]

where \(u_{i_1 \cdots i_s}^{j_1 \cdots j_r}(x)\) are functions of \(x \in U\). The section \(u\) is called \(C^r\)-tensor field or \(C^r\)-section if its components are \(C^r\)-functions.

For any real number \(1 \leq p < \infty\), let \(L^p(E)\) be the space of all \(L^p\)-integrable sections of \(E\):

\[
L^p(E) = \left\{ u : M \to E \mid \int_M |u|^p dx < \infty \right\},
\]

equipped with the norm

\[
||u||_{L^p} = \left[ \int_M |u|^p dx \right]^{1/p} = \left[ \int_M \sum |u_{i_1 \cdots i_s}^{j_1 \cdots j_r}|^p dx \right]^{1/p}.
\]

For \(p = 2\), \(L^2(E)\) is a Hilbert space equipped with the inner product

\[
(u, v) = \int_M \sum g^{i_1 j_1} \cdots g^{i_s j_s} \sqrt{\det(g)} dx,
\]

(2.1)

where \((g_{ij})\) is Riemannian metric, \((g^{ij}) = (g_{ij})^{-1}\), \(g = \det(g_{ij})\), and \(\sqrt{\det(g)} dx\) is the volume element.

For any positive integer \(k\) and any real number \(1 \leq p < \infty\), we can also define the Sobolev spaces \(W^{k,p}(E)\) to be the subspace of \(L^p(E)\) such that all covariant derivatives of \(u\) up to order \(k\) are in \(L^p(E)\). The norm of \(W^{k,p}(E)\) is always denoted by \(\| \cdot \|_{W^{k,p}}\). As \(p = 2\), the spaces \(W^{k,2}(E)\) are Hilbert spaces, and are usually denoted by

\[
H^k(E) = W^{k,2}(E) \quad \text{for } k \geq 0,
\]

(2.2)
equipped with inner product \((\cdot, \cdot)_{H^k}\) and norm \(\| \cdot \|_{H^k}\).

2.2. Gradient and divergent operators. Let \(u : M \to E\) be an \((r, s)\)-tensor field, with the local expression

\[
u = \left\{ u_{i_1 \cdots i_s}^{j_1 \cdots j_r} \right\}.
\]

(2.3)

Then the gradient of \(u\) is defined as

\[
\nabla u = \{ D_k u_{i_1 \cdots i_s}^{j_1 \cdots j_r} \},
\]

(2.4)

where \(D = (D_1, \cdots, D_n)\) is the covariant derivative. It is clear that the gradient \(\nabla u\) defined by (2.4) is an \((r, s + 1)\)-tensor field:

\[
\nabla u : M \to T^r_{s+1} M.
\]

We define \(\nabla^* u\) as

\[
\nabla^* u = \{ g^{kl} D_l u \} : M \to T^r_{s+1} M \quad \text{for } u \text{ as in (2.3)}.
\]

(2.5)

For an \((r + 1, s)\)-tensor field \(u = \{ u_{i_1 \cdots i_s}^{j_1 \cdots j_r} \}\), the divergence of \(u\) is defined by

\[
div u = \{ D_l u_{i_1 \cdots i_s}^{j_1 \cdots j_r} \}.
\]

(2.6)
Therefore, the divergence $\text{div} \ u$ defined by (2.6) is an $(r, s)$-tensor field. Likewise, for an $(r, s + 1)$-tensor field
\[ u = \{ u_{j_1 \ldots j_r}^{i_1 \ldots i_s} \}, \]
the following operator is also called the divergence of $u$,
\[ \text{div} \ u = \{ D^l u_{j_1 \ldots j_r}^{i_1 \ldots i_s} \}, \tag{2.7} \]
where $D^l = g^{lk} D_k$, which is an $(r, s)$-tensor field.

For the gradient operators (2.4)-(2.5) and the divergent operators (2.6)-(2.7), it is well known that the following integral formulas hold true; see among others [9].

**Theorem 2.1.** Let $(M, g_{ij})$ be a closed Riemannian manifold. If $u$ is an $(r - 1, s)$-tensor and $v$ is an $(r, s)$ tensor, then we have
\[ (\nabla^* u, v) = -(u, \text{div} v), \tag{2.8} \]
where $\nabla^* u$ is as in (2.5) and $\text{div} v$ is as in (2.6), the inner product $(\cdot, \cdot)$ is as defined by (2.1). If $u$ is an $(r, s - 1)$-tensor and $v$ is an $(r, s)$ tensor, then
\[ (\nabla u, v) = -(u, \text{div} v), \tag{2.9} \]
where $\nabla u$ is as in (2.4) and $\text{div} v$ is as in (2.7).

**Remark 2.1.** If $M$ is a manifold with boundary $\partial M \neq \emptyset$, and $u|_{\partial M} = 0$ or $v|_{\partial M} = 0$, then the formulas (2.8) and (2.9) still hold true.

2.3. **Acute-angle principle.** Let $H$ be a Hilbert space equipped with inner product $(\cdot, \cdot)$ and norm $\| \cdot \|$, and $G : H \to H$ be a mapping. We say that $G$ is weakly continuous if for any sequence $\{ u_n \} \subset H$ weakly converging to $u_0$, i.e.
\[ u_n \rightharpoonup u_0 \text{ in } H, \]
we have
\[ \lim_{n \to \infty} (G u_n, v) = (G u_0, v) \quad \forall v \in H. \]
If the operator $G$ is linear and bounded, then $G$ is weakly continuous. The following theorem is called the acute-angle principle [16].

**Theorem 2.2.** If a mapping $G : H \to H$ is weakly continuous, and satisfies
\[ (G u, u) \geq \alpha \| u \|^2 - \beta, \]
for some constants $\alpha, \beta > 0$, then for any $f \in H$ there is a $u_0 \in H$ such that
\[ G(u_0) = f. \]

3. **Orthogonal decomposition for tensor fields.**

3.1. **Main theorems.** The aim of this section is to derive an orthogonal decomposition for $(r, s)$-tensor fields with $r + s \geq 1$ into divergence-free and gradient parts. This decomposition plays a crucial role for the theory of gravitational field, dark matter and dark energy developed later in this article.

Let $M$ be a closed Riemannian manifold, and $v \in L^2(T^r_s M)$ $(r + s \geq 1)$. We say that $v$ is divergence-free, i.e., $\text{div} v = 0$, if for all $\nabla \psi \in L^2(T^r_s M),$
\[ (v, \nabla \psi) = 0. \tag{3.1} \]
Here $\psi \in H^1(T^r_s M)$ or $H^1(T^r_{s-1} M)$, and $(\cdot, \cdot)$ is the $L^2$-inner product defined by (2.1).
We remark that if \( v \in H^1(T^r_s M) \) satisfies (3.1), then \( v \) is weakly differentiable, and \( \text{div} v = 0 \) in the \( L^2 \)-sense. If \( v \in L^2(T^r_s M) \) is not differential, then (3.1) means that \( v \) is divergence-free in the distribution sense.

**Theorem 3.1** (Orthogonal Decomposition Theorem). Let \( M \) be a closed Riemannian manifold, and \( u \in L^2(T^r_s M) \) with \( r + s \geq 1 \). The following assertions hold true:

1. The tensor field \( u \) has the following orthogonal decomposition:
   \[
   u = \nabla \varphi + v, \tag{3.2}
   \]
   where \( \varphi \in H^1(T^{r-1}_s M) \) or \( \varphi \in H^1(T^{-1}_s M) \), and \( \text{div} v = 0 \).

2. If \( M \) is compact, then \( u \) can be orthogonally decomposed into
   \[
   u = \nabla \varphi + v + h, \tag{3.3}
   \]
   where \( \varphi \) and \( v \) are as in (3.2), and \( h \) is a harmonic field, i.e.
   \[
   \text{div} v = 0, \quad \text{div} h = 0, \quad \nabla h = 0. \tag{3.4}
   \]

   In particular the subspace of all harmonic tensor fields in \( L^2(T^r_s M) \) is of finite dimensional:
   \[
   H(T^r_s M) = \{ h \in L^2(T^r_s M) \mid \nabla h = 0, \ \text{div} h = 0 \}, \quad \dim H < \infty. \tag{3.5}
   \]

**Remark 3.1.** The above orthogonal decomposition theorem implies that \( L^2(E) \) \((E = T^r_s M)\) can be decomposed into

\[
L^2(E) = G(E) \oplus L_D^2(E) \quad \text{for } \partial M = \emptyset,
\]

\[
L^2(E) = G(E) \oplus H(E) \oplus L_N^2(E) \quad \text{for } M \text{ compact}.
\]

Here \( H \) is as in (3.5), and
\[
G(E) = \{ v \in L^2(E) \mid v = \nabla \varphi, \varphi \in H^1(T^r_s M) \},
\]

\[
L_D^2(E) = \{ v \in L^2(E) \mid \text{div} v = 0 \},
\]

\[
L_N^2(E) = \{ v \in L^2(E) \mid \nabla v \neq 0 \}.
\]

They are orthogonal to each other:
\[
L_D^2(E) \perp G(E), \quad L_N^2(E) \perp H(E), \quad G(E) \perp H(E).
\]

**Remark 3.2.** The dimension of the harmonic space \( H(E) \) is related with the bundle structure of \( E = T^r_s M \). It is conjectured [15] that
\[
\dim H = k = \text{the degree of freedom of } E.
\]

Namely \( k \) is the integer that \( E \) can be decomposed into the Whitney sum of a \( k \)-dimensional trivial bundle \( E^k = M \times \mathbb{R}^k \) and a nontrivial bundle \( E_1 \), i.e.
\[
E = E_1 \oplus E^k.
\]

**Remark 3.3.** We prove in this section only the case where \( M \) is equipped with positive definite Riemannian metric. The case with space-time metric can be proved by using solvability of wave equations, and we omit the details.

**Proof of Theorem 3.1.** We proceed in several steps as follows.

**Step 1 Proof of Assertion (1).** Let \( u \in L^2(E), \ E = T^r_s M \) \((r + s \geq 1)\). Consider the equation
\[
\Delta \varphi = \text{div} u \quad \text{in } M, \tag{3.6}
\]
where \( \Delta \) is the Laplace operator defined by
\[
\Delta = \text{div} \nabla. \tag{3.7}
\]
Without loss of generality, we only consider the case where $\text{div } u \in \tilde{E} = T^{-1}_s M$. It is clear that if the equation (3.6) has a solution $\varphi \in H^1(\tilde{E})$, then by (3.7), the following vector field must be divergence-free
\[ v = u - \nabla \varphi \in L^2(E). \] (3.8)
Moreover, by (3.1) we have
\[ (v, \nabla \varphi) = 0. \] (3.9)
Namely $v$ and $\nabla \varphi$ are orthogonal. Therefore, the orthogonal decomposition $u = v + \nabla \varphi$ follows from (3.8) and (3.9).

It suffices then to prove that (3.6) has a weak solution $\varphi \in H^1(\tilde{E})$:
\[ (\nabla \varphi - u, \nabla \psi) = 0 \quad \forall \psi \in H^1(\tilde{E}). \] (3.10)
To this end, let
\[ H = H^1(\tilde{E}) \setminus \tilde{H}, \]
\[ \tilde{H} = \{ \psi \in H^1(\tilde{E}) | \nabla \psi = 0 \}. \]
Then we define a linear operator $G : H \rightarrow H$ by
\[ (G\varphi, \psi) = (\nabla \varphi, \nabla \psi) \quad \forall \psi \in H. \] (3.11)
It is clear that the linear operator $G : H \rightarrow H$ is bounded, weakly continuous, and
\[ (G\varphi, \varphi) = (\nabla \varphi, \nabla \varphi) = ||\varphi||^2. \] (3.12)
Based on Theorem 2.2, for any $f \in H$, the equation
\[ \Delta \varphi = f \quad \text{in } M \]
has a weak solution $\varphi \in H$. Hence for $f = \text{div } u$ the equation (3.6) has a solution, and Assertion (1) is proved. In fact the solution of (3.6) is unique. We remark that by the Poincaré inequality, for the space $H = H^1(\tilde{E}) \setminus \{ \psi | \nabla \psi = 0 \}$, (3.12) is an equivalent norm of $H$. In addition, by Theorem 2.1, the weak formulation (3.10) for (3.6) is well-defined.

**Step 2 Proof of Assertion (2).** Based on Assertion (1), we have
\[ H^k(E) = H^k_D \oplus G^k, \]
\[ L^2(E) = L^2_D \oplus G, \]
where
\[ H^k_D = \{ u \in H^k(E) | \text{ div } u = 0 \}, \]
\[ G^k = \{ u \in H^k(E) | u = \nabla \psi \}. \]
Define an operator $\tilde{\Delta} : H^2_D(E) \rightarrow L^2_D(E)$ by
\[ \tilde{\Delta} u = P\Delta u, \] (3.13)
where $P : L^2(E) \rightarrow L^2_D(E)$ is the canonical orthogonal projection.

We know that the Laplace operator $\Delta$ can be expressed as
\[ \Delta = \text{div } \nabla = D^k D_k = g^{kl} \frac{\partial^2}{\partial x^k \partial x^l} + B, \] (3.14)
where $B$ is the lower order derivative linear operator. Since $M$ is compact, the Sobolev embeddings $H^2(E) \hookrightarrow H^1(E) \hookrightarrow L^2(E)$ are compact, which implies that the lower order derivative operator
\[ B : H^2(M, \mathbb{R}^N) \rightarrow L^2(M, \mathbb{R}^N) \] is compact,
where the integer \(N\) is the dimension of the tensor bundle \(E\). According to the elliptic operator theory, the elliptic operator in (3.14)

\[
A = g^{kj} \frac{\partial^2}{\partial x^k \partial x^l} : H^2(M, \mathbb{R}^N) \rightarrow L^2(M, \mathbb{R}^N)
\]

is a linear homeomorphism. Therefore the operator in (3.14) is a linear completely continuous field

\[
\Delta : H^2(E) \rightarrow L^2(E),
\]

which implies that the operator of (3.13) is also a linear completely continuous field:

\[
\tilde{\Delta} = P\Delta : H^2_D(E) \rightarrow L^2_D(E).
\]

By the spectral theorem of completely continuous fields [17, 19], the space \(\tilde{H} = \{u \in H^2_D(E) \mid \tilde{\Delta}u = 0\}\) is finite dimensional, and is the eigenspace of the eigenvalue \(\lambda = 0\). By Theorem 2.1, for \(u \in \tilde{H}\)

\[
\int_M (\tilde{\Delta}u, u) \sqrt{-g} \, dx = \int_M (\Delta u, u) \sqrt{-g} \, dx = 0
\]

(by \(\text{div}u = 0\)).

It follows that

\[
u \in \tilde{H} \iff \nabla u = 0 \ \Rightarrow H = \tilde{H},
\]

where \(H\) is the harmonic space as in (3.5). Thus we have

\[
L^2_D(E) = H \oplus L^2_H(E),
\]

\[
L^2_H(E) = \{u \in L^2_D(E) \mid \nabla u \neq 0\}.
\]

The proof of Theorem 3.1 is complete. \(\square\)

3.2. **Uniqueness of the orthogonal decomposition.** In Theorem 3.1, a tensor field \(u \in L^2(T^r_s M)\) with \(r + s \geq 1\) can be orthogonally decomposed into

\[
u = \nabla \varphi + v \quad \text{for} \ \partial M = 0, \quad \nu = \nabla \varphi + v + h \quad \text{for} \ M \text{ compact.} \quad (3.15)
\]

Now we address the uniqueness problem of the decomposition (3.15). In fact, if \(u\) is a vector field or a co-vector field, i.e.

\[
u \in L^2(TM) \text{ or } u \in L^2(T^* M),
\]

then the decomposition of (3.15) is unique.

We can see that if \(u \in L^2(T^r_s M)\) with \(r + s \geq 2\), then there are different types of the decompositions of (3.15). For example, for \(u \in L^2(T^0_2 M)\), the local expression of \(u\) is given by

\[
u = \{ui_j(x)\}.
\]

In this case, \(u\) has two types of decompositions:

\[
u_{ij} = D_i \varphi_j + v_{ij}, \quad D^i v_{ij} = 0, \quad (3.16)
\]

\[
u_{ij} = D_j \psi_i + w_{ij}, \quad D^i w_{ij} = 0. \quad (3.17)
\]
It is easy to see that if \( u_{ij} \neq u_{ji} \), then (3.16) and (3.17) can be two different decompositions of \( u_{ij} \). Namely
\[
\{v_{ij}\} \neq \{w_{ij}\}, \quad (\varphi_1, \cdots, \varphi_n) \neq (\psi_1, \cdots, \psi_n).
\]

If \( u_{ij} = u_{ji} \) is symmetric, \( u \) can be orthogonally decomposed into the following two forms:
\[
\begin{align*}
u_{ij} &= v_{ij} + D_i \varphi_j, \quad D^i v_{ij} = 0, \\
u_{ij} &= w_{ij} + D_j \psi_i, \quad D^j w_{ij} = 0,
\end{align*}
\]
and \( \varphi \) and \( \psi \) satisfy
\[
\begin{align*}
\Delta \varphi_j &= D^k u_{kj}, \quad (3.18) \\
\Delta \psi_j &= D^k u_{jk}. \quad (3.19)
\end{align*}
\]
By \( u_{ij} = u_{ji} \) we have \( D^k u_{kj} = D^k u_{jk} \). Hence, (3.18) and (3.19) are the same, and \( \varphi = \psi \). Therefore, the symmetric tensors \( u_{ij} \) can be written as
\[
\begin{align*}
u_{ij} &= v_{ij} + D_i \varphi_j, \quad D^i v_{ij} = 0, \quad (3.20) \\
u_{ij} &= w_{ij} + D_j \varphi_i, \quad D^j w_{ij} = 0. \quad (3.21)
\end{align*}
\]
From (3.20)-(3.21) we can deduce the following theorem.

**Theorem 3.2.** Let \( u \in L^2(T^0 T^2 M) \) be symmetric, i.e. \( u_{ij} = u_{ji} \), and the first Betti number \( \beta_1(M) = 0 \) for \( M \). Then the following assertions hold true:

1. \( u \) has a unique orthogonal decomposition if and only if there is a scalar function \( \varphi \in H^2(M) \) such that \( u \) can be expressed as
\[
\begin{align*}
u_{ij} &= v_{ij} + D_i D_j \varphi, \\
v_{ij} &= v_{ji}, \quad D^i v_{ij} = 0.
\end{align*}
\]
(3.22)

2. \( u \) can be orthogonally decomposed in the form of (3.22) if and only if \( u_{ij} \) satisfy
\[
\begin{align*}
\frac{\partial}{\partial x^j}(D^k u_{ki}) - \frac{\partial}{\partial x^i}(D^k u_{kj}) &= \frac{\partial}{\partial x^j} \left( R^k_j \partial \varphi \partial x^k \right) - \frac{\partial}{\partial x^i} \left( R^k_i \partial \varphi \partial x^k \right),
\end{align*}
\]
where \( R^k_j = g^{ki} R_{ij} \) and \( R_{ij} \) are the Ricci curvature tensors.
(3.23)

3. If \( u_{ij} \) in (3.20) is symmetric: \( v_{ij} = v_{ji} \), then \( u \) can be expressed by (3.22).

**Proof.** We only need to prove Assertions (2) and (3).

We first prove Assertion (2). It follows from (3.20) that
\[
\frac{\partial}{\partial x^j}(D^k u_{ki}) - \frac{\partial}{\partial x^i}(D^k u_{kj}) = \frac{\partial \Delta \varphi_i}{\partial x^j} - \frac{\partial \Delta \varphi_j}{\partial x^i},
\]
where \( \Delta = D^k D_k \). By the Weitzenböck formula [15],
\[
\Delta \varphi_i = -(\delta d + d \delta) \varphi_i - R_i^k \varphi_k,
\]
and \( (\delta d + d \delta) \) is the Laplace-Beltrami operator. We know that for \( \omega = \varphi_i dx^i \),
\[
\begin{align*}
d\omega &= 0 \quad \Leftrightarrow \quad \varphi_i = \frac{\partial \varphi}{\partial x^i}, \\
d \delta \omega &= \nabla(\Delta \varphi) \quad \Leftrightarrow \quad \varphi_i = \frac{\partial \varphi}{\partial x^i},
\end{align*}
\]
where \( \nabla \) is the gradient operator, and
\[
\Delta \varphi = -\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^i} \left( -\sqrt{-g} g^{ij} \frac{\partial \varphi}{\partial x^j} \right).
\]
Namely
\[(\delta d + d\delta)\varphi_i = \frac{\partial}{\partial x^i} \Delta \varphi \iff \varphi_i = \frac{\partial \varphi}{\partial x^i}.\]
Hence, we infer from (3.25) that
\[\Delta \varphi_i = -\frac{\partial}{\partial x^i} \tilde{\Delta} \varphi - R^k_i \frac{\partial \varphi}{\partial x^k} \iff \varphi_i = \frac{\partial \varphi}{\partial x^i}.\] (3.26)
Thus, by (3.24) and (3.26), we obtain that
\[\frac{\partial}{\partial x^j} \left(D^k_i u_{ki} \right) - \frac{\partial}{\partial x^i} \left(D^k_j u_{kj} \right) = \frac{\partial}{\partial x^i} \left(R^k_j \varphi \right) - \frac{\partial}{\partial x^j} \left(R^k_i \varphi \right)\]
holds true if and only if the tensor \(\psi = (\varphi_1, \cdots, \varphi_n)\) in (3.20) is a gradient \(\psi = \nabla \varphi\).

Assertion (2) is proven.

Now we verify Assertion (3). Since \(v_{ij}\) in (3.20) is symmetric, we have
\[D_i \varphi_j = D_j \varphi_i,\] (3.27)
Note that
\[D_i \varphi_j = \frac{\partial \varphi_j}{\partial x^i} - \Gamma^k_{ij} \varphi_k,\] (3.28)
where \(\Gamma^k_{ij}\) is the Levi-Civita connection, and \(\Gamma^k_{ij} = \Gamma^k_{ji}\). We infer then from (3.27) that
\[\frac{\partial \varphi_i}{\partial x^j} = \frac{\partial \varphi_i}{\partial x^j}.\] (3.29)
By assumption, the 1-dimensional homology of \(M\) is zero,
\[H_1(M) = 0,\]
and it follows from the de Rham theorem and (3.29) that
\[\varphi_k = \frac{\partial \varphi}{\partial x^k},\]
for some scalar function \(\varphi\). Thus Assertion (3) follows and the proof is complete. \(\square\)

**Remark 3.4.** The conclusions of Theorem 3.2 are also valid for second-order contra-variant symmetric tensors \(u = \{u^{ij}\}\), and the decomposition is given as follows:
\[u^{ij} = v^{ij} + g^{ik} g^{jl} D_k D_l \varphi,\]
\[D_i v^{ij} = 0, \quad v^{ij} = v^{ji}, \quad \varphi \in H^2(M).\]


4.1. General theory. Hereafter we always assume that \(M\) is a closed manifold. A Riemannian metric \(G\) on \(M\) is a mapping
\[G : M \to T^*_2 M = T^* M \otimes T^* M,\] (4.1)
which is symmetric and nondegenerate, i.e., in a local coordinate \((v, x)\), \(G\) can be expressed as
\[G = \{g_{ij}(x)\} \quad \text{with} \quad g_{ij} = g_{ji},\] (4.2)
and the matrix \((g_{ij})\) is invertible on \(M\):
\[(g^{ij}) = (g_{ij})^{-1}.\] (4.3)
Therefore, then the set of all metrics \( G \) is the space of Riemannian metrics on \( M \) such that the space for Riemannian metrics on \( M \) are also called the Euler-Lagrange operators of \( F \) where \( W \) is second-order contra-variant and covariant tensor fields respectively, i.e. \( g \).

Expressed as \( W^m,2(M, g) = \{ G \mid G \in W^m,2(T^0_2M), G^{-1} \in W^m,2(T^0_2M), \}

\( G \) is the Riemannian metric on \( M \) as in (4.2)\}, which is a metric space, but not a Banach space. However, it is a subspace of the direct sum of two Sobolev spaces \( W^m,2(T^0_2M) \) and \( W^m,2(T^0_2M) \):

\[ W^m,2(M, g) \subset W^m,2(T^0_2M) \oplus W^m,2(T^0_2M). \]

A functional defined on \( W^m,2(M, g) \):

\[ F: W^m,2(M, g) \to \mathbb{R} \tag{4.6} \]

is called the functional of Riemannian metric. Usually, the functional (4.6) can be expressed as

\[ F(g_{ij}) = \int_M f(x, g_{ij}, \cdots, \partial^m g_{ij}) \sqrt{-g} dx. \tag{4.7} \]

Since \((g^{ij})\) is the inverse of \((g_{ij})\), we have

\[ g_{ij} = \frac{1}{g} \times \text{the cofactor of } g^{ij}. \tag{4.8} \]

Therefore, \( F(g_{ij}) \) in (4.7) also depends on \( g^{ij} \), i.e. putting (4.8) in (4.7) we get

\[ F(g^{ij}) = \int_M \tilde{f}(x, g^{ij}, \cdots, D^m g_{ij}) \sqrt{-g} dx. \tag{4.9} \]

We note that although \( W^m,2(M, g) \) is not a linear space, for a given element \( g_{ij} \in W^m,p(M, g) \) and any symmetric tensor \( X_{ij} \) and \( X^{ij} \), there is a number \( \lambda_0 > 0 \) such that

\[ g_{ij} + \lambda X_{ij} \in W^m,2(M, g) \quad \forall \lambda \leq |\lambda| < \lambda_0, \]

\[ g^{ij} + \lambda X^{ij} \in W^m,2(M, g) \quad \forall \lambda \leq |\lambda| < \lambda_0. \tag{4.10} \]

Due to (4.10), we can define the derivative operators of the functional \( F \), which are also called the Euler-Lagrange operators of \( F \), as follows

\[ \delta_s F: W^m,2(M, g) \to W^{-m,2}(T^0_2M), \]

\[ \delta^* F: W^{-m,2}(M, g) \to W^m,2(T^0_2M), \tag{4.12} \]

where \( W^{-m,2}(E) \) is the dual space of \( W^m,2(E) \), and \( \delta_s F, \delta^* F \) are given by

\[ \langle \delta_s F(g_{ij}), X \rangle = \frac{d}{d\lambda} F(g_{ij} + \lambda X_{ij})|_{\lambda=0}, \tag{4.13} \]

\[ \langle \delta^* F(g^{ij}), X \rangle = \frac{d}{d\lambda} F(g^{ij} + \lambda X^{ij})|_{\lambda=0}. \tag{4.14} \]

For any given metric \( g_{ij} \in W^m,2(M, g) \), the value of \( \delta_s F \) and \( \delta^* F \) at \( g_{ij} \) are second-order contra-variant and covariant tensor fields respectively, i.e.

\[ \delta_s F(g_{ij}): M \to TM \otimes TM, \]

\[ \delta^* F(g_{ij}): M \to T^* M \otimes T^* M. \]
Moreover, the equations
\[ \delta_\ast F(g_{ij}) = 0, \]  
\[ \delta_\ast F(g_{ij}) = 0, \]  
(4.15)  
(4.16)
are called the Euler-Lagrange equations of \( F \), and the solutions of (4.15) and (4.16) are called the extremum points or critical points of \( F \).

**Theorem 4.1.** Let \( F \) be the functionals defined by (4.6) and (4.9). Then the following assertions hold true:

1. For any \( g_{ij} \in W^{m,2}(M,g), \delta_\ast F(g_{ij}) \) and \( \delta_\ast F(g_{ij}) \) are symmetric tensor fields.
2. If \( \{g_{ij}\} \in W^{m,2}(M,g) \) is the extremum point of \( F \), then \( \{g_{ij}\} \) satisfies (4.15) and (4.16) if and only if \( \{g^{ij}\} \) satisfies (4.15) and (4.16).
3. \( \delta_\ast F \) and \( \delta_\ast F \) have the following relation
\[ (\delta_\ast F(g_{ij}))^{kl} = -g^{kr}g^{ls}(\delta_\ast F(g_{ij}))_{rs}, \]
where \( (\delta_\ast F)^{kl} \) and \( (\delta_\ast F)_{kl} \) are the components of \( \delta_\ast F \) and \( \delta_\ast F \) respectively.

**Proof.** We only need to verify Assertion (3). Noting that
\[ g_{ik}g_{kj} = \delta_{ij}, \]
we have the variational relation
\[ \delta(g_{ik}g_{kj}) = g_{ik}\delta g_{kj} + g_{kj}\delta g_{ik} = 0. \]
It implies that
\[ \delta g^{kl} = -g^{ki}g^{lj}\delta g_{ij}, \]  
(4.17)
In addition, in (4.13) and (4.14),
\[ \lambda X_{ij} = \delta g_{ij}, \quad \lambda X^{ij} = \delta g^{ij}, \quad \lambda \neq 0 \text{ small}. \]
Therefore, by (4.17) we get
\[ ((\delta_\ast F)_{kl}, \delta g^{kl}) = -((\delta_\ast F)_{kl}, g^{ki}g^{lj}\delta g_{ij}) = \langle g^{ki}g^{lj}(\delta_\ast F)_{kl}, \delta g_{ij} \rangle = ((\delta_\ast F)^{ij}, \delta g_{ij}). \]
Hence
\[ (\delta_\ast F)^{ij} = -g^{ki}g^{lj}(\delta_\ast F)_{kl}. \]
Thus Assertion (3) follows and the proof is complete.

**4.2. Scalar potential theorem for constraint variations.** We know that the critical points of the functional \( F \) in (4.6) are the solution
\[ \delta F(g_{ij}) = 0, \]  
(4.18)
in the following sense
\[ (\delta F(g_{ij}), X) = \frac{d}{d\lambda} F(g^{ij} + \lambda X^{ij})|_{\lambda=0} \]
\[ = \int_M (\delta F(g_{ij}))_{kl}X^{kl}\sqrt{-g}dx \]
\[ = 0 \quad \forall X^{kl} = X^{lk} \text{ in } L^2(E), \]
where \( E = TM \otimes TM \). Hence, the critical points of functionals of Riemannian metrics are not solutions of (4.18) in the usual sense.
It is easy to see that $L^2(TM \otimes TM)$ can be orthogonally decomposed into the direct sum of the symmetric and contra-symmetric spaces, i.e.

$$
L^2(E) = L^2_s(E) \oplus L^2_c(E),
$$

(4.20)

$$
L^2_s(E) = \{ u \in L^2(E) \mid u_{ij} = u_{ji} \},
$$

$$
L^2_c(E) = \{ v \in L^2(E) \mid v_{ij} = -v_{ji} \}.
$$

Since $\delta F$ is symmetric, by (4.20) the extremum points $\{g_{ij}\}$ of $F$ satisfy the more general equality

$$
(\delta F(g_{ij}), X) = 0 \quad \forall X = \{X_{ij}\} \in L^2(E).
$$

(4.21)

Thus, we can say that the extremum points of functionals of the Riemannian metrics are solutions of (4.18) in the usual sense of (4.21), or are zero points of the variational operators

$$
\delta F : W^{m,2}(M, g) \to W^{-m,2}(E).
$$

Now we consider the variations of $F$ under the divergence-free constraint. In this case, the Euler-Lagrangian equations with symmetric divergence-free constraints are equivalent to the Euler-Lagrangian equations with general divergence-free constraints. Hence we have the following definition.

**Definition 4.1.** Let $F : W^{m,2}(M, g) \to \mathbb{R}$ be a functional of Riemannian metric. A metric tensor $\{g_{ij}\} \in W^{m,2}(M, g)$ is called an extremum point of $F$ with divergence-free constraint, if $\{g_{ij}\}$ satisfies

$$
(\delta F(g_{ij}), X) = 0 \quad \forall X = \{X_{ij}\} \in L^2(E),
$$

(4.22)

where $L^2(D)(E)$ is the space of all divergence-free tensors:

$$
L^2(D)(E) = \{ X \in L^2(E) \mid \text{div } X = 0 \}.
$$

It is clear that an extremum point satisfying (4.22) is not a solution of (4.18). Instead, we have the scalar potential theorem for the extremum points of divergence-free constraint (4.22), which is based on the orthogonal decomposition theorems. This result is also crucial for the gravitational field equations and the theory of dark matter and dark energy developed later.

**Theorem 4.2** (Scalar Potential Theorem). Assume that the first Betti number of $M$ is zero, i.e. $\beta_1(M) = 0$. Let $F$ be a functional of the Riemannian metric. Then there is a $\varphi \in H^2(M)$ such that the extremum points $\{g_{ij}\}$ of $F$ with divergence-free constraint satisfy

$$
(\delta F(g_{ij}))(kl) = D_k D_l \varphi.
$$

(4.23)

**Proof.** Let $\{g_{ij}\}$ be an extremum point of $F$ under the constraint (4.22). Namely, $\delta F(g_{ij})$ satisfies

$$
\int_M (\delta F(g_{ij}))(kl) X^{kl} \sqrt{-g} dx = 0 \quad \forall X = \{X_{kl}\} \text{ with } D_k X^{kl} = 0.
$$

(4.24)

By Theorem 3.1, $\delta F(g_{ij})$ can be orthogonally decomposed as

$$
(\delta F(g_{ij}))(kl) = v_{kl} + D_k \psi_l , \quad D^k v_{kl} = 0.
$$

(4.25)

By Theorem 2.1, for any $D_k X^{kl} = 0$,

$$
(D_k \psi_l, X^{kl}) = \int_M D_k \psi_l X^{kl} \sqrt{-g} dx = - \psi_l D_k X^{kl} \sqrt{-g} dx = 0.
$$

(4.26)
Therefore it follows from (4.24)-(4.26) that
\[
\int_M v_{kl} X^{kl} \sqrt{-g} dx = 0 \quad \forall D_k X^{kl} = 0.
\] (4.27)

Let \( X^{kl} = g^{ki} g^{lj} v_{ij} \). Since
\[
D_k g_{ij} = D_k g^{ij} = 0,
\]
thanks to \( D_i v_{ij} = 0 \). Inserting \( X^{kl} = g^{ki} g^{lj} v_{ij} \) into (4.27) leads to
\[
||v||^2_{L^2} = \int_M g^{ki} g^{lj} v_{kl} v_{ij} \sqrt{-g} dx = 0,
\]
which implies that \( v = 0 \). Thus, (4.25) becomes
\[
(\delta F(g_{ij}))_{kl} = D_k \psi_l.
\] (4.28)

By Theorem 4.1, \( \delta F \) is symmetric. Hence we have
\[
D_k \psi_l = D_l \psi_k.
\]

It follows from (3.28) that
\[
\frac{\partial \psi_l}{\partial x^k} = \frac{\partial \psi_k}{\partial x^l}.
\] (4.29)

By assumption, the first Betti number of \( M \) is zero, i.e. the 1-dimensional homology of \( M \) is zero: \( H_1(M) = 0 \). It follows from the de Rham theorem that if
\[
d(\psi_k dx^k) = \left( \frac{\partial \psi_k}{\partial x^l} - \frac{\partial \psi_l}{\partial x^k} \right) dx^l \wedge dx^k = 0,
\]
then there exists a scalar function \( \varphi \) such that
\[
d \varphi = \frac{\partial \varphi}{\partial x^k} dx^k = \psi_k dx^k.
\]

Thus, we infer from (4.29) that
\[
\psi_l = \frac{\partial \varphi}{\partial x^l} \text{ for some } \varphi \in H^2(M).
\]

Therefore we get (4.23) from (4.28). The theorem is proved.

If the first Betti number \( \beta_1(M) \neq 0 \), then there are \( N = \beta_1(M) \) number of 1-forms:
\[
\omega_j = \psi^j_k dx^k \in H_1^d(M) \quad \text{for} \quad 1 \leq j \leq N,
\] (4.30)
which constitute a basis of the 1-dimensional de Rham homology \( H_1^d(M) \). We know that the components of \( \omega_j \) are co-vector fields:
\[
\psi^j = (\psi^j_1, \cdots, \psi^j_n) \in H^1(T^*M) \quad \text{for} \quad 1 \leq j \leq N,
\] (4.31)
which possess the following properties:
\[
\frac{\partial \psi^j_k}{\partial x^l} = \frac{\partial \psi^j_l}{\partial x^k} \quad \text{for} \quad 1 \leq j \leq N,
\]
or equivalently,
\[
D_l \psi^j_k = D_k \psi^j_l \quad \text{for} \quad 1 \leq j \leq N.
\]

Namely, \( \nabla \psi^j \in L^2(T^*M \otimes T^*M) \) are symmetric second-order contra-variant tensors. Hence Theorem 4.2 can be extended to the non-vanishing first Betti number case as follows.
Theorem 4.3. Let the first Betti number $\beta_1(M) \neq 0$ for $M$. Then for the functional $F$ of Riemannian metrics, the extremum points $\{g_{ij}\}$ of $F$ with the constraint (4.22) satisfy the equations

\[
(\delta F(g_{ij}))_{kl} = D_k D_l \varphi + \sum_{j=1}^{N} \alpha_j D_k \psi_j^l,
\]

(4.32)

where $N = \beta_1(M)$, $\alpha_j$ are constants, $\varphi \in H^2(M)$, and the tensors $\psi^j = (\psi_1^j, \cdots, \psi_n^j) \in H^1(T^*M)$ are as given by (4.31).

The proof of Theorem 4.3 is similar to Theorem 4.2, and is omitted here.

Remark 4.1. By the Hodge decomposition theory, the 1-forms $\omega_j$ in (4.30) are harmonic:

\[
d\omega_j = 0, \quad \delta \omega_j = 0 \quad \text{for} \quad 1 \leq j \leq N,
\]

which implies that the tensors $\psi^j$ in (4.32) satisfy

\[
(\delta d + d \delta)\psi^j = 0 \quad \text{for} \quad 1 \leq j \leq N.
\]

(4.33)

According to the Weitzenböck formula (3.25), we obtain from (4.33) that

\[
D^k D_k \psi^j_l = -R^k_l \psi^j_k \quad \text{for} \quad 1 \leq j \leq N,
\]

(4.34)

for $\psi^j = (\psi_1^j, \cdots, \psi_n^j)$ in (4.32).

Remark 4.2. Theorem 4.2 is derived for deriving new gravitational field equations in the next section for explaining the phenomena of dark matter and dark energy. The condition that $\beta_1(M) = 0$ means that any loops in the manifold $M$ can shrink to a point. Obviously, our universe can be considered as a 4-dimensional manifold satisfying this condition.

Part 2. Physics

5. Gravitational field equations.

5.1. Einstein-Hilbert functional. The general theory of relativity is based on three basic principles: the principle of equivalence, the principle of general relativity, and the principle of Lagrangian dynamics. The first two principles tell us that the spatial and temporal world is a 4-dimensional Riemannian manifold $(M, g_{ij})$, where the metric $\{g_{ij}\}$ represents gravitational potential, and the third principle determines that the Riemannian metric $\{g_{ij}\}$ is an extremum point of the Lagrangian action, which is the Einstein-Hilbert functional.

Let $(M, g_{ij})$ be an $n$-dimensional Riemannian manifold. The Einstein-Hilbert functional\(^1\)

\[
F : W^{2,2}(M, g) \to \mathbb{R}
\]

is defined by

\[
F(g_{ij}) = \int_M \left( R + \frac{8\pi G}{c^4} g^{ij} S_{ij} \right) \sqrt{-g} dx,
\]

(5.2)

where $W^{2,2}(M, g)$ is defined by (4.5), $R = g^{ij} R_{ij}$ and $R_{ij}$ are the scalar and the Ricci curvatures, $S_{ij}$ is the stress tensor, $G$ is the gravitational constant, and $c$ is the speed of light.

\(^1\)The matter tensor is included here as well.
The Euler-Lagrangian of the Einstein-Hilbert functional $F$ is given by
\[
\delta F(g_{ij}) = R_{ij} - \frac{1}{2}g_{ij}R + \frac{8\pi G}{c^4}T_{ij},
\] (5.3)
where $T_{ij}$ is the energy-momentum tensor given by
\[
T_{ij} = S_{ij} - \frac{1}{2}g_{ij}S + g_{kl}\frac{\partial S_{kl}}{\partial g_{ij}},
\]
and the Ricci curvature tensor $R_{ij}$ is given by
\[
R_{ij} = \frac{1}{2}g^{kl}\left(\frac{\partial^2 g_{kl}}{\partial x^i \partial x^j} + \frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} - \frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} - \frac{\partial^2 g_{kl}}{\partial x^i \partial x^j}\right)
\] + $g_{rl}g_{rs}\left(\Gamma^r_{kl}\Gamma^s_{ij} - \Gamma^r_{il}\Gamma^s_{kj}\right),
\] (5.4)
\[
\Gamma^k_{ij} = \frac{1}{2}g^{kl}\left(\frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l}\right).
\] (5.5)
By (5.3)-(5.5), the Euler-Lagrangian $\delta F(g_{ij})$ of the Einstein-Hilbert functional is a second order differential operator on $\{g_{ij}\}$, and $\delta F(g_{ij})$ is symmetric.

5.2. Einstein field equations. The General Theory of Relativity consists of two main conclusions:

1) The space-time of our world is a 4-dimensional Riemannian manifold $(M^4, g_{ij})$, and the metric $\{g_{ij}\}$ represents gravitational potential.

2) The metric $\{g_{ij}\}$ is the extremum point of the Einstein-Hilbert functional (5.2). In other words, gravitational field theory obeys the principle of Lagrange dynamics.

The principle of Lagrange dynamics is a universal principle, stated as:

**Principle of Lagrange Dynamics.** For any physical system, there are a set of state functions
\[ u = (u_1, \ldots, u_N), \]
which describe the state of this system, and there exists a functional $L$ of $u$, called the Lagrange action:
\[
L(u) = \int_0^T \int_\Omega L(u, Du, \ldots, D^m u) dx dt,
\] (5.6)
such that the state $u$ is an extremum point of $L$. Usually the function $L$ in (5.6) is called the Lagrangian density.

Based on this principle, the gravitational field equations are the Euler-Lagrange equations of the Einstein-Hilbert functional:
\[
\delta F(g_{ij}) = 0,
\] (5.7)
which are the classical Einstein field equations:
\[
R_{ij} - \frac{1}{2}g_{ij}R = -\frac{8\pi G}{c^4}T_{ij},
\] (5.8)
By the Bianchi identities, the left hand side of (5.8) is divergence-free, i.e.
\[
D^i (R_{ij} - \frac{1}{2}g_{ij}R) = 0.
\] (5.9)
Therefore it is required in the general theory of relativity that the energy-momentum tensor $\{T_{ij}\}$ in (5.8) satisfies the following energy-momentum conservation law:
\[
D^i T_{ij} = 0 \quad \text{for } 1 \leq j \leq n.
\] (5.10)
5.3. **New gravitational field equations.** Motivated by the mystery of dark energy and dark matter and the other difficulties encountered by the Einstein field equations as mentioned in Introduction, we introduce in this section a new set of field equations, still obeying the three basic principles of the General Theory of Relativity.

Our key observation is a well-known fact that the Riemannian metric $g_{ij}$ is divergence-free. This suggests us two important postulates for deriving the new gravitational field equations:

- The energy-momentum tensor of matter need not to be divergence-free due to the presence of dark energy and dark matter; and
- The field equation obeys the Euler-Lagrange equation of the Einstein-Hilbert functional under the natural divergence-free constraint.

Under these two postulates, by the Scalar Potential Theorem, Theorem 4.2, if the Riemannian metric $\{g_{ij}\}$ is an extremum point of the Einstein-Hilbert functional (5.2) with the divergence-free constraint (4.22), then the gravitational field equations are taken in the following form:

$$R_{ij} - \frac{1}{2} g_{ij} R = -\frac{8\pi G}{c^4} T_{ij} - D_i D_j \varphi,$$

(5.11)

where $\varphi \in H^2(M)$ is called the scalar potential. We infer from (5.9) that the conservation laws for (5.11) are as follows

$$\text{div} \ (D_i D_j \varphi + \frac{8\pi G}{c^4} T_{ij}) = 0.$$

(5.12)

Using the contraction with $g^{ij}$ in (5.11), we have

$$R = \frac{8\pi G}{c^4} T + \Phi,$$

(5.13)

where

$$T = g^{ij} T_{ij}, \quad \Phi = g^{ij} D_i D_j \varphi,$$

represent respectively the energy-momentum density and the scalar potential density. Physically this scalar potential density $\Phi$ represents potential energy caused by the non-uniform distribution of matter in the universe. One important property of this scalar potential is

$$\int_M \Phi \sqrt{-g} dx = 0,$$

(5.14)

which is due to the integration by parts formula in Theorem 2.1. This formula demonstrates clearly that the negative part of this quantity $\Phi$ represents the dark matter, which produces attraction, and the positive part represents the dark energy, which drives the acceleration of expanding galaxies. We shall address this important issue in the next section.

5.4. **Field equations for closed universe.** The topological structure of closed universe is given by

$$M = S^1 \times S^3,$$

(5.15)

where $S^1$ is the time circle and $S^3$ is the 3-dimensional sphere representing the space. We note that the radius $R$ of $S^3$ depends on time $t \in S^1$,

$$R = R(t), \quad t \in S^1,$$

and the minimum time $t_0$,

$$t_0 = \min_t R(t)$$
is the initial time of the Big Bang.

For a closed universe as \((5.15)\), by Theorem 4.3, the gravitational field equations are in the form

\[
R_{ij} - \frac{1}{2} g_{ij} R = -\frac{8\pi G}{c^4} T_{ij} - D_i D_j \varphi + \alpha D_i \psi_j,
\]

\[
\Delta \psi_j + g^{ik} R_{ij} \psi_k = 0,
\]

\[
D_i \psi_j = D_j \psi_i,
\]

where \(\Delta = D_k D_k\), \(\varphi\) the scalar potential, \(\psi = (\psi_0, \psi_1, \psi_2, \psi_3)\) the vector potential, and \(\alpha\) is a constant. By (5.9), the conservation laws of (5.16) are as follows

\[
\Delta \psi_j = \frac{1}{\alpha} \Delta \left( \frac{\partial \varphi}{\partial x^j} \right) + \frac{8\pi G}{\alpha c^4} D^k T_{kj},
\]

(5.17)

6. Interaction in a central gravitational field.

6.1. Schwarzschild solution. We know that the metric of a central gravitational field is in a diagonal form [1]:

\[
ds^2 = g_{00} c^2 dt^2 + g_{11} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2),
\]

(6.1)

and physically \(g_{00}\) is given by

\[
g_{00} = - \left( 1 + \frac{2}{c^2} \psi \right),
\]

(6.2)

where \(\psi\) is the Newton gravitational potential; see among others [1].

If the central matter field has total mass \(M\) and radius \(r_0\), then for \(r > r_0\), the metric (6.1) is the well known Schwarzschild solution for the Einstein field equations (5.8), and is given by

\[
ds^2 = - \left( 1 - \frac{2MG}{c^2 r} \right) c^2 dt^2 + \frac{dr^2}{\left( 1 - \frac{2MG}{c^2 r} \right)} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2).
\]

(6.3)

We derive from (6.2) and (6.3) the classical Newton gravitational potential

\[
\psi = - \frac{MG}{r}.
\]

(6.4)

6.2. New gravitational interaction model. We now consider the metric determined by the new field equations (5.11), from which we derive a gravitational potential formula replacing (6.4).

Using (5.13), the field equations (5.11) can be equivalently expressed as

\[
R_{ij} = -\frac{8\pi G}{c^4} \left( T_{ij} - \frac{1}{2} g_{ij} T \right) - (D_i D_j \varphi - \frac{1}{2} g_{ij} \Phi),
\]

(6.5)

where

\[
T = g^{kl} T_{kl}, \quad \Phi = g^{kl} D_k D_l \varphi,
\]

For the central matter field with total mass \(M\) and radius \(r_0\), by the Schwarzschild assumption, for \(r > r_0\), there exists no matter, i.e.

\[
T_{ij} = 0.
\]

(6.6)

Therefore for \(r > r_0\), the conservation laws of (6.5) are

\[
\Delta \left( \frac{\partial \varphi}{\partial x^k} \right) = 0 \quad \text{for } k = 0, 1, 2, 3.
\]

(6.7)
The tensors $g_{ij}$ in (6.1) are written as

$$
\begin{align*}
g_{00} &= -e^u, & g_{11} &= e^v, & g_{22} &= r^2, & g_{33} &= r^2 \sin^2 \theta, \\
u &= u(r), & v &= v(r).
\end{align*}
$$

(6.8)

Noting that the central field is spherically symmetric, we assume that

$$
\phi = \phi(r) \text{ is independent of } t, \theta, \phi.
$$

(6.9)

$$
r \gg \frac{2MG}{c^2}.
$$

(6.10)

For the metric (6.7), the non-zero components of the Levi-Civita connection are as follows

$$
\begin{align*}
\Gamma^1_{00} &= \frac{1}{2} e^{u-v} u', & \Gamma^1_{11} &= \frac{1}{2} v', & \Gamma^1_{22} &= -re^{-v}, \\
\Gamma^1_{33} &= -re^{-v} \sin^2 \theta, & \Gamma^0_{10} &= \frac{1}{2} u', & \Gamma^2_{12} &= \frac{1}{v}, \\
\Gamma^2_{33} &= -\sin \theta \cos \theta, & \Gamma^3_{13} &= \frac{1}{r}, & \Gamma^3_{23} &= \frac{\cos \theta}{\sin \theta}.
\end{align*}
$$

(6.11)

Hence the Ricci tensor

$$
R_{ij} = \frac{\partial \Gamma^k_{ij}}{\partial x^j} - \frac{\partial \Gamma^k_{ij}}{\partial x^k} + \Gamma^k_{ir} \Gamma^r_{jk} - \Gamma^k_{ij} \Gamma^r_{kr}
$$

are given by

$$
\begin{align*}
R_{00} &= -e^{u-v} \left[ \frac{u''}{2} + \frac{u'}{r} + \frac{u'}{4} (u' - v') \right], \\
R_{11} &= \frac{u''}{2} - \frac{v'}{r} + \frac{u'}{4} (u' - v'), \\
R_{22} &= e^{-v} \left[ 1 - e^v + \frac{r}{2} (u' - v') \right], \\
R_{33} &= R_{22} \sin^2 \theta, \\
R_{ij} &= 0 \quad \forall i \neq j.
\end{align*}
$$

(6.12)

Furthermore, we infer from (6.7), (6.9) and (6.11) that

$$
\begin{align*}
D_i D_j \phi - \frac{1}{2} g_{ij} \Phi &= 0, \quad \forall i \neq j, \\
D_0 D_0 \phi - \frac{1}{2} g_{00} \Phi &= \frac{1}{2} e^{u-v} \left[ \phi'' - \frac{1}{2} (u' + v' - \frac{4}{r}) \phi' \right], \\
D_1 D_1 \phi - \frac{1}{2} g_{11} \Phi &= \frac{1}{2} \left[ \phi'' - \frac{1}{2} (v' + u' + \frac{4}{r}) \phi' \right], \\
D_2 D_2 \phi - \frac{1}{2} g_{22} \Phi &= -\frac{r^2}{2} e^{-v} \left[ \phi'' + \frac{1}{2} (u' - v') \phi' \right], \\
D_3 D_3 \phi - \frac{1}{2} g_{33} \Phi &= \sin^2 \theta \left( D_2 D_2 \phi - \frac{1}{2} g_{22} \Phi \right).
\end{align*}
$$

(6.13)
Thus, by (6.12) and (6.13), the equations (6.5) are as follows

\[ u'' + \frac{2u'}{r} + \frac{u'}{2}(u' - v') = \phi'' - \frac{1}{2}(u' + v' - \frac{4}{r})\phi', \quad (6.14) \]

\[ u'' - \frac{2v'}{r} + \frac{u'}{2}(u' - v') = -\phi'' + \frac{1}{2}(u' + v' - \frac{4}{r})\phi', \quad (6.15) \]

\[ u' - v' + \frac{2}{r}(1 - e^v) = r(\phi'' + \frac{1}{2}(u' - v')\phi'). \quad (6.16) \]

6.3. Consistency. We need to consider the existence and uniqueness of solutions of the equations (6.14)-(6.16). First, in the vacuum case, the classical Einstein equations are in the form

\[ R_{kk} = 0 \quad \text{for } k = 1, 2, 3, \quad (6.17) \]

two of which are independent. The system contains two unknown functions, and therefore for a given initial value (as \( u' \) is basic in (6.17)):

\[ u'(r_0) = \sigma_1, \quad v(r_0) = \sigma_2, \quad r_0 > 0, \quad (6.18) \]

the problem (6.17) with (6.18) has a unique solution, which is the Schwarzschild solution

\[ u' = \frac{\varepsilon_2}{r^2} \left(1 - \frac{\varepsilon_1}{r}\right)^{-1}, \quad v = -\ln \left(1 - \frac{\varepsilon_1}{r}\right), \]

where

\[ \varepsilon_1 = r_0(1 - e^{-\sigma_2}), \quad \varepsilon_2 = r_0^2 \left(1 - \frac{\varepsilon_1}{r_0}\right)\sigma_1. \]

Now if we consider the influence of the cosmic microwave background (CMB) for the central fields, then we should add a constant energy density in equations (6.17):

\[ (T_{ij}) = \begin{pmatrix} -\rho_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (6.19) \]

Namely,

\[ R_{00} = \frac{4\pi G}{c^4} \rho_0, \quad R_{11} = 0, \quad R_{22} = 0, \quad (6.20) \]

where \( \rho \) is the density of the microwave background, whose value is \( \rho = 4 \times 10^{-31} \text{kg/m}^3 \). Then it is readily to see that the problem (6.20) with (6.18) has no solution. In fact, the divergence-free equation \( D^i T_{ij} = 0 \) yields that

\[ \Gamma^0_{10} T_{00} = \frac{1}{2} u' \rho = 0, \]

which implies that \( u' = 0 \). Hence \( R_{00} = 0 \), a contradiction to (6.20). Furthermore, if we regard \( \rho \) as an unknown function, then the equations (6.20) still have no solutions.

On the other hand, the new gravitational field equations (6.14)-(6.16) are solvable for the microwave background as the number of unknowns are the same as the number of independent equations.
Equations (6.14)-(6.16) have the following equivalent form:

\begin{align*}
  u'' &= \left( \frac{1}{r} + \frac{u'}{2} \right) (v' - u') + \frac{\varphi'}{r}, \\
  \varphi'' &= -\frac{1}{r^2} (e^v - 1) + \frac{1}{2} \varphi' v' + \frac{1}{r} u' \\
  v' &= -\frac{1}{r} (e^v - 1) - \frac{r}{2} \varphi' u',
\end{align*}

(6.21)

equipped with the following initial values:

\begin{align*}
  u' (r_0) &= \alpha_1, \quad v (r_0) = \alpha_2, \quad \varphi' (r_0) = \alpha_3, \quad r_0 > 0. \quad (6.22)
\end{align*}

It is classical that (6.21) with (6.22) possesses a unique local solution. In fact, we can prove that the solution exists for all \( r > r_0 \).

6.4. Gravitational interaction. We now derive the gravitational interaction formula from the basic model (6.14)-(6.16).

First, we infer from (6.14)-(6.16) that

\begin{align*}
  u' + v' &= \frac{r \varphi''}{1 + \frac{r}{2} \varphi'}, \\
  u' - v' &= \frac{1}{1 - \frac{r}{2} \varphi'} \left[ \frac{2}{r} (e^v - 1) + r \varphi'' \right].
\end{align*}

(6.23)

Consequently,

\begin{align*}
  u' &= \frac{1}{1 - \frac{r}{2} \varphi'} \left( e^v - 1 \right) + \frac{r \varphi''}{1 - \left( \frac{r}{2} \varphi' \right)^2}.
\end{align*}

(6.24)

By (6.2) and (6.7), we have

\begin{equation}
  \psi = \frac{c^2}{2} (e^u - 1).
\end{equation}

(6.25)

As the interaction force \( F \) is given by

\[ F = -m \nabla \psi, \]

it follows from (6.23) and (6.25) that

\begin{align*}
  F &= \frac{mc^2}{2} e^v \left[ -\frac{1}{1 - \frac{r}{2} \varphi'} \left( e^v - 1 \right) - \frac{r \varphi''}{1 - \left( \frac{r}{2} \varphi' \right)^2} \right], \\
  \varphi'' &= -e^v R + \frac{1}{2} (u' - v' + \frac{4}{r}) \varphi'.
\end{align*}

(6.26)

(6.27)

Of course, the following energy balance and conservation law hold true as well:

\begin{equation}
  R = \frac{8\pi G}{c^4} T + \Phi, \quad \int_0^\infty e^{u+v} r^2 \Phi dr = 0.
\end{equation}

(6.28)

where \( R \) is the scalar curvature and \( \Phi = g^{kl} D_k D_l \varphi \). Equation (6.27) is derived by solving \( \varphi'' \) using (6.28). Namely, for \( r > r_0 \) where \( T = 0 \),

\[ R = \Phi = g^{kl} D_k D_l \varphi = e^{-v} [-\varphi'' + \frac{1}{2} (u' - v' + \frac{4}{r}) \varphi']. \]
6.5. Simplified formulas. We now consider the region: \( r_0 < r < r_1 \). Physically, we have
\[
|\varphi'|, |\varphi''| << 1. \tag{6.29}
\]
Hence \( u \) and \( v \) in (6.26) can be replaced by the Schwarzschild solution:
\[
u_0 = \ln \left(1 - \frac{\delta}{r}\right), \quad v_0 = -\ln \left(1 - \frac{\delta}{r}\right), \quad \delta = \frac{2GM}{c^2}. \tag{6.30}
\]
As \( \delta/r \) is small for \( r \) large, by (6.29), the formula (6.26) can be expressed as
\[
F = \frac{mc^2}{2} \left[-\frac{\delta}{r^2} - \varphi''r\right]. \tag{6.31}
\]
This is the interactive force in a central symmetric field. The first term in the parenthesis is the Newton gravity term, and the added second term \(-r\varphi''\) is the scalar potential energy density, representing the dark matter and dark energy.

In addition, replacing \( u \) and \( v \) in (6.27) by the Schwarzschild solution (6.30), we derive the following approximate formula:
\[
\varphi'' = \left(\frac{2}{r} + \frac{\delta}{r^2}\right)\varphi' - R. \tag{6.32}
\]
Consequently we infer from (6.31) that
\[
F = mMG \left[-\frac{1}{r^2} - \frac{1}{\delta} \left(2 + \frac{\delta}{r}\right)\varphi' + \frac{Rr}{\delta}\right], \quad R = \Phi \quad \text{for } r > r_0. \tag{6.33}
\]
The first term is the classical Newton gravitation, the second term is the coupling interaction between matter and the scalar potential \( \varphi \), and the third term is the interaction generated by the scalar potential energy density \( \Phi \). In this formula, the negative and positive values of each term represent respectively the attracting and repelling forces.

Integrating (6.32) yields (omitting \( e^{-\delta/r} \))
\[
\varphi' = -\varepsilon_2 r^2 - r^2 \int r^{-2}Rdr, \tag{6.34}
\]
where \( \varepsilon_2 \) is a free parameter. Hence the interaction force \( F \) is approximated by
\[
F = mMG \left[-\frac{1}{r^2} + \left(2 + \frac{\delta}{r}\right)\varepsilon r^2 + \frac{Rr}{\delta} + \frac{1}{\delta} \left(2 + \frac{\delta}{r}\right) r^2 \int r^{-2}Rdr\right], \tag{6.35}
\]
where \( \varepsilon = \varepsilon_2 \delta^{-1} \), \( R = \Phi \) for \( r > r_0 \), and \( \delta = 2MG/c^2 \). We note that based on (6.28), for \( r > r_0 \), \( R \) is balanced by \( \Phi \), and the conservation of \( \Phi \) suggests that \( R \) behaves like \( r^{-2} \) as \( r \) sufficiently large. Hence for \( r \) large, the second term in the right hand side of (6.35) must be dominate and positive, indicating the existence of dark energy.

We note that the scalar curvature is infinite at \( r = 0 \): \( R(0) = \infty \). Also \( R \) contains two free parameters determined by \( u' \) and \( v \) respectively. Hence if we take a first order approximation as
\[
R = -\varepsilon_1 + \frac{\varepsilon_0}{r} \quad \text{for } r_0 < r < r_1 = 10^{21} \text{ km}, \tag{6.36}
\]
where \( \varepsilon_1 \) and \( \varepsilon_0 \) are free yet to be determined parameters. Then we deduce from (6.34) and (6.36) that
\[
\varphi' = -\varepsilon \delta r^2 - \varepsilon_1 r + \frac{\varepsilon_0}{2}.
\]
Therefore,

\[ F = mMG \left[ -\frac{1}{r^2} - \frac{\varepsilon_0}{2} + \varepsilon_1 + (\varepsilon\delta + \varepsilon_1\delta^{-1})r + 2\varepsilon r^2 \right]. \] (6.37)

Physically it is natural to choose \( \varepsilon_0 > 0, \varepsilon_1 > 0, \varepsilon > 0 \).

Also, \( \varepsilon_1 \) and \( 2\varepsilon_2\delta^{-1}r^2 \) are much smaller than \( (\varepsilon\delta + \varepsilon_1\delta^{-1})r \) for \( r \leq r_1 \). The justification of these approximations is based on properly choosing (initial) conditions for \( u', \varphi' \) and \( v \) at certain \( r = r_0 \) in (6.21). Hence

\[ F = mMG \left[ -\frac{1}{r^2} - \frac{k_0}{r} + k_1r \right]. \] (6.38)

where \( k_0 \) and \( k_1 \) can be estimated using the Rubin law of rotating galaxy and the acceleration of the expanding galaxies:

\[ k_0 = 4 \times 10^{-18} \text{km}^{-1}, \quad k_1 = 10^{-57} \text{km}^{-3}. \] (6.39)

We emphasize here that the formula (6.38) is only a simple approximation for illustrating some features of both dark matter and dark energy.


7.1. Dark matter and dark energy. Dark matter and dark energy are two of most remarkable discoveries in astronomy in recent years, and they are introduced to explain the acceleration of the expanding galaxies. In spite of many attempts and theories, the mystery remains. As mentioned earlier, this article is an attempt to develop a unified theory for the dark matter and dark energy.

A strong support to the existence of dark matter is the Rubin law for galactic rotational velocity. The rotation curve of a galaxy is the rotational velocity of the visible stars or gas in the galaxy on their radial distance from the center of the galaxy. The Rubin law amounts to saying that most stars in spiral galaxies orbit at roughly the same speed. If a galaxy had a mass distribution as the observed distribution of stars, the rotational velocity would decrease at large distances. Hence the Rubin law demonstrates the existence of additional gravitational effect than the visible stars in the galaxy.

More precisely, the orbital velocity \( v(r) \) of the stars located at radius \( r \) from the center of galaxies is almost a constant:

\[ v(r) = \text{constant for a given galaxy.} \] (7.1)

Typical galactic rotation curves [13] are illustrated by Figure 7.1(a), where the vertical axis represents the velocity (km/s), and the horizontal axis is the distance from the galaxy center (extending to the galaxy radius).

However, observational evidence shows discrepancies between the mass of large astronomical objects determined from their gravitational effects, and the mass calculated from the visible matter they contain, and Figure 7.1 (b) gives a calculated curve. The missing mass suggests the presence of dark matter in the universe.

In astronomy and cosmology, dark energy is a hypothetical form of energy, which spherically symmetrically permeates all of space and tends to accelerate the expansion of the galaxies.

The High-Z Supernova Search Team in 1998 and the Supernova Cosmology Project in 1998 published their observations which reveal that the expansion of
the galaxies is accelerating. In 2011, a survey of more than $2 \times 10^5$ galaxies from Austrian astronomers confirmed the fact. Thus, the existence of dark energy is accepted by most astrophysicist.

7.2. **Nature of dark matter and dark energy.** With the new gravitational field equations with the scalar potential energy, and we are now in position to derive the nature of the dark matter and dark energy. More precisely, using the revised Newton formula derived from the new field equations:

$$ F = mMG \left( -\frac{1}{r^2} - \frac{k_0}{r} + k_1r \right), \tag{7.2} $$

we determine an approximation of the constants $k_0, k_1$, based on the Rubin law and the acceleration of expanding galaxies.

First, let $M_r$ be the total mass in the ball with radius $r$ of the galaxy, and $V$ be the constant galactic rotation velocity. By the force equilibrium, we infer from (7.2) that

$$ \frac{V^2}{r} = M_r G \left( \frac{1}{r^2} + \frac{k_0}{r} - k_1r \right), \tag{7.3} $$

which implies that

$$ M_r = \frac{V^2}{G} \frac{r}{1 + k_0r - k_1r^3}. \tag{7.4} $$

This matches the observed mass distribution formula of the galaxy, which can explain the Rubin law (7.3).

Second, if we use the classical Newton formula

$$ F = -\frac{mMG}{r^2}, $$

to calculate the galactic rotational velocity $v_r$, then we have

$$ \frac{v_r^2}{r} = \frac{M_r G}{r^2}. \tag{7.5} $$

Inserting (7.4) into (7.5) implies

$$ v_r = \frac{V}{\sqrt{1 + k_0r - k_1r^3}}. \tag{7.6} $$

As $1 \gg k_0 \gg k_1$, (7.6) can approximatively written as

$$ v_r = V \left( 1 - \frac{1}{2} k_0r + \frac{1}{4} k_0^2 r^2 \right), $$
which is consistent with the theoretic rotational curve as illustrated by Figure 7.1(b). It implies that the distribution formula (7.4) can be used as a test for the revised gravitational field equations.

Third, we now determine the constants $k_0$ and $k_1$ in (7.2). According to astro-
nomic data, the average mass $M_{r_1}$ and radius $r_1$ of galaxies is about

\begin{equation}
M_{r_1} = 10^{11} M_{\odot} \simeq 2 \times 10^{41} \text{kg},
\end{equation}

\begin{equation}
r_1 = 10^4 \sim 10^5 \text{pc} \simeq 10^{18} \text{km},
\end{equation}

where $M_{\odot}$ is the mass of the Sun.

Taking $V = 300\text{km/s}$, then we have

\begin{equation}
\frac{V^2}{G} = 8 \times 10^{23} \text{kg/km}.
\end{equation}

Based on physical considerations,

\begin{equation}
k_0 \gg k_1 r_1 \quad (r_1 \text{ as in (7.7)})
\end{equation}

By (7.7)-(7.9), we deduce from (7.4) that

\begin{equation}
k_0 = \frac{V^2}{G} \frac{1}{M_{r_1}} - \frac{1}{r_1} = 4 \times 10^{-18} \text{km}^{-2}.
\end{equation}

Now we consider the constant $k_1$. Due to the accelerating expansion of galaxies, the interaction force between two clusters of galaxies is repelling, i.e. for (7.2),

\begin{equation}
F \geq 0, \quad r \geq \bar{r},
\end{equation}

where $\bar{r}$ is the average distance between two galactic clusters. It is estimated that

\begin{equation}
\bar{r} = 10^8 \text{pc} \simeq 10^{20} \sim 10^{21} \text{km}.
\end{equation}

We take

\begin{equation}
\bar{r} = \frac{1}{\sqrt{2}} \times 10^{20} \text{km}
\end{equation}

as the distance at which $F = 0$. Namely,

\begin{equation}
k_1 \bar{r} - k_0 \bar{r} - \frac{1}{\bar{r}^2} = 0.
\end{equation}

Hence we derive from (7.10) and (7.11) that

\begin{equation}
k_1 = k_0 \bar{r}^{-2} = 10^{-57} \text{km}^{-3}.
\end{equation}

Thus, the constants $k_0$ and $k_1$ are estimated by

\begin{equation}
k_0 = 4 \times 10^{-18} \text{km}^{-2}, \quad k_1 = 10^{-57} \text{km}^{-3}.
\end{equation}

In summary, for the formula (7.2) with (7.12), if the matter distribution $M_r$ is in the form

\begin{equation}
M_r = \frac{V^2}{G} \frac{r}{1 + k_0 r},
\end{equation}

then the Rubin law holds true. In particular, the mass $\tilde{M}$ generated by the revised gravitation is

\begin{equation}
\tilde{M} = M_T - M_{r_1} = \frac{V^2}{G} r_1 - \frac{V^2}{G} \frac{r_1}{1 + k_0 r_1}, \quad r_1 \text{ as in (7.7)},
\end{equation}

where $M_T = \frac{V^2 r_1}{G}$ is the theoretic value of total mass. Hence

\begin{equation}
\frac{\tilde{M}}{M_T} = \frac{k_0 r_1}{1 + k_0 r_1} = \frac{4}{5}.
\end{equation}
Namely, the revised gravitational mass $\tilde{M}$ is four times of the visible matter $M_{\text{v}} = M_{\text{T}} - \tilde{M}$. Thus, it gives an alternative explanation for the dark matter.

In addition, the formula (7.2) with (7.12) also shows that for a central field with mass $M$, an object at $r > \bar{r}$ ($\bar{r}$ as in (7.11)) will be exerted a repelling force, resulting the acceleration of expanding galaxies at $r > \bar{r}$.

Thus the new gravitational formula (7.2) provides a unified explanation of dark matter and dark energy.

**7.3. Effects of non-homogeneity.** In this section, we prove that if the matter is homogeneously distributed in the universe, then the scalar potential $\phi$ is a constant, and consequently the scalar potential energy density is identically zero: $\Phi \equiv 0$.

It is known that the metric for an isotropic and homogeneous universe is given by

$$ds^2 = -c^2 dt^2 + a^2 \left[ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right],$$  \hspace{1cm} (7.14)

which is called the Friedmann-Lemaître-Robertson-Walker (FLRW) metric, where the scale factor $a = a(t)$ represents the cosmological radius, and $k$ takes one of the three numbers: $-1, 0, 1$.

For the FLRW metric (7.14), the energy-momentum tensor $\{T_{ij}\}$ is given by

$$T_{ij} = \text{diag}(\rho c^2, g_{11} p, g_{22} p, g_{33} p),$$  \hspace{1cm} (7.15)

where $\rho$ is the mass density, $p$ is pressure, and

$$g_{11} = \frac{a^2}{1 - kr^2}, \quad g_{22} = a^2 r^2, g_{33} = a^2 r^2 \sin^2 \theta.$$

By (7.14) and (7.15), the Einstein field equations (1.1) are reduced to two equations:

$$a'' - \frac{4\pi G}{3} \left( \rho + \frac{3p}{c^2} \right) a = 0,$$  \hspace{1cm} (7.16)

$$aa'' + 2(a')^2 + 2kc^2 = 4\pi G \left( \rho - \frac{p}{c^2} \right) a^2,$$  \hspace{1cm} (7.17)

and the conservation law $\text{div} T_{ij} = 0$ gives

$$\frac{d\rho}{dt} + \frac{3}{R} \frac{dR}{dt} \left( \rho + \frac{p}{c^2} \right) = 0.$$  \hspace{1cm} (7.18)

Then only two of the above three equations (7.16)-(7.18) are independent, and are called the Friedman equations.

On the other hand, for the metric (7.14) with (7.15), the new gravitational field equations (1.7) with scalar potential are reduced to

$$a'' = -\frac{4\pi G}{3} \left( \rho + \frac{3p}{c^2} \right) a + \frac{1}{6} \varphi'' a - \frac{1}{2} a' \varphi',$$  \hspace{1cm} (7.19)

$$\frac{a''}{a} + 2 \left( \frac{a'}{a} \right)^2 + 2kc^2 = 4\pi G \left( \rho - \frac{p}{c^2} \right) - \frac{\varphi''}{2} - \frac{\varphi'}{2} \frac{a'}{a},$$  \hspace{1cm} (7.20)

and the conservation equation

$$\varphi''' + 3 \frac{a'}{a} \varphi'' = 8\pi G \left( \rho' + \frac{3a'}{a} \rho + \frac{3}{a} \frac{p}{c^2} \right).$$  \hspace{1cm} (7.21)
Again two of the above three equations (7.19)-(7.21) are independent. It follows from (7.19) and (7.20) that

\[(a')^2 = \frac{8\pi G}{3}a^2\rho - \frac{1}{3}\varphi''a^2 - kc^2.\] (7.22)

In the following we shall prove that

\[\nabla\varphi = 0,\] (7.23)

Namely, \(\varphi = \text{constant}\). In fact, let \(\varphi''\) and \(\rho\) have the form:

\[\rho = \frac{\theta}{a^3}, \quad \varphi'' = \frac{\psi}{a^3}.\] (7.24)

Inserting (7.24) in (7.19), (7.21) and (7.22), we arrive at

\[a'' = -\frac{4\pi G}{3}\frac{\theta}{a^2} + \frac{1}{6}\frac{\psi}{a^2} - \frac{4\pi G}{c^2}\theta a - \frac{1}{2}a'\varphi',\] (7.25)

\[(a')^2 - \frac{8\pi G}{3}\frac{\theta}{a} + \frac{1}{3}\frac{\psi}{a} = -kc^2,\] (7.26)

\[\psi' = 8\pi G\theta + 24\pi Ga^2a'/c^2.\] (7.27)

Multiplying both sides of (7.25) by \(a'\) we obtain

\[\frac{1}{2}\frac{d}{dt}\left[(a')^2 - \frac{8\pi G}{3}\frac{\theta}{a} + \frac{1}{3}\frac{\psi}{a}\right] + \frac{4\pi G}{3}\frac{\theta'}{a} - \frac{1}{6}\frac{\psi'}{a} = -\frac{4\pi G}{c^2}paa' - \frac{1}{2}(a')^2\varphi'.\] (7.28)

It follows then from (7.26)-(7.28) that

\[\frac{1}{2}(a')^2\varphi' = 0,\]

which implies that (7.23) holds true.

The conclusion (7.23) indicates that if the universe is in the homogeneous state, then the scalar potential energy density \(\Phi\) is identically zero: \(\Phi \equiv 0\). This fact again demonstrates that \(\varphi\) characterizes the non-uniform distribution of matter in the universe.

8. Conclusions. We have discovered new gravitational field equations (1.7) with scalar potential under the postulate that the energy momentum tensor \(T_{ij}\) needs not to be divergence-free due to the presence of dark energy and dark matter:

\[R_{ij} - \frac{1}{2}g_{ij}R = -\frac{8\pi G}{c^4}T_{ij} - D_iD_j\varphi,\]

With the new field equations, we have obtained the following physical conclusions:

First, gravitation is now described by the Riemannian metric \(g_{ij}\), the scalar potential \(\varphi\) and their interactions, unified by the new gravitational field equations (1.7).

From quantum field theoretic point of view, the vector field \(\Phi_{\mu} = D_{\mu}\varphi\) on the right-hand side of the field equations (1.7) is a spin-1 massless bosonic particle field. The field equations induce a natural duality between the spin-2 massless bosonic particle, called graviton, and this spin-1 massless bosonic particle. Both particles can be considered as gravitational force carriers, and as they are massless, the induced forces are long-range forces. The (nonlinear) interaction between these bosonic particle fields leads to a unified theory for dark energy and dark matter. Although, we do not exclude the existence of exotic particles in the universe, the
main constituent of dark matter and dark energy is the new massless spin-1 particle field. This duality was first introduced by the authors in [20, 21] for all four interactions.

SECOND, associated with the scalar potential $\varphi$ is the scalar potential energy density $\frac{c^4}{8\pi G} \Phi$, which represents a new type of energy/force caused by the non-uniform distribution of matter in the universe. This scalar potential energy density varies as the galaxies move and matter of the universe redistributes. Like gravity, it affects every part of the universe as a field.

This scalar potential energy density $\frac{c^4}{8\pi G} \Phi$ consists of both positive and negative energies. The negative part of this potential energy density produces attraction, and the positive part produces repelling force. Also, this scalar energy density is conserved with mean zero:

$$\int_M \Phi \, dM = 0.$$ 

THIRD, using the new field equations, for a spherically symmetric central field with mass $M$ and radius $r_0$, the force exerted on an object of mass $m$ at distance $r$ is given by (see (6.33)):

$$F = mMG \left[ -\frac{1}{r^2} - \frac{1}{\delta} \left(2 + \frac{\delta}{r}\right) \varphi' + \frac{R r}{\delta}\right], \quad R = \Phi \quad \text{for } r > r_0.$$ 

where $\delta = 2MG/c^2$.

FOURTH, the sum $\varepsilon = \varepsilon_1 + \varepsilon_2$ of this new potential energy density

$$\varepsilon_1 = \frac{c^4}{8\pi G} \Phi$$

and the coupling energy between the energy-momentum tensor $T_{ij}$ and the scalar potential field $\varphi$

$$\varepsilon_2 = -\frac{c^4}{8\pi G} \left(\frac{2}{r} + \frac{2MG}{c^2 r^2}\right) \frac{d\varphi}{dr},$$

gives rise to a new unified theory for dark matter and dark energy: The negative part of $\varepsilon$ represents the dark matter, which produces attraction, and the positive part represents the dark energy, which drives the acceleration of expanding galaxies.

FIFTH, the scalar curvature $R$ of space-time obeys:

$$R = \frac{8\pi G}{c^4} T + \Phi.$$ 

Consequently, when there is no normal matter present (with $T = 0$), the curvature $R$ of space-time is balanced by $R = \Phi$. Therefore, there is no real vacuum in the universe.

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