Unified Field Equations Coupling Four Forces and Theory of Dark Matter and Dark Energy

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Supported in part by NSF and ONR
http://www.indiana.edu/~fluid
Outline

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I. Motivations

Mystery of dark matter and dark energy
• In 1998, published observations of Type Ia supernovae by the High-z Supernova Search Team followed in 1999 by the Supernova Cosmology Project suggested that

  the expansion of the universe is accelerating.

**Saul Perlmutter, Brian P. Schmidt, Adam G. Riess.**

• **Dark energy is a hypothetical form of energy** that permeates all of space and tends to accelerate the expansion of the universe.

  Unfortunately, no theory can explain the nature of dark energy, which is now considered the cosmos greatest mystery.
Possible explanations:

- **Cosmological constant**, which would be a property of the vacuum and would stretch space.

- **Quintessence**: Dark energy could be a new type of force field that occupies space.
Grand challenges for coupling all four interactions
### Three Generations of Matter (Fermions)

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II. Principle of Interaction Dynamics (PID)

Let $u \in L^2(T^r_s M)$. We define the operators $D_A$ and $\text{div}_A$ as

$$D_A u = Du + u \otimes A,$$
$$\text{div}_A u = \text{div}u - A \cdot u,$$

where $A$ is a vector or covector field, $D$ and $\text{div}$ are the usual gradient and divergent operators.

**Orthogonal Decomposition Theorem:** Let $A$ is a given vector field or covector field, and $u \in L^2(T^r_s M)$ with $r + s \geq 1$. Then $u$ can be orthogonally decomposed into

$$u = D_A \varphi + v \quad \text{with} \quad \text{div}_A v = 0,$$

where $\varphi \in H^1(T^{r-1}_s M)$ or $\varphi \in H^1(T^{r}_{s-1} M)$. 


The classical Einstein equations are derived based on three principles:

- **principle of equivalence and principle of general relativity**, which amount to saying that space-time is a Riemannian manifold \((M, g_{ij})\), and gravity is described by the metric \(g_{ij}\).

- Lagrangian least action principle:

**Einstein-Hilbert functional:**

\[
L_{EH} = \int_{M} \left( R + \frac{8 \pi G}{c^4} S \right) \sqrt{-g} dx
\]

\[
\delta L_{EH} = 0 \quad \implies \quad R_{ij} - \frac{1}{2} g_{ij} R + \frac{8 \pi G}{c^4} T_{ij} = 0
\]
\[ T_{ij} = S_{ij} - \frac{1}{2}g_{ij}S + g^{kl} \frac{\partial S_{kl}}{\partial g_{ij}}, \quad S = g^{kl}S_{kl}, \]

\[ R_{ij} = \frac{1}{2}g^{kl} \left( \frac{\partial^2 g_{kl}}{\partial x^i \partial x^j} + \frac{\partial^2 g_{ij}}{\partial x^k \partial x^l} - \frac{\partial^2 g_{il}}{\partial x^j \partial x^k} - \frac{\partial^2 g_{kj}}{\partial x^i \partial x^l} \right) \]

\[ + g^{kl}g_{rs} \left( \Gamma^r_{kl} \Gamma^s_{ij} - \Gamma^r_{il} \Gamma^s_{kj} \right), \]

\[ \Gamma^k_{ij} = \frac{1}{2}g^{kl} \left( \frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right). \]

For example, for an ideal fluid case with pressure \( p \), energy density \( \varepsilon \), and 4D velocity \( u^\mu \)

\[ T^{\mu\nu} = (g^{00}p + \varepsilon)u^\mu u^\nu - pg^{\mu\nu} \]
Due to the presence of dark energy and dark matter, the energy-momentum tensor $T_{ij}$ of normal matter is no longer conserved: $D^i(T_{ij}) \neq 0$.

By an orthogonal decomposition theorem, there is a scalar function $\varphi : M \to R$ such that

$$T_{ij} = \tilde{T}_{ij} - \frac{c^4}{8\pi G} D_i D_j \varphi,$$

$$D^i \tilde{T}_{ij} = 0$$

$$L_{EH} = \int_M \left( R + \frac{8\pi G}{c^4} S \right) \sqrt{-g} dx,$$

$$D^i \left[ R_{ij} - \frac{1}{2} g_{ij} R \right] = 0 \implies R_{ij} - \frac{1}{2} g_{ij} R + \frac{8\pi G}{c^4} \tilde{T}_{ij} = 0$$

Namely

$$R_{ij} - \frac{1}{2} g_{ij} R = -\frac{8\pi G}{c^4} T_{ij} - D_i D_j \varphi,$$

$$D^i \left( D_i D_j \varphi + \frac{8\pi G}{c^4} T_{ij} \right) = 0$$
Equivalently: The new gravitational field equations (2) can be derived with constraint least action principle:

\[
\lim_{\lambda \to 0} \frac{1}{\lambda} [L_{EH}(g_{ij} + \lambda X_{ij}) - L_{EH}(g_{ij})] = (\delta L_{EH}(g_{ij}), X) = 0 \quad \forall D^i X_{ij} = 0
\]

This constraint least action leads us to postulate:

**Principle of Interaction Dynamics (PID):** For all physical interactions there are Lagrangian actions

\[
L(g, A, \psi) = \int_M L(g_{ij}, A, \psi) \sqrt{-g} dx,
\]

where \( A \) is a set of vector fields representing the gauge potentials, and \( \psi \) are the wave functions of particles. The states \((g, A, \psi)\) are the extremum points of (3) with the \((D + A)\)-free constraint.
III. Unified Field Equations Coupling Four Forces

The unified field equations are derived based on the following principles:

- the Einstein principle of general relativity (or Lorentz invariance) and the principle of equivalence, which amount to saying that the space-time is a 4-dimensional Riemannian manifold \((M, g_{ij})\)

- the principle of gauge invariance

- the principle of interaction dynamics (PID)
Recapitulation of Maxwell Equations: Let

\[(E^1, E^2, E^3) = -\nabla A^0 - \frac{\partial (A^1, A^2, A^3)}{\partial t}, \quad B = \text{curl } A\]

\[F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}\]

Lagrangian action:

\[\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2} (E^2 - B^2)\]

Maxwell equation (with no source):

\[\partial_\mu F^{\mu\nu} = 0\]
Coupling Dirac theory to Maxwell–Quantum Electrodynamics (QED):

Dirac equation: \((i\gamma^\mu \partial_\mu - m)\psi = 0\), where \(\psi\) is a 4-vector wave function. Here \(\gamma^\mu\) are Dirac matrices:

\[
\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ -\sigma^\mu & 0 \end{pmatrix} \\
\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

Then the Lagrangian action

\[
\mathcal{L} = \psi^\dagger [i\gamma^\mu (\partial_\mu + ieA_\mu) - m] \psi - \frac{1}{4} F_{\mu\nu}F^{\mu\nu}
\]

is invariant under the \(\text{U}(1)\) gauge transformation:

\[
\psi(x) \rightarrow e^{ie\Lambda(x)}\psi(x), \quad A_\mu \rightarrow A_\mu - \partial_\nu \Lambda(x).
\]
Both the field equations below and the Lagrangian action above preserve both the gauge symmetry and the Lorentz symmetry:

\[
\partial^\mu (\partial_\mu A_\nu - \partial_\nu A_\mu) - e\psi^\dagger \gamma^\nu \psi = 0,
\]

\[
[i\gamma^\mu (\partial_\mu + ieA_\mu) - m]\psi = 0
\]

- U(1) gauge theory works very well for QED

- But for the $U(1) \otimes SU(2)$ electroweak gauge theory, Glashow-Weinberg-Salam needed to add to the Lagrangian the Higgs terms

\[
\mathcal{L}_H = D_\mu \phi^\dagger D^\mu \phi - \lambda (\phi^\dagger \phi - a)^2 + G_e(\bar{L}\phi R + \bar{R}\phi^\dagger L)
\]

for energy creation and mass generation. Here $\phi = (\phi^+, \phi^0)^t$.

- Higgs mechanism: The added Higgs terms spontaneously breaks the gauge symmetry in the field equations, but not in the action.
Field Equations Coupling Four Forces

Lagrangian action functional $L$ is the sum of the Einstein-Hilbert $\mathcal{L}_{EH}$, the GWS $\mathcal{L}_{GWS}$ without Higgs field terms, and the standard QCD $\mathcal{L}_{QCD}$.

\begin{equation}
L = \int \left[ \mathcal{L}_{EH} + \mathcal{L}_{GWS} + \mathcal{L}_{QCD} \right] \sqrt{-g} dx
\end{equation}

$\mathcal{L}_{EH} = R + \frac{8\pi G}{c^4} S,$

$\mathcal{L}_{GWS} = -\frac{1}{4} W^a_{\mu\nu} W^{a\mu\nu} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

$$ + \sum_k \left\{ \bar{L}_k (i\gamma^\mu \tilde{D}_\mu - m_k) L_k + \bar{\psi}^R_k (i\gamma^\mu \tilde{D}_\mu - m^l_k) \psi^R_k \right\},$$

$\mathcal{L}_{QCD} = -\frac{1}{4} F^b_{\mu\nu} F^{b\mu\nu} + \sum_{k=1}^{6} \bar{q}_k (i\gamma^\mu \tilde{D}_\mu - m^q_k) q_k$
\[ W^a_{\mu\nu} = D_\mu W^a_\nu - D_\nu W^a_\mu + g_1 f^{abc} W^b_\mu W^c_\nu, \]
\[ F_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu, \]
\[ F^b_{\mu\nu} = D_\mu B^b_\nu - D_\nu B^b_\mu + g_2 g^{bcd} B^c_\mu B^d_\nu, \]
\[ \tilde{D}_\mu L_k = (\nabla_\mu - ig_1^2 W^a_\mu \sigma_a + i\frac{e}{2} A_\mu) L_k, \]
\[ \tilde{D}_\mu \psi^R_k = (\nabla_\mu + ie A_\mu) \psi^R_k, \]
\[ \tilde{D}_\mu q_k = (\nabla_\mu + i\frac{g_2}{2} B^b_\mu \lambda_b) q_k, \]

where \( g_1 \) and \( g_2 \) are constants, \( f^{abc} \) and \( g^{bcd} \) are the structure constants of \( SU(2) \) and \( SU(3) \), \( \sigma_a \) are the Pauli matrices, \( \lambda_b \) are the Gell-Mann matrices, \( L_k = (\psi_{\nu_k}, \psi^L_k)^t \) are the wave functions of left-hand lepton and quark pairs (each has 3 generations), \( \psi^R_k \) are the right-hand leptons and quarks, \( q_k = (q_{k1}, q_{k2}, q_{k3})^t \) is the k-th flavored quark, and \( \nabla_\mu \) is the Lorentz Vierbein covariant derivative.
• Thanks to the $\text{div}_A$-free constraint, the Euler-Lagrangian of the action functional is balanced by gradient fields, resulting the unified field equations (6) coupling all four forces.

• The gradient fields break spontaneously the gauge-symmetries. This gives rise to a complete new mechanism for spontaneous gauge symmetry-breaking and for energy and mass generation, which provides similar outcomes as the Higgs mechanism.
Unified Field Equations Coupling Four Forces:

\[
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \frac{8\pi G}{c^4}T_{\mu\nu} = (D_\mu + \alpha_0 A_\mu + \alpha W_a W^a_\mu + \alpha B^i_j B^i_j)\Phi_\nu,
\]

\[
\partial^\mu (\partial_\mu A_\nu - \partial_\nu A_\mu) - e \sum_{k=1}^6 \left[ \frac{1}{2} L_k \gamma_\nu L_k + \bar{\psi}_k \gamma_\nu \psi_k^R \right] = (D_\nu + \beta_0 A_\nu + \beta W_a W^a_\nu + \beta B^i_j B^i_j)\phi^0,
\]

\[
\partial^\mu W^a_{\mu\nu} - \frac{g_1}{2} \sum_{b=1}^3 f^{bca} g^{\mu\alpha} W^a_{\mu\nu} W^c_\alpha + \frac{g_1}{2} \sum_{k=1}^6 \bar{L}_k \gamma_\nu \sigma_a L_k
\]

\[
= (D_\nu + \kappa^a_0 A_\nu + \kappa^a_{Wb} W^b_\nu + \kappa^a_{B^i_j} B^i_j)\phi^a_W, \quad a = 1, 2, 3,
\]

\[
\partial^\mu F^b_{\mu\nu} - \frac{g_2}{2} \sum_{k=1}^8 g^{k\ell b} g^{\mu\alpha} B^b_{\mu\nu} B^\ell_\alpha - \frac{g_2}{2} \sum_{k=1}^6 \bar{q}_k \gamma_\nu \lambda_b q_k
\]

\[
= (D_\nu + \theta^b_0 A_\nu + \theta^b_{Wl} W^l_\nu + \theta^b_{Bl} B^l_\nu)\phi^b_B, \quad b = 1, \ldots, 8,
\]

\[
(i\gamma^\mu \widetilde{D}_\mu - m_k) L_k = 0
\]

\[
(i\gamma^\mu \widetilde{D}_\mu - m^l_k) \psi^R_k = 0
\]

\[
(i\gamma^\mu \widetilde{D}_\mu - m^q_k) q_k = 0.
\]
\[ T_{\mu\nu} = \frac{\delta S}{\delta g_{ij}} + \frac{c^4}{16\pi G} g^{\alpha\beta} [ W^a_{\alpha\mu} W^a_{\beta\nu} + F_{\alpha\mu} F_{\beta\nu} + F^b_{\alpha\mu} F^b_{\beta\nu} ] - \frac{c^4}{16\pi G} g_{\mu\nu} (\mathcal{L}_{GWs} + \mathcal{L}_{QCD}) . \]
By the Bianachi identity, taking divergence on the field equations, we have

\[ D^\mu D_\mu \Phi + (\text{div} V_1) \Phi = o(1), \]
\[ D^\mu D_\mu \phi^0 + (\text{div} V_2) \phi^0 = o(1), \]
\[ D^\mu D_\mu \phi^a_W + (\text{div} V_3^a) \phi^a_W = o(1) \quad \text{for} \ a = 1, 2, 3, \]
\[ D^\mu D_\mu \phi^b_B + (\text{div} V_3^b) \phi^b_B = o(1) \quad \text{for} \ b = 1, \ldots, 8, \]

which imply that \( \Phi \) represents a \textbf{vector boson}, and \( \phi^0, \phi^a_W, \phi^b_B \) represent \textbf{scalar bosons}. Here

\[ V_1 = \alpha_0 A_\mu + \alpha_{W_a} W^{a}_\mu + \alpha_{B_j} B^{j}_\mu, \]
\[ V_2 = \beta_0 A_\nu + \beta_{W_a} W^{a}_\nu + \beta_{B_j} B^{j}_\nu, \]
\[ V_3^a = \kappa^a_0 A_\nu + \kappa^a_{W_b} W^{b}_\nu + \kappa^a_{B_j} B^{j}_\nu, \]
\[ V_3^b = \theta^j_0 A_\nu + \theta^j_{W_b} W^{b}_\nu + \theta^j_{B_l} B^{l}_\nu. \]
IV. Duality

The unified field equations (6) coupling all four forces induce a natural duality:

\[
\begin{align*}
\{g_{\mu\nu}\} & \longleftrightarrow \Phi_\mu, \\
A_\mu & \longleftrightarrow \phi^0, \\
W^a_\mu & \longleftrightarrow \phi^a_W \quad \text{for } a = 1, 2, 3, \\
B^b_\mu & \longleftrightarrow \phi^b_B \quad \text{for } b = 1, \cdots, 8.
\end{align*}
\]

Equivalently the graviton $g$, the photon $\gamma$, vector bosons $W^\pm$ and $Z$, and the gluons $g_b$, represented by the left-hand side of the above duality corresponds to

the bosonic fields: a vector massless spin-1 boson $\Phi_\mu$, a scalar massless spin-0 boson $\phi^0$, and scalar massive spin-0 bosons $\phi^a_W, \phi^b_B$.  

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**Zero-point energy:** The field $\phi^0$, adjoint to the electromagnetic potential $A_\mu$, is a massless field with spin $s = 0$, and it needs to be confirmed experimentally.

We think it is this particle field $\phi^0$ that causes the vacuum fluctuation or zero-point energy.

**Eightfold Way mesons:** We conjecture that the combination of the eight scalar bosons $\phi^b_B$ corresponding to gluons $g_b$ ($1 \leq b \leq 8$) leads to the Eightfold Way mesons: $\pi^+, \pi^-, \pi^0, K^+, K^-, K^0, \bar{K}^0, \eta^0$. 
**Higgs Bosons:** The combination of the three Higgs bosonic fields on the right hand side of the duality induces three Higgs boson particles given by

\[ \phi^\pm = \frac{1}{\sqrt{2}} (\phi_W^1 \pm i\phi_W^2), \quad \phi_W^3 \]

with \( \phi^\pm \) having \( \pm \) electric charges, and with \( \phi_W^3 \) being neutral.

- Note that the classical Weinberg-Salam theory induces one neutral Higgs boson particle. All three Higgs bosons deduced here possess masses, generated by the new mechanism.

- In fact, consider only the electromagnetic and weak interactions and ignore the effect of other interactions, we derive a totally different electroweak theory.

Again this electroweak theory produces the three vector bosons \( W^\pm \) and \( Z \), as well as the three Higgs bosons. The spontaneous gauge-symmetry breaking is achieved by the constraint action **without Higgs terms** in the action functional.
V. Electroweak Theory

Glashow-Weinberg-Salam electroweak theory: The action density is given by

\[ \mathcal{L} = \mathcal{L}_G + \mathcal{L}_F + \mathcal{L}_H, \]
\[ \mathcal{L}_G = -\frac{1}{4} W^a_{\mu\nu} W^{a\mu\nu} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \]
\[ \mathcal{L}_F = i \bar{R} \gamma^\mu D_\mu R + i \bar{L} \gamma^\mu D_\mu L, \]
\[ \mathcal{L}_H = D_\mu \phi^\dagger D^\mu \phi - \lambda (\phi^\dagger \phi - a)^2 + G_e (\bar{L}\phi R + \bar{R}\phi^\dagger L) \]

Here \( \phi = (\phi^+, \phi^0)^t \) is the scalar field.

Three components of this \( \phi \) induce masses for bosonic force careers \( W^\pm \) and \( Z \), and the remaining one defines the neutral Higgs boson.
New electroweak theory based on PID: The Lagrangian action

\( L = \int \left[ -\frac{1}{4} W^{a \mu \nu} W^{a \mu \nu} - \frac{1}{4} F_{\mu \nu} F^{\mu \nu} 
+ \bar{L}(i\gamma^\mu D'_\mu - m)L + \bar{R}(i\gamma^\mu D_\mu - m_e)R \right] dx, \)

where

\[
W^k_{\mu \nu} = \partial_\mu W^k_\nu - \partial_\nu W^k_\mu + \lambda f^{kij} W^i_\mu W^j_\nu,
F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,
D'_\mu = \partial_\mu + ieA_\mu + i\lambda W^k_\mu \sigma_k,
D_\mu = \partial_\mu + ieA_\mu.
\]

**Note:** When no weak interaction is present, \( W^k_{\mu \nu} = 0, \psi^L_\nu = 0, \) and the action (8) reduces to the classical QED action.
$SU(2)$ gauge transformation:

\[
L \rightarrow e^{(i/2)\theta_k \cdot \sigma_k} L, \\
\phi \rightarrow e^{-(i/2)\theta_k \cdot \sigma_k} \phi, \\
R \rightarrow R, \\
W^k_\mu \rightarrow W^k_\mu - \frac{2}{g_1} \partial_\mu \theta_k + f^{kij} \theta_i W^j_\mu, \\
B_\mu \rightarrow B_\mu
\]

$U(1)$ gauge transformation:

\[
L \rightarrow e^{(i/2)\beta} L, \\
\phi \rightarrow e^{-(i/2)\beta} \phi, \\
R \rightarrow e^{i\beta} R, \\
W^k_\mu \rightarrow W^k_\mu - \frac{2}{g_2} \partial_\mu \beta, \\
B_\mu \rightarrow B_\mu + \frac{2}{g_2} \partial_\mu \beta.
\]
Electroweak interacting field equations (based on PID):

\[
\partial^\mu W^{k}_{\mu\nu} - \frac{\lambda}{2} \sum_{l=1}^{3} f^{lik} W^{k}_{\mu\nu} W^{i}_\mu + J^{k}_\nu, \tag{9}
\]

\[
= (\partial_\nu + \alpha_k W^{k}_\nu + \alpha_0 A_\nu) \phi^{k}_1 \quad \text{for } k = 1, 2, 3,
\]

\[
\partial^\mu F^{L}_{\mu\nu} + J^{L}_{\nu} + J^{R}_{\nu} = (\partial_\nu + \beta_j W^{j}_\nu + \beta_0 A_\nu) \phi^{2}_{2}, \tag{10}
\]

\[
(i\gamma^\mu D^\prime_\mu - m)L = 0, \tag{11}
\]

\[
(i\gamma^\mu D^\mu - M_e)R = 0, \tag{12}
\]

where

\[
J^{k}_{\nu} = -\lambda \bar{L}\gamma_\nu\sigma_k L \quad \text{weak neutral current}
\]

\[
J^{L}_{\nu} = -e \bar{L}\gamma_\nu L,
\]

\[
J^{R}_{\nu} = -e \bar{R}\gamma_\nu R.
\]
Mass generation for $W^\pm$ and $Z$:

Let the scalar function $\phi_1^k$ be in the form

$$\phi_1 = \rho^k + \phi^k, \quad \rho^k = \text{constant}, \phi^k \simeq 0.$$  

Then omitting higher order terms, equations (9) become

(13) $$\partial^\mu W_{\mu \nu}^k - m_k W_{\nu}^k = 0, \quad m_k = \rho^k \alpha_k, \quad \text{for } k = 1, 2, 3.$$  

Let $m_1 = m_2 = M_W$, $m_3 = M_Z$. Then we derive three vector bosons with respective masses as follows:

$$W_\mu^\pm = \frac{1}{\sqrt{2}} (W_\mu^1 \pm iW_\mu^2), \quad Z_\mu = W_\mu^3,$$

which are the same as those derived from the Weinberg-Salam theory.
Higgs bosons: Taking divergence on both sides of (9) and noticing that 
\[ \partial^\nu \partial^\mu W^{k}_{\nu \mu} = 0, \]
we obtain that

\[ (14) \quad \partial^\mu \partial_\mu \phi^k_1 = m_k \phi^k_1 + \text{higher order terms}, \]

where \( m_k \) is the constant term in

\[ \alpha_k \partial^\mu W^k_\mu + \alpha_0 \partial^\mu A_\mu = m_k + \text{higher order terms}. \]

Hence (14) can be regarded as the Higgs field equations, and \( \phi^k_1 \) are the Higgs particles with mass \( m_k \).

Weak neutral current: \( J^k_\nu \).
VI. Unified Theory of Dark Energy and Dark Matter

- The new vector particle field $\Phi_\mu$ is massless with spin $s = 1$.

- The interaction between this particle field $\Phi_\mu$ and the graviton leads to a unified theory of dark matter and dark energy and explains the acceleration of expanding universe.
Consider the gravitational field equations alone:

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{c^4} T_{\mu\nu} - D_\mu D_\nu \varphi, \quad D^\mu \left( \frac{8\pi G}{c^4} T_{\mu\nu} + D_\mu D_\nu \varphi \right) = 0 \]

\[ R = \frac{8\pi G}{c^4} T + \Phi, \quad \int_M \Phi \sqrt{-g} dx = 0 \]

- The new spin-1 bosonic massless particle field is now given by the \( \Phi_\mu = D_\mu \varphi \), satisfying the Klein-Gordon equation in the vacuum (\( T_{\mu\nu} = 0 \)):

\[ D^\nu D_\nu (D_\mu \varphi) = 0. \]

- If the matter in the universe is uniformly distributed, we can show that \( \Phi_\mu = D_\mu \varphi = 0 \). Hence the new particle field will have no effect.

- This energy density \( \frac{c^4}{8\pi G} \Phi \) consists of both positive and negative energies. The negative part produces attraction, and the positive part produces repelling force.
Consider a spherically symmetric central matter field with mass $M$ and radius $r_0$. In this case,

\begin{equation}
    ds^2 = g_{00}c^2dt^2 + g_{11}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),
\end{equation}

and physically $g_{00}$ is given by

\begin{equation}
    g_{00} = -\left(1 + \frac{2}{c^2}\psi\right),
\end{equation}

where $\psi$ is the Newton gravitational potential. The tensors $g_{ij}$ in (15) are written as

\begin{equation}
    g_{00} = -e^u, \quad g_{11} = e^v, \quad g_{22} = r^2, \quad g_{33} = r^2 \sin^2\theta,
\end{equation}

where $u = u(r), \quad v = v(r)$.
The field equations become:

\begin{align*}
(18) \quad u'' + \frac{2u'}{r} + \frac{u'}{2}(u' - v') &= \varphi'' - \frac{1}{2}(u' + v' - \frac{4}{r})\varphi', \\
(19) \quad u'' - \frac{2v'}{r} + \frac{u'}{2}(u' - v') &= -\varphi'' + \frac{1}{2}(u' + v' + \frac{4}{r})\varphi', \\
(20) \quad u' - v' + \frac{2}{r}(1 - e^v) &= r(\varphi'' + \frac{1}{2}(u' - v')\varphi').
\end{align*}
With the new field equations, we derive that the force exerted on an object with mass \( m \) is given by

\[
F = mMG \left[ - \frac{1}{r^2} - \frac{c^2}{2GM} \left( 2 + \frac{2GM}{c^2} \frac{1}{r} \right) \frac{d\varphi}{dr} + \frac{c^2}{2GM} \Phi \right] \quad \text{for } r > r_0.
\]

The first term is the classical Newton gravitation, the second term is the coupling interaction between matter and the scalar potential \( \varphi \), and the third term is the interaction generated by the scalar potential energy density \( \Phi \) (\( R = \Phi \) for \( r > r_0 \)).
The sum $\varepsilon = \varepsilon_1 + \varepsilon_2$ of the scalar potential energy density

$$\varepsilon_1 = \frac{c^4}{8\pi G} \Phi$$

and the coupling interaction energy between the matter field and the scalar potential field $\varphi$

$$\varepsilon_2 = -\frac{c^4}{8\pi G} \left( \frac{2}{r} + \frac{2MG}{c^2 r^2} \right) \frac{d\varphi}{dr},$$

unifies dark matter and dark energy:

- $\varepsilon > 0$ represents dark energy,
- $\varepsilon < 0$ represents dark matter.
The interaction force formula \((21)\) can be further simplified to derive the following approximate formula for \(r_0 < r < r_1 \approx 10^{21} - 10^{22} km:\)

\[
(22) \quad F = mMG \left[ -\frac{1}{r^2} - \frac{k_0}{r} + k_1 r \right],
\]

where \(k_0 = 4 \times 10^{-18} km^{-1}\) and \(k_1 = 10^{-57} km^{-3}\), which are estimated using rotation curves of galactic motion.

Again, in \((22)\), the first term represents the Newton gravitation, the attracting second term stands for dark matter and the repelling third term is the dark energy.
VII. Concluding Remarks


- **Mechanism of energy creation and mass generation**, completely different from the Higgs mechanism.

- **Unified field equations coupling all four forces**, leading to results agreeable with those produced by the standard model and hence experimentally tested.

- **Duality** between the force carriers and the new bosonic particle fields.

- New predictions such as the **charged Higgs bosons**.

- The interaction of the new massless, spin-1 bosonic particle $\Phi_\mu$, and the graviton offers a unified theory of **dark matter and dark energy**.