Unified Field Theory of Four Interactions

Tian Ma, Shouhong Wang

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I. Two Roads to a Unified Field Theory

Four fundamental forces/interactions in Nature:
1. **Einstein's Unification**: To unify all four forces with larger symmetry groups such as SU(5), SO(10), ....

In fact, it appears that most, if not all, models with larger groups violate **PRI**.

2. Field Theory based on two new principles (**Ma-Wang, 2012**):

- **Principle of Interaction dynamics (PID)**: Variation of the Lagrangian action under energy-momentum conservation constraint.

- **Principle of Representation Invariance (PRI)**: Physical laws should be independent of representations of the gauge groups.
Challenges and Open Questions

Great achievements and insights have been made for last 100 years or so on the understanding of the structure of subatomic particles. There are still many longstanding open questions and challenges:

- Dark energy and dark matter

- What is the nature of strong interaction:
  - Why cannot we observe free quarks? – quark confinement
  - What is the mechanism of strong interaction decays?
  - Why the strong force is short-ranged? or why does not the massless gluon fields result in a long range-force, like gravity or the electromagnetic force?
  - As a force, what is the force formula/potential for strong interactions?

- The same questions above can also be asked for weak interactions
• Why is our universe as it is? – stability and structure of matter

• Is there a unified field theory for all four interactions?

These questions are longstanding questions, which touch the deepest secrets of Nature.
Principle of Interaction Dynamics (PID)

The classical Einstein equations are derived based on three principles:

• principle of equivalence and principle of general relativity, which amount to saying that space-time is a Riemannian manifold \((M, g_{ij})\), and gravity is described by the metric \(g_{ij}\).

• Lagrangian least action principle:

Einstein-Hilbert functional:

\[
L_{EH} = \int_{M} \left( R + \frac{8\pi G}{c^4} S \right) \sqrt{-g} dx
\]

\[
\delta L_{EH} = 0 \quad \Rightarrow \quad R_{ij} - \frac{1}{2} g_{ij} R + \frac{8\pi G}{c^4} T_{ij} = 0
\]
Due to the presence of dark energy and dark matter, the energy-momentum tensor $T_{ij}$ of normal matter is no longer conserved: $D^i(T_{ij}) \neq 0$.

By an orthogonal decomposition theorem, there is a scalar function $\varphi : M \rightarrow \mathbb{R}$ such that

\[
T_{ij} = \tilde{T}_{ij} - \frac{c^4}{8\pi G} D_i D_j \varphi, \quad D^i \tilde{T}_{ij} = 0
\]

\[
L_{EH} = \int_M \left( R + \frac{8\pi G}{c^4} S \right) \sqrt{-g} dx,
\]

\[
D^i \left[ R_{ij} - \frac{1}{2} g_{ij} R \right] = 0 \quad \implies \quad R_{ij} - \frac{1}{2} g_{ij} R + \frac{8\pi G}{c^4} \tilde{T}_{ij} = 0
\]

Namely

(1) \[
R_{ij} - \frac{1}{2} g_{ij} R = -\frac{8\pi G}{c^4} T_{ij} - D_i D_j \varphi, \quad D^i \left( D_i D_j \varphi + \frac{8\pi G}{c^4} T_{ij} \right) = 0
\]
Equivalently: The new gravitational field equations (1) can be derived with constraint least action principle:

\[
\lim_{\lambda \to 0} \frac{1}{\lambda} [L_{EH}(g_{ij} + \lambda X_{ij}) - L_{EH}(g_{ij})] = (\delta L_{EH}(g_{ij}), X) = 0 \quad \forall \ D^i X_{ij} = 0
\]

This constraint least action leads us to postulate the PID

**Principle of Interaction Dynamics (Ma-Wang, 2012):** For the four interactions in Nature, states \((g, A, \psi)\) are the extremum points of the Lagrangian action

\[
L(g, A, \psi) = \int_M L(g_{ij}, A, \psi) \sqrt{-g} dx,
\]

with the \((D + A)\)-free constraint (energy-momentum conservation).

Here \(A\) is a set of vector fields representing the gauge potentials, and \(\psi\) are the wave functions of particles.
**Principle of Representation Invariance (PRI)**

PRI (Ma-Wang, 2012): Physical laws for an $SU(N)$ gauge theory should be independent of different representations of $SU(N)$.

$SU(N)$ gauge theory: Certain physical properties of fermionic particles $\Psi$ are not distinguishable under the $SU(N)$ transformations:

\[
\Psi(x) = U(x)\tilde{\Psi}(x), \quad U(x) = e^{i\theta^a(x)\tau_a} \in SU(N) \quad \forall x \in M,
\]

where $M$ is the space-time manifold, the $K = N^2 - 1$ generators $\tau_a$ of $SU(N)$ are $N \times N$, traceless Hermitian matrices:

\[
\tau_a^\dagger = \tau_a, \quad \text{Tr } \tau_a = 0, \quad [\tau_a, \tau_b] = \tau_a\tau_b - \tau_b\tau_a = i\lambda^c_{ab}\tau_c,
\]

where $\theta^a = \theta^a(x) \ (1 \leq a \leq K)$ are real parameters, and $\lambda^c_{ab}$ are structure constants of generators.
Gauge fields $A^a_\mu (a = 1, \cdots, K)$ and $N$ Dirac spinor fields $\psi^j$: $\Psi = (\psi^1, \cdots, \psi^N)^t$.

Connection and Curvature:

$D_\mu = \partial_\mu + igA^a_\mu \tau_a$, $F_{\mu\nu} = F^a_{\mu\nu} \tau_a = \frac{i}{g}[D_\mu, D_\mu] = (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\lambda^a_{bc}A^b_\mu A^c_\nu)\tau_a$
PRI (Ma-Wang, 2012) Each family of generators $\tau_a = \{\tau_1, \cdots, \tau_K\} = \{\tau^1, \cdots, \tau^K\}$ can be regarded as a basis of $T_eSU(N)$. Take a basis transformation

$$\tilde{\tau}_a = x^b_a \tau_b, \quad X = (x^b_a), \quad X^{-1} = (\tilde{x}^b_a)$$

**Thm. 1.** $\tilde{\theta}^a = \tilde{x}^a_b \theta^b$, $\tilde{A}_\mu^a = \tilde{x}^a_b A^b_\mu$, $\tilde{\lambda}^c_{ab} = x^f_a x^g_b \tilde{x}^c_d \lambda^d_{fg}$. Thus, the quantities $\theta^a$, $A^a_\mu$, and $\lambda^c_{ab}$ are called $SU(N)$-tensors.

2. $G_{ab} = \frac{1}{4N} \lambda^c_{ad} \lambda^d_{cb}$ is a symmetric positive definite 2nd-order covariant $SU(N)$-tensor, which can be regarded as a Riemannian metric on $SU(N)$.

3. For $SU(2)$ using Pauli matrices as generators and for $SU(3)$ using the Gell-Mann matrices as generators, $G_{ab} = \delta_{ab}$.
We postulate the following new physical principle:

**Principle of Representation Invariance (PRI).** All $SU(N)$ gauge theories are invariant under the transformation (4). The representation invariant action and gauge field equations are

$$
L = \int_M -\frac{1}{4} G_{ab} g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu}^a F_{\alpha\beta}^b + \bar{\Psi} \left[ i\gamma^\mu (\partial_\mu + ig A^a_\mu \tau_a) - m \right] \Psi,
$$

$$
G_{ab} \left[ \partial^\nu F^b_{\nu\mu} - g\lambda^b_{\alpha\beta} F^c_{\alpha\mu} A^d_\beta \right] - g\bar{\Psi} \gamma_\mu \tau_a \Psi = 0,
$$

$$
(i\gamma^\mu D_\mu - m)\Psi = 0 \quad \text{Dirac eqs for fermions}
$$
II. Unified Field Theory of Four Interactions

The unified field model is derived based on the following principles:

• principle of general relativity (or Lorentz invariance) and the principle of equivalence, postulated by Albert Einstein (1905, 1915)


• principle of interaction dynamics (PID)

• principle of representation invariance (PRI)
Unified field model (Ma-Wang, 2012):

(5) \[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \frac{8\pi G}{c^4} T_{\mu\nu} = \left[ \nabla_\mu - \frac{e\alpha^E}{\hbar c} A_\mu - \frac{g_w\alpha^w_a}{\hbar c} W^a_\mu - \frac{g_s\alpha^s_k}{\hbar c} S^k_\mu \right] \Phi_\nu, \]

(6) \[ \partial^\nu F_{\nu\mu} - e\bar{\psi}\gamma_\mu\psi = \left[ \nabla_\mu - \frac{e\alpha^E}{\hbar c} A_\mu - \frac{g_w\alpha^w_a}{\hbar c} W^a_\mu - \frac{g_s\alpha^s_k}{\hbar c} S^k_\mu \right] \phi^E, \]

(7) \[ G^{w}_{ab} \left[ \partial^\nu W^b_{\nu\mu} - \frac{g_w}{\hbar c} \lambda^b_{\alpha\beta} W^c_{\alpha\mu} W^d_{\beta} \right] = g_w \bar{L}\gamma_\mu\sigma_a L \]
\[ = \left[ \nabla_\mu + \frac{1}{4} \left( \frac{m_H c}{\hbar} \right)^2 x_\mu - \frac{e\alpha^E}{\hbar c} A_\mu - \frac{g_w\alpha^w_a}{\hbar c} W^a_\mu - \frac{g_s\alpha^s_k}{\hbar c} S^k_\mu \right] \phi^w_a, \]

(8) \[ G^{s}_{kj} \left[ \partial^\nu S^j_{\nu\mu} - \frac{g_s}{\hbar c} \Lambda^j_{\alpha\beta} S^c_{\alpha\mu} S^d_{\beta} \right] = g_s \bar{q}\gamma_\mu\tau_k q \]
\[ = \left[ \nabla_\mu + \frac{1}{4} \left( \frac{m_\pi c}{\hbar} \right)^2 x_\mu - \frac{e\alpha^E}{\hbar c} A_\mu - \frac{g_w\alpha^w_a}{\hbar c} W^a_\mu - \frac{g_s\alpha^s_j}{\hbar c} S^j_\mu \right] \phi^s_k, \]

(9) \[ (i\gamma^\mu \tilde{D}_\mu - \tilde{m})\Psi = 0. \]
Conclusions and Predictions of the Unified Field Model

1. Duality: The unified field model induces a natural duality:

\[
\begin{align*}
\{g_{\mu\nu}\} \text{ (massless graviton)} & \quad \longleftrightarrow \quad \Phi_\mu, \\
A_\mu \text{ (photon)} & \quad \longleftrightarrow \quad \phi^E, \\
W^a_\mu \text{ (massive bosons } W^\pm \text{ & } Z) & \quad \longleftrightarrow \quad \phi^w_a \quad \text{for } a = 1, 2, 3, \\
S^k_\mu \text{ (massless gluons)} & \quad \longleftrightarrow \quad \phi^s_k \quad \text{for } k = 1, \ldots, 8.
\end{align*}
\]
2. Decoupling and Unification:

- An important characteristics is that the unified model can be easily decoupled.
- Both PID and PRI can be applied directly to individual interactions.
- For gravity alone, we have derived modified Einstein equations, leading to a unified theory for dark matter and dark energy.

3. The two $SU(2)$ and $SU(3)$ constant vectors $\{\alpha_w^a\}$ and $\{\alpha_s^k\}$, containing 11 parameters, represent the portions distributed to the gauge potentials by the weak and strong charges.

4. Origin of mass: We obtained a much simpler mechanism for mass generation and energy creation, completely different from the classical Higgs mechanism. This new mechanism offers new insights on the origin of mass.
III. Potential and Force Formulas for Strong Interactions

Observations demonstrate the following mysteries for strong interaction:

- Why is strong interaction finite between hadrons, but ”infinite” between quarks

- Two hadrons are repelling when $r < 0.6\, fm$, and attracting between $0.6\, fm < r < 2\, fm$. However, two quarks are still attracting for $r < 0.6\, fm$, and are repelling only for much smaller $r$.

What is the reason why different particles possess different properties of strong forces.

- The strong interaction between nucleons exchanges Yukawa meson, but strong interaction between quark exchanges Higgs particle with much larger mass. It is puzzling that the exchange particles are different.
Why can strong decay, achieved by repelling force, occur in quark attracting region?

Each interaction in Nature has its source, which we call charge, generating the corresponding force:

- gravitation: mass charge $m$ Newton, Einstein
- electromagnetism: electric charge $e$ Coulomb, Maxwell
- weak interaction: weak charge $g_w$
- strong interaction: strong charge $g_s$

The interaction force is the negative gradient of the charge potential $\Phi$:

$$F = -K\nabla \Phi$$

with $K$ being the corresponding charge.
Recall electromagnetic potential \( A_\mu = (A_0, A_1, A_2, A_3) \), \( A_0 \) represents its charge potential: \( \Phi_E = A_0 \), and \( \vec{A} = (A_1, A_2, A_3) \) represents the magnetic potential:

\[
F_E = -e \nabla \Phi_E \quad \text{the Coulomb force,} \quad F_M = \frac{1}{c} e \vec{v} \times \text{curl} \vec{A} \quad \text{Lorentz force.}
\]

**Strong interaction potentials (Ma-Wang, 2012):** QCD fields are the \( SU(3) \) gauge potentials \( \{ S^k_\mu \mid k = 1, \ldots, 8 \} \) (gluons). We define the total potential \( S_\mu \), which is \( SU(3) \) representation invariant:

\[
S_\mu = \alpha^s_k S^k_\mu = \{ S_0, \quad S_1, S_2, S_3 \} = \{ \text{strong charge potential}, \quad \text{strong rotational potential} \}.
\]
For the first time, we derive three levels of strong interaction potentials: the quark potential $S_q$, the nucleon potential $S_n$ and the atom/molecule potential $S_a$:

\[
S_q = g_s \left[ \frac{1}{r} - \frac{Bk_0^2}{\rho_0} e^{-k_0 r} \varphi(r) \right], \tag{11}
\]

\[
S_n = 3 \left( \frac{\rho_0}{\rho_1} \right)^3 g_s \left[ \frac{1}{r} - \frac{B_n k_1^2}{\rho_1} e^{-k_1 r} \varphi(r) \right], \tag{12}
\]

\[
S_a = 3N \left( \frac{\rho_0}{\rho_1} \right)^3 \left( \frac{\rho_1}{\rho_2} \right)^3 g_s \left[ \frac{1}{r} - \frac{B_n k_1^2}{\rho_2} e^{-k_1 r} \varphi(r) \right], \tag{13}
\]

where $\varphi(r) \sim r/2$, $B, B_n$ are constants, $k_0 = mc/\hbar$, $k_1 = m_\pi c/\hbar$, $m$ is mass of the strong interaction Higgs particle, $m_\pi$ is the mass of the Yukawa meson, $\rho_0$ is the effective quark radius, $\rho_1$ is the radius of a nucleon, $\rho_2$ is the radius of an atom/molecule, and $N$ is the number of nucleons in an atom/molecule.

These potentials match very well with experimental data, and offer a number of physical conclusions.
IV. Quark Confinement

The potential $\Phi$ has a minimum at $\bar{r}$ ($\rho_0 < \bar{r} < r_0 \simeq \rho_1 \leq 10^{-16}\text{cm}$), where the quark acting force $F$ is zero. Namely the corresponding force satisfies

$$F = \begin{cases} 
> 0 & \text{for } 0 < r < \bar{r}, \\
= 0 & \text{for } r = \bar{r}, \\
< 0 & \text{for } \bar{r} < r < r_0, \\
> 0 & \text{for } r > r_0.
\end{cases}$$

We infer from (14) the following conclusions:

- Two close enough quarks are repelling.

- Near $r = \bar{r}$, there are no interactions between quarks—the interactions are weak. This explains the quark asymptotic freedom phenomena.
• In the region $\bar{r} < r < r_0$, the quark acting force is attracting. In particular, the attracting potential energy has the order of magnitude as

$$
\Phi \sim \frac{-c\tau}{r_0^2\rho_0}.
$$

It implies that

$$
\Phi \to -\infty \text{ as } \rho_0 \to 0,
$$

which explains the quark confinement!

In fact, the ratio of binding energies of quark and nucleon is

$$
\frac{E_q}{E_n} = \left( \frac{B}{\rho_0} \right) / \left( 3 \left( \frac{\rho_0}{\rho_1} \right)^3 \frac{B_n}{\rho_1} \right) \sim \left( \frac{\rho_1}{\rho_0} \right)^4 \sim 10^{20},
$$

which is in the Planck level. This is the reason why we do not see free quarks!
V. Short-Range Nature of Strong Interaction

The short-range nature of strong interaction is due to the factors in the strong potentials:

\[
\left( \frac{\rho_0}{\rho_1} \right)^3, \quad \left( \frac{\rho_0}{\rho_2} \right)^3
\]
VI. Weakton Model and Mechanism of Sub-Atomic Decays
• Sub-atomic decays implies **interior structure of leptons, quarks and mediators.**

• We propose six elementary particles, which we call weaktons, and their anti-particles:

\[
 w^*, \ w_1, \ w_2, \ \nu_e, \ \nu_\mu, \ \nu_\tau, \\
 \bar{w}^*, \ \bar{w}_1, \ \bar{w}_2, \ \bar{\nu}_e, \ \bar{\nu}_\mu, \ \bar{\nu}_\tau,
\]

where

– \( \nu_e, \nu_\mu, \nu_\tau \) are the three generation neutrinos, and
– \( w^*, w_1, w_2 \) are three new particles, which we call \( w \)-weaktons.
– All weaktons are massless, spin-\( \frac{1}{2} \) particles with one unit of weak charge \( g_w \).
– Both \( w^* \) and \( \bar{w}^* \) are the only weaktons carrying strong charge \( g_s \).
Weakton constituents of charged leptons and quarks:

\[ e = \nu_e w_1 w_2, \quad \mu = \nu_\mu w_1 w_2, \quad \tau = \nu_\tau w_1 w_2 \]

\[ u = w^* w_1 \bar{w}_1 (\uparrow \downarrow, \downarrow \uparrow, \uparrow \uparrow, \downarrow \downarrow), \quad c = w^* w_2 \bar{w}_2 (\uparrow \downarrow, \downarrow \uparrow), \quad t = w^* w_2 \bar{w}_2 (\uparrow \downarrow, \downarrow \uparrow, \downarrow \downarrow) \]

\[ d = w^* w_1 w_2 (\uparrow \downarrow, \downarrow \uparrow), \quad s = w^* w_1 w_2 (\uparrow \downarrow, \downarrow \uparrow), \quad b = w^* w_1 w_2 (\uparrow \uparrow, \downarrow \downarrow) \]
Weakton constituents of mediators:

\[ \gamma = \cos \theta_w w_1 \bar{w}_1 - \sin \theta_w w_2 \bar{w}_2 \ (\uparrow\downarrow, \downarrow\uparrow), \]
\[ Z^0 = \cos \theta_w w_2 \bar{w}_2 + \sin \theta_w w_1 \bar{w}_1 \ (\uparrow\downarrow, \downarrow\uparrow), \]
\[ W^- = w_1 w_2 (\uparrow\downarrow, \downarrow\uparrow), \quad W^+ = \bar{w}_1 \bar{w}_2 (\uparrow\downarrow, \downarrow\uparrow), \]
\[ g^k = w^* \bar{w}^* (\uparrow\downarrow, \downarrow\uparrow), \quad k = \text{color index}, \]

**dual bosons:**

\[ \phi_\gamma = \cos \theta_w w_1 \bar{w}_1 - \sin \theta_w w_2 \bar{w}_2 (\uparrow\downarrow, \downarrow\uparrow), \]
\[ \phi_0^Z = \cos \theta_w w_2 \bar{w}_2 + \sin \theta_w w_1 \bar{w}_1 (\uparrow\downarrow, \downarrow\uparrow), \]
\[ \phi^-_W = w_1 w_2 (\uparrow\downarrow, \downarrow\uparrow), \quad \phi^+_W = \bar{w}_1 \bar{w}_2 (\uparrow\downarrow, \downarrow\uparrow), \]
\[ \phi^k_g = w^* \bar{w}^* (\uparrow\downarrow, \downarrow\uparrow), \]

**an additional mediator:**

\[ \phi^0_\nu = \sum_l \alpha_l \nu_l \bar{\nu}_l (\downarrow\uparrow), \quad \sum_l \alpha^2_l = 1 \]
Decay Mechanism

- Electro-positron annihilation/creation:

\[
2\gamma \rightarrow e^+ + e^-, \quad e^+ + e^- \rightarrow 2\gamma.
\]

In fact, the reaction formulas in (17) are not complete, and the correct formulas should be as follows

\[
2\gamma + \phi^0_\nu \rightleftharpoons e^+ + e^-.
\]

and, in their weakton constituents, the formula (18) can be written as

\[
w_1\bar{w}_1(\uparrow\uparrow) + w_2\bar{w}_2(\downarrow\downarrow) + \nu_e\bar{\nu}_e(\downarrow\uparrow) \rightleftharpoons \nu_e w_1 w_2(\downarrow\uparrow\downarrow) + \bar{\nu}_e \bar{w}_1 \bar{w}_2(\uparrow\downarrow\downarrow).
\]
• **β-decay:** Consider the classical β-decay process

\[ n \rightarrow p + e^- + \bar{\nu}_e, \quad n = udd, \quad p = uud, \]

which is equivalent to the following \(d\)-quark decay:

\[ d \rightarrow u + e^- + \bar{\nu}_e, \]

whose complete form should be given by

\[
    w^* w_1 w_2 (d) + \nu_e \bar{\nu}_e (\phi_\nu) + w_1 \bar{w}_1 (\gamma) \rightarrow w^* w_1 \bar{w}_1 (u) + w_1 w_2 (W^-) + \nu_e \bar{\nu}_e (\phi_\nu) \\
    \rightarrow w^* w_1 \bar{w}_1 (u) + \nu_e w_1 w_2 (e^-) + \bar{\nu}_e.
\]

Here the exchange of \(w_2\) and \(\bar{w}_1\) in \(d\) quark and photon \(\gamma\) form a \(u\) quark and a charged vector boson \(W^-\), then \(W^-\) captures a \(\nu_e\) from \(\phi_\nu\) to yield an electron \(e^-\) and a \(\bar{\nu}_e\).
VII. Summary

A unified field model for four interactions is derived based on two basic principles: PID and PRI. This model

• offers a unified theory for dark energy and dark matter,

• induces a natural duality between the mediators: graviton, photon, $W^\pm, Z^0$, gluons, and their dual fields;

• explains consistently quark confinement, asymptotic freedom, short-ranged nature of strong interaction, and Yukawa potential;

• introduces weak and strong charges, leading to strong and weak interaction potentials/force formulas;

The new weakton model of elementary particles explains sub-atomic decays, and is mutually supportive with the unified field model.
VIII. References
