Theory of Modeling Inhomogeneity Domains in GFLOW1

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Inhomogeneity domains in GFLOW

Inhomogeneity domains in GFLOW allow for discontinuities in hydraulic conductivity, porosity, aquifer base and recharge rate due to precipitation. Discontinuities in the porosity do not influence the groundwater flow solution, they are only used for calculation of average groundwater flow velocities, e.g. during particle tracking (pathline tracing). Discontinuities in the areal recharge rate due to precipitation cause the inhomogeneity domains to serve as areal sinks (with negative sink density). A discussion on areal sinks is found in Haitjema (1995), see page 220 through 223, and more extensively in Strack (1989), see pages 426 through 434. Discontinuities in the hydraulic conductivity lead to jumps in the discharge potential across the domain boundary, which is modeled by use of second order line-doublets, see Strack and Haitjema (1981), Strack (1989), pages 411 and 412, and Haitjema (1995), pages 215 through 220. Changes in the aquifer base elevation also lead to jumps in the discharge potential across the domain boundary, see Strack (1989), pages 412 through 414. The mathematical details of the implementation of inhomogeneity domains in GFLOW are summarized below.

Jump in hydraulic conductivity and aquifer base elevation

With reference to Figure 1 the following parameters are defined: Hydraulic conductivity k_i and k_o, inside and outside the domain, respectively. Aquifer base elevation b_i and b_o, inside and outside the domain, respectively. The aquifer top elevation is h_t, which is the same inside and outside the domain. The discharge potential at a point on the domain boundary, just inside and outside the domain, is Φ_i and Φ_o, respectively. The line-doublet strength at some point on the boundary is denoted by s, which varies parabolically along the straight line-doublet:

$$s(x) = -s_j \frac{(x-1)}{2} + s_{j+1} \frac{(x+1)}{2} + \mu_j (x^2 - 1) \tag{1}$$

where s_j and s_{j+1} are the real line-doublet strengths at the beginning and end node of the jth line-doublet and where μ_j is a parabolic strength component

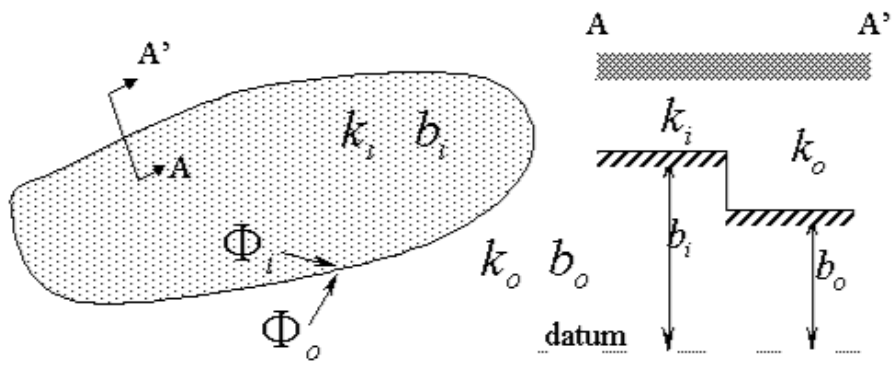

Figure 1: Closed domain with a jump in hydraulic conductivity and aquifer base elevation across its boundary.

at the center of the jth line-doublet, see Strack and Haitjema (1981). The parameter x defines a point on the line-doublet between the beginning node ($x = -1$) and the end node ($x = +1$). The line-doublet strength s equals the jump in the discharge potential across the line-doublet as follows:

$$s = \Phi_i - \Phi_o \tag{2}$$

For *unconfined flow* conditions expression (2) may be expanded as,

$$s = \tfrac{1}{2}k_i(\phi - b_i)^2 - \tfrac{1}{2}k_o(\phi - b_o)^2 \tag{3}$$

where ϕ is the head at the point on the boundary. Equation (3) can be rewritten in terms of the transmissivities T_i and T_o, which are defined as:

$$T_i = k_i h_i \qquad ; \qquad T_o = k_o h_o \tag{4}$$

where for unconfined flow conditions:

$$h_i = \phi - b_i \qquad ; \qquad h_o = \phi - b_o \qquad \phi \leq h_t \tag{5}$$

or for confined flow conditions:

$$h_i = h_t - b_i \qquad ; \qquad h_o = h_t - b_o \qquad \phi > h_t \tag{6}$$

With (6) the jump condition (2) can be written for *confined flow* conditions as

$$s = k_i h_i(\phi - b_i) - \tfrac{1}{2}k_i h_i^2 - (k_o h_o(\phi - b_o) - \tfrac{1}{2}k_o h_o^2 \tag{7}$$

Both (3) and (7) may be written in terms of Φ_i as:

$$s = \frac{T_i - T_o}{T_i}\Phi_i + \tfrac{1}{2}T_o(b_i - b_o) \tag{8}$$

or alternatively in terms of Φ_o as:

$$s = \frac{T_i - T_o}{T_o}\Phi_o + \tfrac{1}{2}T_i(b_i - b_o) \tag{9}$$

In writing (3) and (7) it has been tacitly assumed that the head ϕ is the same just inside and outside the domain boundary. In other words, the head is continuous. This is true as long as the head is larger than the largest aquifer base elevation. If not, either Φ_i or Φ_o is zero because the head is at the aquifer base inside or outside the domain, depending on which aquifer base is the highest. It follows from (8) that the jump s is equal to the inside potential in case the head is below the outside aquifer base:

$$s = \lim_{T_o \to 0}\left(\frac{T_i - T_o}{T_i}\Phi_i + \tfrac{1}{2}T_o(b_i - b_o)\right) = \Phi_i \tag{10}$$

Similarly, it follows from (9) that the jump s is equal to minus the outside potential in case the head is below the inside aquifer base:

$$s = \lim_{T_i \to 0}\left(\frac{T_i - T_o}{T_o}\Phi_o + \tfrac{1}{2}T_i(b_i - b_o)\right) = -\Phi_o \tag{11}$$

The jump condition may be satisfied at a discrete number of collocation points distributed along the inhomogeneity boundary. At the jth collocation point the inside potential may be written as:

$$(\Phi_i)_j = s_1F_{1j} + s_2F_{2j} + ...s_jF_{jj} + ... + \sum \text{other functions} \tag{12}$$

Where the terms F_{ij} represent the influence functions of the line-doublets associated with the strength parameters s_i, and where the last term in (12) represents the sum of the strength parameters times influence functions for the remainder of the line-doublets and all other analytic elements in the flow problem. Combining (8) with (12) yields:

$$-\tfrac{1}{2}\frac{T_oT_i}{T_i - T_o}(b_i - b_o) = s_1F_{1j} + s_2F_{2j} + ...s_j\left(F_{jj} - \frac{T_i}{T_i - T_o}\right) + ... + \sum \text{other functions} \tag{13}$$

In GFLOW the equations are formulated as the difference between two successive iterations (*iterative refinement*), as follows:

$$-\tfrac{1}{2}\frac{T_oT_i}{T_i - T_o}(b_i - b_o) - \overset{old}{\Phi}_i + \overset{old}{s}_j\frac{T_i}{T_i - T_o} = (s_1 - \overset{old}{s}_1)F_{1j} + (s_2 - \overset{old}{s}_2)F_{2j} + ...$$
$$(s_j - \overset{old}{s}_j)\left(F_{jj} - \frac{T_i}{T_i - T_o}\right) + ... + \sum \text{other functions} \tag{14}$$

By use of (2),

$$s_j = (\Phi_i)_j - (\Phi_o)_j \tag{15}$$

it may be shown that when using (9) instead of (8) the same matrix equation (13) or (14) will result.

As written the matrix coefficient for s_j depends on the head for the case of unconfined flow, since the transmissivities T_i and T_o depend on the head, see (4) and (5). Consequently, the matrix equations are non-linear in that some of its coefficients depend on the solution. It may be seen that in case the aquifer base elevation does not jump, hence only the hydraulic conductivity jumps, the matrix coefficient of s_j reduces to $k_i/(k_i - k_o)$, which is independent of the solution. Hence, hydraulic conductivity inhomogeneities are easier to solve than inhomogeneities in the aquifer base elevation.

Jump in the recharge rate

The inhomogeneity domain may be given an areal source or sink distribution to add or subtract recharge due to precipitation. In GFLOW all withdrawals of water are defined as positive, hence a jump in recharge rate is implemented by adding an areal sink distribution to the inside of the inhomogeneity domain. The theory of area sinks is presented in detail in Strack (1989), see section 37 page 426 through 442. The idea is to add a solution to Poisson's equation inside the boundary of the inhomogeneity (omitting it on the outside) and apply both line-doublets and line-dipoles along the boundary to eliminate the resulting jumps in the potential and flow across the boundary, respectively. The inhomogeneity domains in GFLOW already have line-doublets along their boundary to model the jump in the potential associated with the jump in hydraulic conductivity and aquifer bottom. By adding a complex strength parameter to these line-doublets the desired line-doublet (real part of the strength parameter) and line-dipole (imaginary part of the strength parameter) distribution can be created for the added recharge in the inhomogeneity domain. In GFLOW the following potential is added to the inside of the inhomogeneity:

$$\Phi = \frac{\gamma}{4}(r^2 - R^2) \tag{16}$$

The Laplacian of (16) is γ, which is the sink density (exfiltration rate) of the inhomogeneity. The implementation in GFLOW is carried out as follows.

With reference to Figure 2 the potential added to the inside of the polygon (inhomogeneity) is:

$$\Phi = \frac{\gamma}{4}\left(|z - z_0|^2 - R^2\right) \tag{17}$$

where z_0 is the average of all complex vertices z_n of the polygon:

$$z_0 = \frac{\sum_{n=1}^{N} z_n}{N} \tag{18}$$

The distance R in (17) is chosen such that the area of the circle with radius R equals that of the polygon:

$$R^2 = \frac{\sum_{n=1}^{N} A_n}{\pi} \tag{19}$$

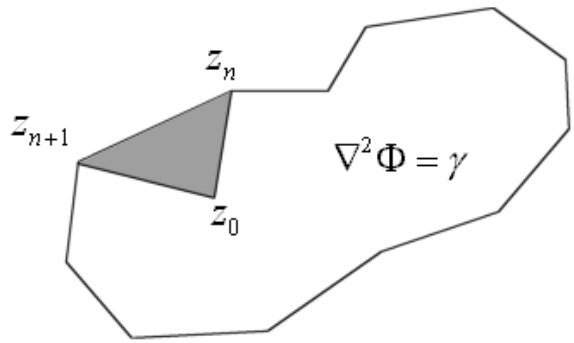

Figure 2: Adding recharge to an inhomogeneity domain.

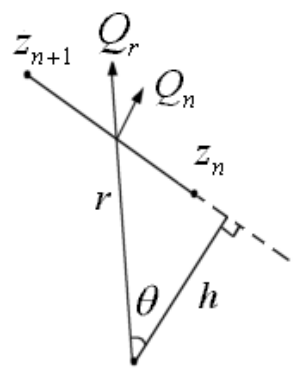

Figure 3: The discharge Q_n across the polygon side is constant.

where A_n is the area of the triangle formed by the line elements $z_0 - z_n$, $z_0 - z_{n+1}$ and $z_n - z_{n+1}$, see Figure 2. The area A_n is calculated by evaluating the vector product of the two vectors u_i and v_i that span the shaded triangle in Figure 2:

$$A_n = \tfrac{1}{2}(u_1 v_2 - u_2 v_1) \tag{20}$$

where u_i and v_i are defined as:

$$u_i = (\Re\{z_n - z_0\}, \Im\{z_n - z_0\}) \tag{21}$$

and

$$v_i = (\Re\{z_{n+1} - z_0\}, \Im\{z_{n+1} - z_0\}) \tag{22}$$

The flow across a polygon side, due to the added potential (17) is constant along the side, as may be seen as follows. The discharge Q_r due to (17) is radially toward z_0, hence with (16):

$$Q_r = -\frac{d\Phi}{dr} = -\frac{\gamma}{2} r \tag{23}$$

In Figure 3 Q_r is shown as positive and thus pointing away from z_0. The component Q_n normal to the polygon side is:

$$Q_n = Q_r \cos\theta = -\frac{\gamma}{2} r \cos\theta \tag{24}$$

With $h = r\cos\theta$ equation (24) becomes:

$$Q_n = -\frac{\gamma}{2} h \tag{25}$$

which is indeed constant for the polygon side. The minus signs in equations (23) through (25) indicate that the directions of flow due to the sink distribution inside the polygon are inward, opposite the direction shown in Figure 3.

In GFLOW line-doublet functions are used with a parabolic strength distribution, which is composed of a linear component with strength s_n and s_{n+1} at the vertices z_n and z_{n+1}, respectively, and a parabolic component with strength μ_n at the line-doublet center and zero at the vertices, see (1). In order to eliminate the jump in both the potential and the normal flow across the polygon sides, which are caused by adding (17) to the inside of the domain, complex strength parameters are added at the vertices (c_n) and at the line-doublet centers (d_n):

$$c_n = s_n^* + it_n^* \tag{26}$$

and

$$d_n = \mu_n^* + i\nu_n^* \tag{27}$$

Since the potential (17) is only added to the inside of the domain, the jump across the polygon side equals (17) evaluated at that side. Hence, the real part of c_n and d_n must be minus the inside potential in order to eliminate the jump:

$$s_n^* = -\frac{\gamma}{4}\left(|z_n - z_0|^2 - R^2\right) \tag{28}$$

and

$$\mu_n^* = -\frac{\gamma}{4}\left(|\frac{z_n + z_{n+1}}{2} - z_0|^2 - R^2\right) - \frac{s_n^* + s_{n+1}^*}{2} \tag{29}$$

The imaginary parts of the strength parameters provide a line-dipole distribution, which increase in strength represents the inflow in the line-dipole, see Strack, 1989 page 291 - 297. A constant inflow rate is obtained by a linearly varying line-dipole strength. Hence, in view of (25) the parabolic strength component ν_n^* vanishes:

$$\nu_n^* = 0 \tag{30}$$

The linear strength component t_n^* at each successive vertex increases with the amount of flow that crosses the polygon side, which is equal to the area of the shaded triangle in Figure 2 times the recharge rate γ:

$$t_{n+1}^* = t_n^* - \gamma A_n \tag{31}$$

where t_1^* at the first vertex is zero. The minus sign in (31) compensates the minus sign generated by it_n^* times the line-doublet function, which is $-t_n^*$ times a line-dipole function. Note, that the last vertex is also the first vertex in a closed domain. Consequently, at that vertex there is both an imaginary strength component of zero and an imaginary strength component equal to the total withdrawal of groundwater by the inhomogeneity domain. This jump in the line-dipole strength at the first (and last) vertex is realized by adding another potential function to both the inside and outside domain:

$$\Phi = \frac{\gamma \sum_{n-1}^{N} A_n}{2\pi} \ln(z - z_1) \tag{32}$$

which represents a well with a total discharge equal to the total withdrawal of groundwater by the inhomogeneity domain, see also Strack, 1989 page 433 equation (37.51).

References

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