$\mathcal{A}$: The logic of All x are y

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Probably the key point of logic is that there is a distinction between

syntax and semantics.

The idea is that syntax is the raw symbols.
The semantics is where we get the meaning.
So in our examples, we need some context or model to give a meaning.

In our examples, the syntax will start with some atoms p, q, n, n_1, ...

Then our sentences are expressions of the form *All p are q*.

We'll let **P** be the set of all atoms,
and sometimes we call these atoms nouns.

Syntax: We start with **P**, and then we make sentences *All p are q*,

We do not use any of the traditional logic symbols $\wedge$, $\vee$, $\neg$, $\forall$, $\exists$.
These could be added, however.

We use Greek letters φ, ψ, *chi*, etc. for sentences:

Syntax: We start with **P**, and then we make sentences *All p are q*,

Semantics: A model $\mathcal{M}$ is a set M,
and for each atom p we have an interpretation $[\![p]\!] \subseteq M$.

$$\mathcal{M} \models \textit{All } p \textit{ are } q \qquad \text{iff} \qquad [\![p]\!] \subseteq [\![q]\!]$$

The symbols $\mathcal{M} \models \varphi$ is read as $\mathcal{M}$ satisfies φ.

A statement like $\mathcal{M} \models \textit{All } p \textit{ are } q$ could also be read as

All p are q is true in $\mathcal{M}$

One fine point on the definition is that if $[\![x]\!]$ is the empty set $\emptyset$, then our sentence All x are y is *true*!
So in this room now,

All people in the room over 7 feet tall are standing

is (on this definition) true.

This strange point will lead us to various issues over the week.

For now, it might be best to say that it's true because there are *no exceptions*.

But we again admit that our semantics of All x are y
is not what most people would agree to in cases where $[\![x]\!] = \emptyset$.

Another fine point: what is this $\mathbf{P}$?

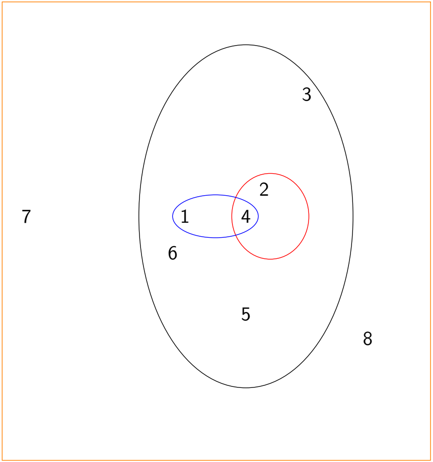
The name of our logic is $\mathcal{A}$, standing for *All*.
Recall that $\mathcal{A}$ is built from a set $\mathbf{P}$ of atoms.

This set could be anything.
In examples from English, it would be common nouns, usually in the plural.

In a more mathematical setting, we would usually take it to be some set of Roman letters (with subscripts if we need them).

Practice with subsets and with models

Let $M = \{1, 2, 3, 4, 5, 6, 7, 8\}$: the orange rectangle.
Let $[\![a]\!] = \{1, 2, 3, 4, 5, 6\}$, in the black oval.
Let $[\![x]\!] = \{1, 4\}$, shown in the blue oval.
Let $[\![y]\!] = \{2, 4\}$, in the red oval.



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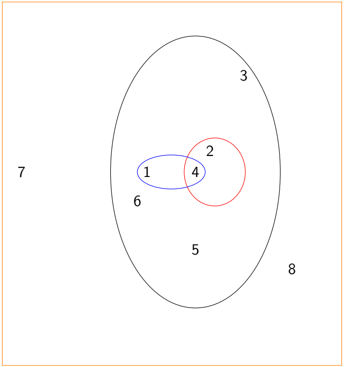
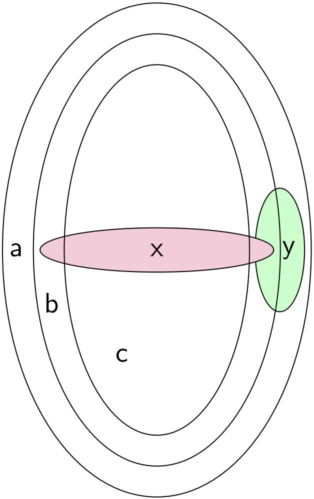
$\mathcal{M} \models$ All x are a
$\mathcal{M} \not\models$ All a are x
$\mathcal{M} \not\models$ All y are x
$\mathcal{M} \models$ All y are a
$\mathcal{M} \models$ All a are a



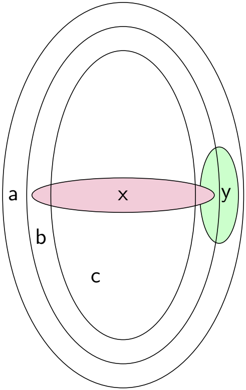
My convention is that each letter represents the biggest region that it is in.



More practice with subsets and with models

$c \subseteq b$ and $b \subseteq a$
$c \subseteq a$ (but this follows)
For all sets q in the picture, $q \subseteq q$ (again, this is by logic).
$y \subseteq a$
$x \subseteq b$

No other inclusion relations hold.



$\varphi_1, \ldots, \varphi_n \models \psi$ MEANS

Every model of the premises $\varphi_1, \ldots, \varphi_n$ is a model of the conclusion ψ.

We read this as ψ follows from $\varphi_1, \ldots, \varphi_n$.

To argue that $\varphi_1, \ldots, \varphi_n \models \psi$ we need some reasoning.

Usually, we do this in English and in an informal way,
just as one would do ordinary mathematics.

$\varphi_1, \ldots, \varphi_n \models \psi$ MEANS

Every model of the premises $\varphi_1, \ldots, \varphi_n$ is a model of the conclusion ψ.

$\varphi_1, \ldots, \varphi_n \not\models \psi$ MEANS

Some model of the premises $\varphi_1, \ldots, \varphi_n$ is not a model of the conclusion ψ.

To argue that $\varphi_1, \ldots, \varphi_n \not\models \psi$ we can produce a counterexample.

In all of this work, the main thing is that we have a rigorous definition.

All x are y, All y are z $\models$ All x are z

This means:
Every model $\mathcal{M}$ of the two premises

$$\begin{aligned}\varphi_1 &= \text{All } x \text{ are } y\\ \varphi_2 &= \text{All } y \text{ are } z\end{aligned}$$

is also a model of the conclusion $\psi =$ All x are z.

All x are y, All y are z $\models$ All x are z

Here is the reasoning:

Let $\mathcal{M}$ be any model, and assume that the premises are true in $\mathcal{M}$. We show that the conclusion is also true in $\mathcal{M}$.

So we know that $[\![x]\!] \subseteq [\![y]\!]$, and also $[\![y]\!] \subseteq [\![z]\!]$. Thus by basic facts about sets, $[\![x]\!] \subseteq [\![z]\!]$.

This shows that $\mathcal{M} \models$ All x are z.

And since $\mathcal{M}$ was arbitrary, we are done.

ALL x ARE y, ALL y ARE $z \not\models$ ALL z ARE x

This means:
There is some model $\mathcal{M}$ of the two premises

$$\begin{aligned}\varphi_1 &= \text{All } x \text{ are } y \\ \varphi_2 &= \text{All } y \text{ are } z\end{aligned}$$

which is not a model of the conclusion $\psi =$ All z are x.

ALL x ARE y, ALL y ARE z $\not\models$ ALL z ARE x

We have to give a concrete counterexample.
Let's take $M = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$,
$[\![x]\!] = \{1, 2, 3, 4\}$,
$[\![y]\!] = \{1, 2, 3, 4, 5, 9\}$,
$[\![z]\!] = \{1, 2, 3, 4, 5, 7, 9\}$.

Then the premises are true in this model,
but the conclusion is false.
(Any model would do, I picked a "random" one.)

So we are done.

Throughout the week, we'll use letters like φ, ψ, χ for sentences.

We use letters like Γ (Greek letter Gamma) for sets of sentences.

And then we would write $\Gamma \models \varphi$ to mean
that every model of all the sentences in Γ is also a model of φ.

However, if Γ is a set that we have listed out, say

$$\Gamma \quad = \quad \{\varphi_1, \varphi_2, \ldots, \varphi_{104}\}$$

Then usually we would write $\Gamma \models \varphi$ as

$$\varphi_1, \varphi_2, \ldots, \varphi_{104} \models \varphi$$

rather than as

$$\{\varphi_1, \varphi_2, \ldots, \varphi_{104}\} \models \varphi.$$

That is, we drop the set braces on the left of the $\models$ symbol.
We do this to make things a little more readable.

$$\overbrace{\varphi_1, \varphi_2, \dots, \varphi_n}^{\text{premises}} \quad \models \quad \overbrace{\varphi}^{\text{conclusion}}$$

The intuition is that

$$\varphi_1, \varphi_2, \dots, \varphi_n \models \varphi$$

should mean that

> any circumstance in which the premises $\varphi_1, \varphi_2, \dots, \varphi_n$ are all true is also a circumstance in which the conclusion φ is true

Of course, the "official" meaning is in terms of models.

Formal proofs: a preliminary point

A formal proof is like a caricature of human reasoning.

It is very common in introductory logic classes to
present one or another kind of formal proof systems.
(There are probably hundreds of them.)

Working with a formal proof system is usually a tedious and boring experience.

Let Γ be a set of sentences $\{\varphi_1, \ldots, \varphi_n\}$.
A proof tree over Γ is a tree following properties:

1. The leaves are either labeled with sentences from Γ, or with sentences of the form *All x are x*.
2. The interior leaves match one of the rules of our system (see the next slide).

The trees are drawn with the root at the bottom and the leaves at the top.

$\Gamma \vdash \varphi$ MEANS

There is a proof tree over Γ whose root is labeled φ.

This is read "Γ proves φ."
or "there is a proof of φ from Γ,"
or "φ is provable from Γ".

$$\frac{}{\textsf{All } p \textsf{ are } p}\ \textsc{axiom}$$

$$\frac{\textsf{All p are n} \quad \textsf{All n are q}}{\textsf{All p are q}}\ \textsc{barbara}$$

The (AXIOM) rule means that we can have All p are p
provided that there is nothing above it,
and this All p are p is justified as an "axiom".

The second rule is pretty straightforward.

Let Γ be the set

$$\{\textit{All a are b}, \textit{All q are a}, \textit{All b are d}, \textit{All c are d}, \textit{All a are q}\}$$

Let φ be *All q are d*.

EXAMPLE

Here is a proof tree showing that $\Gamma \vdash \varphi$:

$$\dfrac{\textit{All q are a} \qquad \dfrac{\textit{All a are b} \quad \textit{All b are d}}{\textit{All a are d}}\ \text{BARBARA}}{\textit{All q are d}}\ \text{BARBARA}$$

All three leaves belong to Γ.
Note also that two elements of Γ are not used as leaves.

This is permitted according to our definition.
The proof tree above shows that $\Gamma \vdash \varphi$.

What are we doing here?

The idea is that proof trees are our model of basic reasoning using the very limited kind of sentences that we have in this lecture:
All x are y.

It can be examined (and even constructed) by a person or computer
who has no understanding of anything but the rules!

There are several hopes about this work:
⋆ The whole thing will "scale up" to include many more words. (This would call on linguistic semantics to provide the correct notion of model.)
⋆ The formal relation $\vdash$ should have something to do with $\models$ (logic)
⋆ The proof system $\vdash$ should have something to do with actual human reasoning (psychology)
⋆ A computer should be able to work with $\vdash$ without understanding anything.

$$\Gamma = \left\{ \begin{array}{l} \text{All a are b,} \\ \text{All a are c,} \\ \text{All b are c,} \\ \text{All c are b,} \\ \text{All c are d,} \\ \text{All b are e,} \\ \text{All d are g,} \\ \text{All f are g,} \\ \text{All g are f} \end{array} \right\}$$

Do you think that $\Gamma \vdash$ *All b are g*?

Do you think that $\Gamma \vdash$ *All d are e*?

A computer could check whether a purported tree actually satisfies our definition, even if it didn't "understand" *All*.

So one important question is: what is the relation between

$$\Gamma \vdash \varphi \quad \text{and} \quad \Gamma \models \varphi \quad ?$$

Soundness

If $\Gamma \vdash \varphi$, then $\Gamma \models \varphi$.

This means that proof trees do not lead us astray:
if $\Gamma \vdash \varphi$, then in any model where the sentences of Γ all hold, φ too must hold.

Our proof system will not lead us to believe that bogus syllogisms are in fact valid.

Here is the basic idea of why the Soundness Lemma holds.

The two most basic facts about $\subseteq$ are:

1. $x \subseteq x$ for all sets x.
2. For all sets x, y, and z: if $x \subseteq y$ and $y \subseteq x$, then $x \subseteq z$.

(Probably the third basic fact would be that $\emptyset \subseteq x$ for all x.)

Soundness Sketch, Continued

Let's go back to our example proof tree. $\Gamma \vdash \varphi$:

$$\dfrac{\textit{All q are a} \qquad \dfrac{\textit{All a are b} \qquad \textit{All b are d}}{\textit{All a are d}}\,\mathrm{B}}{\textit{All q are d}}\,\mathrm{B}$$

Take any model, say $\mathcal{M}$.
Assume that $\mathcal{M} \models \Gamma$.
That is, we assume that in $\mathcal{M}$, $[\![a]\!] \subseteq [\![b]\!]$, etc.
We have to show that in this same model $\mathcal{M}$, $[\![q]\!] \subseteq [\![d]\!]$.

The idea is to use our proof tree and
read it downwards, talking about subsets of this one model $\mathcal{M}$:

$$\dfrac{[\![q]\!] \subseteq [\![a]\!] \qquad \dfrac{[\![a]\!] \subseteq [\![b]\!] \qquad [\![b]\!] \subseteq [\![d]\!]}{[\![a]\!] \subseteq [\![d]\!]}}{[\![q]\!] \subseteq [\![d]\!]}$$

And then going downward mirrors *intuitively valid reasoning in* $\mathcal{M}$.

Since the model $\mathcal{M}$ was arbitrary (had no special features),
the conclusion $\Gamma \models \varphi$ holds.

Educational point

Of course it goes by induction on proof trees.

SOUNDNESS

If $\Gamma \vdash \varphi$, then $\Gamma \models \varphi$.
If $\Gamma \not\models \varphi$, then $\Gamma \nvdash \varphi$.

EXAMPLE

$$\text{All } x \text{ are } y \nvdash \text{All } x \text{ are } z$$

To show this, we'll show that

$$\text{All } x \text{ are } y \not\models \text{All } x \text{ are } z$$

and then use the soundness of the proof system.

And for this last assertion, we can make a model:
$M = \{1, 2, 3\}$, $[\![x]\!] = \{1\}$, $[\![y]\!] = \{1, 2\}$, $[\![z]\!] = \{2, 3\}$.

$\Gamma \models \varphi$ means

Every model of all sentences in Γ is also a model of φ.

$\Gamma \vdash \varphi$ means

There is a proof tree over Γ whose root is φ and whose leaves are in Γ.

It's important to see that these two concepts

$$\Gamma \models \varphi \quad \text{and} \quad \Gamma \vdash \varphi$$

are different.

It will turn out that

$$\Gamma \models \varphi \quad \text{if and only if} \quad \Gamma \vdash \varphi$$

but this should not be obvious!

At this point, we know that our system is sound:

$$\text{If } \Gamma \vdash \varphi, \text{ then } \Gamma \models \varphi.$$

Perhaps more important is the *converse* of this:

Completeness

If $\Gamma \models \varphi$, then $\Gamma \vdash \varphi$.

Before we turn to the proof, it is important to see what this says.

Soundness says that the proof system will not lead us astray.

Completeness tells us that if Γ semantically implies ψ,
then we can find one of our (semantics-free!) proof trees showing
$\Gamma \vdash \psi$.

DEFINITION

A preorder is a pair $(P, \leq)$,
where P is a set
and $\leq$ is a relation on it with the following properties:

REFLEXIVE $p \leq p$

TRANSITIVE If $p \leq q$ and $q \leq r$, then $p \leq r$.

CAUTION

We need not have the following property:

ANTI-SYMMETRIC if $p \leq q$ and $q \leq p$, then $p = q$.

An anti-symmetric preorder is a partially ordered set (poset).

A simple example of a preorder

Let N be the positive and negative numbers, and let

$$n \leq m \qquad \text{iff} \qquad n \text{ divides } m \text{ with no remainder}$$

We have facts like

$$\begin{array}{lll} 2 \leq 4 & 2 \leq -4 & -5 \leq 500 \\ 3 \not\leq 4 & 4 \leq 4 & 74 \not\leq 4 \\ 3 \leq 6 & 4 \leq -2900 & -3 \not\leq -2 \end{array}$$

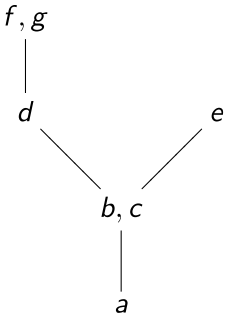
We would in addition have $6 \equiv -6$, because $6 \leq -6$ and $-6 \leq 6$.

Of course 6 and -6 are not equal,
so this preorder is not a partial order.

We draw the preorder with $\leq$ indicated by "going up (or equal)" and equivalent elements drawn right next to each other.

Downsets in preorders

In any preorder, $\downarrow p = \{x : x \leq p\}$.



$\downarrow f = \downarrow g = \{a, \ldots, g\} \setminus \{e\}$

$\downarrow d = \{a, b, c, d\}, \downarrow e = \{a, b, c, e\}$

$\downarrow b = \{a, b, c\} = \downarrow c$

$\downarrow a = \{a\}$

$\downarrow$ is monotone: if $p \leq q$, then $\downarrow p \subseteq \downarrow q$.

Proof of completeness: the canonical model

Suppose that $\Gamma \models$ *All x are y*.
At this point, we're going to make up a special model called the canonical model.

Let the universe M be the set of nouns (!).

Definition

Define $a \leq_\Gamma b$ to mean that $\Gamma \vdash$ *All a are b*.

Check that $\leq_\Gamma$ is reflexive and transitive, using the logic.
So we have a preorder, called the canonical preorder of Γ.

To make a model, we must interpret the nouns.
We get to interpret the nouns any way we like.

We use the downsets in the canonical preorder of Γ:

$$[\![a]\!] \quad = \quad \downarrow a \quad = \quad \{b : b \leq_\Gamma a\}$$

Claim: $\mathcal{M} \models \Gamma$.

Here is the proof that $\mathcal{M} \models \Gamma$

Suppose Γ contains All c are d.
We must check that $[\![c]\!] \subseteq [\![d]\!]$ in the model which we defined.

Let $w \in [\![c]\!]$. So $w \leq_\Gamma c$.
Thus $\Gamma \vdash$ All w are c.

But look:

$$\dfrac{\begin{array}{c}\vdots \\ \text{All w are c}\end{array} \quad \text{All c are d}}{\text{All w are d}}\ \textsc{barbara}$$

All the leaves are in Γ, and the tree shows that $\Gamma \vdash$ All w are d.
Thus $w \in [\![d]\!]$.

Since w is arbitrary, we have shown that $[\![c]\!] \subseteq [\![d]\!]$.

The $\vdots$ notation means that we have some proof tree over Γ whose root is All w are c.

We are therefore showing how to extend some such tree with one more step.

Where are we?

We assumed that $\Gamma \models$ All x are y.
We then made up a model $\mathcal{M}$.

By the Lemma, our model $\mathcal{M}$ makes all of the sentences in Γ true.
So in this model, $[\![x]\!] \subseteq [\![y]\!]$.
(Why is this true? It's a key point!)

But $x \in [\![x]\!]$, since we can prove *All x are x* from (any) Γ.
(This is where we use the (AXIOM) rule in our system.)
So $x \in [\![y]\!]$.
(Again, why is this true?)

This means that $\Gamma \vdash$ *All x are y*, as desired.


"That's all Folks!"

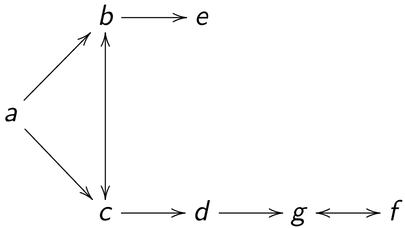
We have proved the completeness of our logical system A.

But we have not done it in a way that is algorithmically meaningful.

That is, we have not given an algorithm to tell whether or not $\Gamma \vdash \varphi$.

Our next order of business is to do this, using graphs.

$$\Gamma = \left\{ \begin{array}{l} \text{All a are b}, \\ \text{All a are c}, \\ \text{All b are c}, \\ \text{All c are b}, \\ \text{All c are d}, \\ \text{All b are e}, \\ \text{All d are g}, \\ \text{All f are g}, \\ \text{All g are f} \end{array} \right\}$$



This is called the All-graph of Γ.

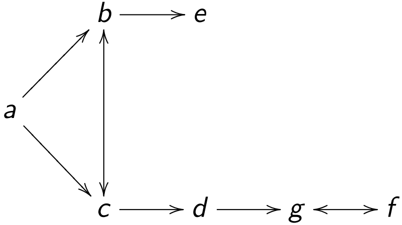
Let $(G, \to)$ be any graph.

We get a preorder $(G, \xrightarrow{*})$ with the same points by

$g \xrightarrow{*} h$ iff we can go from g to h
in 0 or more steps, following the arrows

We have already seen that every set Γ gives a graph

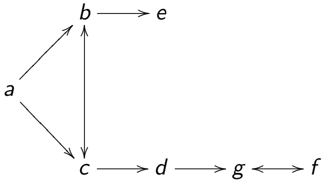
$$\Gamma = \left\{ \begin{array}{l} \text{All a are b,} \\ \text{All a are c,} \\ \text{All b are c,} \\ \text{All c are b,} \\ \text{All c are d,} \\ \text{All b are e,} \\ \text{All d are g,} \\ \text{All f are g,} \\ \text{All g are f} \end{array} \right\}$$



The graph just goes by looking at Γ.
If All x are y is in Γ, we put an edge from x to y in the graph.
In this case, we would write $x \xrightarrow[\Gamma]{} y$.

We have already seen that every set Γ gives a graph

$$\Gamma \;=\; \left\{ \begin{array}{l} \text{All a are b,} \\ \text{All a are c,} \\ \text{All b are c,} \\ \text{All c are b,} \\ \text{All c are d,} \\ \text{All b are e,} \\ \text{All d are g,} \\ \text{All f are g,} \\ \text{All g are f} \end{array} \right\}$$



We use $\xrightarrow[\Gamma]{*}$ for the preorder relation determined by the graph.

In the preorder, $a \xrightarrow[\Gamma]{*} b$, $b \xrightarrow[\Gamma]{*} c$, $c \xrightarrow[\Gamma]{*} d$, $b \xrightarrow[\Gamma]{*} e$, $a \xrightarrow[\Gamma]{*} e$, $b \xrightarrow[\Gamma]{*} d$, $d \xrightarrow[\Gamma]{*} g$, $d \xrightarrow[\Gamma]{*} f$, $g \xrightarrow[\Gamma]{*} f$, $f \xrightarrow[\Gamma]{*} g$, $a \xrightarrow[\Gamma]{*} e$, $a \xrightarrow[\Gamma]{*} c$, $a \xrightarrow[\Gamma]{*} d$, $a \xrightarrow[\Gamma]{*} g$, $a \xrightarrow[\Gamma]{*} f$, $b \xrightarrow[\Gamma]{*} c$, $b \xrightarrow[\Gamma]{*} d$, $b \xrightarrow[\Gamma]{*} g$, $b \xrightarrow[\Gamma]{*} f$, $c \xrightarrow[\Gamma]{*} g$, $c \xrightarrow[\Gamma]{*} f$.

Also $x \xrightarrow[\Gamma]{*} x$ for all x.

Fix a set Γ.

Let x and y be any atoms.

Then the following are equivalent:

1. $x \leq_\Gamma y$.
   (That is, $\Gamma \vdash$ All x are y.)

2. $x \xrightarrow[\Gamma]{*} y$.
   (That is, there is a path from x to y in the All-graph of Γ.)

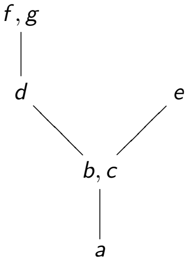
THE POINT

There are algorithms for finding the reachability relation
in any graph,
in particular, we can build the All-graph of Γ and then find $\xrightarrow[\Gamma]{*}$.

Γ induces a graph, the graph induces a preorder

Here is a picture of that preorder

$$\Gamma = \left\{ \begin{array}{l} \text{All a are b}, \\ \text{All a are c}, \\ \text{All b are c}, \\ \text{All c are b}, \\ \text{All c are d}, \\ \text{All b are e}, \\ \text{All d are g}, \\ \text{All f are g}, \\ \text{All g are f} \end{array} \right\}$$



The set P here is $\{a, \ldots, g\}$.
The order is $\xrightarrow[\Gamma]{*}$, which we just saw listed in full.
We draw the order with $\xrightarrow[\Gamma]{*}$ indicated by going upwards.
Note that $b \equiv c$ and $f \equiv g$.

A challenge question from before: Does $\Gamma \vdash All\ d\ are\ e$?

All a are b
All a are c
All b are c
All c are b
All c are d
All b are e
All d are g
All f are g
All g are f

$$
\begin{aligned}
[\![a]\!] &= \{a\} \\
[\![b]\!] &= \{a, b, c\} \\
[\![c]\!] &= \{a, b, c\} \\
[\![d]\!] &= \{a, b, c, d\} \\
[\![e]\!] &= \{a, b, c, e\} \\
[\![f]\!] &= \{a, \ldots, g\} \setminus \{e\} \\
&= \{a, b, c, d, f, g\} \\
[\![g]\!] &= \{a, \ldots, g\} \setminus \{e\}
\end{aligned}
$$

Γ is listed on the left.
We construct the canonical model of Γ.
It is shown on the right.

Since $[\![d]\!]$ is not a subset of $[\![e]\!]$, $\Gamma \not\models All\ d\ are\ e$.

By soundness, $\Gamma \not\vdash All\ d\ are\ e$.

Given Γ and also nouns u and v, the following are equivalent:

(1) $\Gamma \vdash$ *All u are v*.
(2) $\Gamma \models$ *All u are v*.
(3) $u \in [\![v]\!]$ in the canonical model of Γ.
(4) $u \leq_\Gamma v$ in the preorder determined from Γ.
(5) $u \xrightarrow[\Gamma]{*} v$: there is a path from u to v in the all-graph of Γ

(1) $\Rightarrow$ (2) is soundness of the proof system.
(2) $\Rightarrow$ (3) is by the argument in the completeness theorem.
(3) $\Rightarrow$ (4) is by the structure of the canonical model of Γ.
(4) $\Rightarrow$ (5) is by induction on proof trees.
(5) $\Rightarrow$ (1) by induction on the length of paths in the All-graph.

Given Γ and also nouns u and v, the following are equivalent:

(1) $\Gamma \vdash$ *All u are v*.
(2) $\Gamma \models$ *All u are v*.
(3) $u \in [\![v]\!]$ in the canonical model of Γ.
(4) $u \leq_\Gamma v$ in the preorder determined from Γ.
(5) $u \xrightarrow[\Gamma]{*} v$: there is a path from u to v in the all-graph of Γ

The point of this is that (5) is efficiently computable.

In fact, given Γ as an input,
we can find the full table of which pairs (u, v) have
$\Gamma \vdash$ *All u are v*
in one fell swoop, in polynomial time.

The algorithm is easily implemented on a computer.

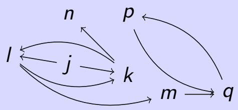
You can see something similar at
this web site, for example.

Another example of everything

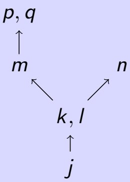
A set Γ

All j are k,
All j are l,
All k are l,
All l are k,
All l are m,
All k are n,
All m are q,
All p are q,
All q are p

the all-graph of Γ



the associated preorder



the canonical model of Γ

$$
\begin{aligned}
[\![j]\!] &= \{j\}\\
[\![k]\!] &= \{j,k,l\}\\
[\![l]\!] &= \{j,k,l\}\\
[\![m]\!] &= \{j,k,l,m\}\\
[\![n]\!] &= \{j,k,l,n\}\\
[\![p]\!] &= \{j,k,l,m,p,q\}\\
[\![q]\!] &= \{j,k,l,m,p,q\}
\end{aligned}
$$

Completeness: For all Γ and φ,
if $\Gamma \not\vdash \varphi$,
then there is some $\mathcal{M} \models \Gamma$ where φ is false.

Characteristic Model Property:
For all Γ, there is a model $\mathcal{M} \models \Gamma$
such that for all φ,
if $\Gamma \not\vdash \varphi$, then φ is false in $\mathcal{M}$.

The main facts are

1. $\mathcal{M} \models \Gamma$
2. If $\mathcal{M} \models$ all x are y, then $\Gamma \vdash$ *all x are y*.

Completeness: For all Γ and φ,
if $\Gamma \not\vdash \varphi$,
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Characteristic Model Property:
For all Γ, there is a model $\mathcal{M} \models \Gamma$
such that for all φ,
if $\Gamma \not\vdash \varphi$, then φ is false in $\mathcal{M}$.

This model is the canonical model of Γ.

The universe of the model is the set of nouns (=variables) that occur in Γ.

The definition of the model is

$$[\![a]\!] \quad = \quad \downarrow a \quad = \quad \{b : b \leq_\Gamma a\} \quad = \quad \{b : \Gamma \vdash \textit{All } b \textit{ are } a\}$$

The main facts are

1. $\mathcal{M} \models \Gamma$
2. If $\mathcal{M} \models$ all x are y, then $\Gamma \vdash$ *all x are y*.