Natural Logic
Homework 1

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Problem 1

The semantics of *All X are Y* in a model is that $[\![X]\!] \subseteq [\![Y]\!]$.

What should the semantics be for the following:

All X which are Y are Z.

Answer: $X \cap Y \subseteq Z$.

What should the semantics be for the following:

Of the X, the only ones which are Y are Z.

Answer: $X \cap Z \subseteq Y$.

We define

$$\mathcal{M} \models \mathsf{All}^* \; x \; y \quad \text{iff} \quad [\![x]\!] \neq \emptyset \text{ and } [\![x]\!] \subseteq [\![y]\!]$$

Notice that if we have a model $\mathcal{M}$ and an atom x where $[\![x]\!] = \emptyset$, then automatically $\mathcal{M} \not\models \mathsf{All}^* \; x \; y$, no matter what y is.

Problem 2

1. True or false? $\models \mathsf{All}^* \; x \; x$.
   False: Consider any model where $[\![x]\!] = \emptyset$.
2. Show that $\mathsf{All}^* \; x \; y \models \mathsf{All}^* \; x \; x$.
3. Show that $\mathsf{All}^* \; x \; y \models \mathsf{All}^* \; y \; y$.
4. Show that $\mathsf{All}^* \; x \; y, \mathsf{All}^* \; y \; z \models \mathsf{All}^* \; x \; z$.
   These are all easy.

This is a continuation of the last problem. Let's make a logical system for our logic using the following rules:

$$\frac{\mathsf{All}^*\ x\ y}{\mathsf{All}^*\ x\ x}\ \mathrm{R} \qquad \frac{\mathsf{All}^*\ x\ y}{\mathsf{All}^*\ y\ y}\ \mathrm{S} \qquad \frac{\mathsf{All}^*\ x\ y \quad \mathsf{All}^*\ y\ z}{\mathsf{All}^*\ x\ z}\ \mathrm{B}$$

Fix a set Γ. Define the canonical model of Γ to be the model $\mathcal{M}$, where M is the set of all atoms, and with the interpretation function given as follows: For all atoms w,

$$[\![w]\!] \;=\; \{x : \Gamma \vdash \mathsf{All}^*\ x\ w\}.$$

Problem 3

1. Show that $\mathcal{M} \models \Gamma$. That is, if the sentence $\mathsf{All}^*\ u\ v$ belongs to Γ, show that it is true in $\mathcal{M}$.
   Take a sentence in Γ, say $\mathsf{All}^*\ a\ b$. Using (R),
   $$\Gamma \vdash \mathsf{All}^*\ a\ a$$
   and so $a \in [\![a]\!]$. In particular, $[\![a]\!] \neq \emptyset$.
   Let $x \in [\![a]\!]$.
   Then $\Gamma \vdash \mathsf{All}^*\ x\ a$.
   And using (B), $\Gamma \vdash \mathsf{All}^*\ x\ b$. Thus $x \in [\![b]\!]$.

2. Show that every sentence true in $\mathcal{M}$ can be proved from Γ.
   That is, if $\mathcal{M} \models \mathsf{All}^*\ a\ b$, then $\Gamma \vdash \mathsf{All}^*\ a\ b$.
   We know that $[\![a]\!] \neq \emptyset$.
   So let $c \in [\![a]\!]$. Thus, $\Gamma \vdash \mathsf{All}^*\ c\ a$.
   By (S), $\Gamma \vdash \mathsf{All}^*\ a\ a$. So $a \in [\![a]\!]$.
   By our assumption that $\mathcal{M} \models \mathsf{All}^*\ a\ b$, $a \in [\![b]\!]$, etc.

3. Use the last two parts to prove that our logic is complete.
   [This should be quite familiar.]