Chapter 1

# TOPOLOGY AND EPISTEMIC LOGIC

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**Abstract** We present the main ideas behind a number of logical systems for reasoning about points and sets that incorporate knowledge-theoretic ideas, and also the main results about them. Some of our discussions will be about applications of modal ideas to topology, and some will be on applications of topological ideas in modal logic, especially in epistemic logic.

In the former area, we would like to present the basic ideas and results of topologic, the study of two-sorted bimodal logical systems interpreted on subset spaces; these are arbitrary sets with collections of subsets called *opens*. Many of the papers in this field deal with questions of axiomatizing the logics of particular classes of subset spaces determined by conditions on the "opens", such as being closed under intersection, being topologies, or satisfying various chain conditions.

In the area of applications of topological ideas in epistemic logic, we include a section on the following topics: a topological semantics and completeness proof for the logic of belief $KD45$.

## 1. Introduction

This handbook chapter explores some themes which relate general topology and epistemic logic. The leading ideas are: (1) to review the connection between the modal logic $S4$ and topology going back to the work of Alfred Tarski and J. C. C. McKinsey in the 1940's; (2) to discuss the epistemic interpretation of topology (3) to present the two-sorted semantics of topologic and to mention what is known about it, including some of the main completeness and decidability results; (4) to present a topological semantics for the logic of belief $KD45$ based on the derived set operation; and (5) to briefly mention related work in a number of directions.

**Topology and modal logic: a first look.** One of the things which strikes one when studying elementary (set-theoretic) topology is how easy it is. Notions like *open, closed, dense*, seem intuitively transparent: their basic properties easy to prove. Contrasting this fact is that topology uses second order notions as it reasons with both points and sets. This would imply that like second order logic, topology ought to be computationally very difficult.

This intuitive tension between the two paradigms vanishes when one realizes that a large part of topology can be seen as a *modal logic*, i.e., as an epistemic logic which combines the notion of knowledge and effort. Recall that modal logics tend to be much easier than first order logic, let alone second order.

Suppose that we have made a *measurement* – for example, that some velocity $v$ is 50 $\pm .5$. We interpret this as saying that $v$ is in the open set (49.5,50.5) and therefore anything which we *know* about $v$ must hold not only of $v$ itself, but also of any $v'$ in the same interval. (49.5,50.5) thus becomes an equivalence class for an appropriate $S5$ logic of knowledge. Thus the connection with modal logic.

However, the notion of *effort* can also enter, as $v$ might be measured more accurately with more effort. Following up on this basic intuition, the first two authors of this chapter developed a bimodal logic called topologic for studying elementary topology. This logic turned out to have a nice axiomatization and to be decidable. The original work was followed up then by further work by Georgatos and also by Dabrowski in conjunction with the two original writers (see Georgatos, 1993; Georgatos, 1994a; Georgatos, 1994b; Georgatos, 1997; Heinemann, 1997; Heinemann, 1999b; Heinemann, 2001; Weiss, 1999). We should mention that the original logic was defined for arbitrary subset spaces. The subsequent logics consider extensions of this logic where restricted families of sets were considered, closed under union or intersection. The bulk of our chapter, Sections 4–8 is about topologic.

Another large part of the chapter comes from Steinsvold's discovery that the notion of *belief* can also be given a topological meaning via the notion of derived set. This is covered in Section 10; the main reference is his dissertation (Steinsvold, 2006).

This chapter may be considered a continuation of van Benthem and Bezhanishvili's Chapter **??** of this handbook. Although we have written this chapter to be read on its own, readers would certainly benefit from a look at the many related discussions about different logics that can be found in Chapter **??**.

## 2. Perspectives

This handbook deals with *logic* and *space*, more precisely with various logics formulated with different goals in mind. But in a certain sense, the goal is usually to take a common mathematical model of space and then to fashion logical tools to work with it. This chapter works in a different way. The overall points are to investigate notions such as *knowledge*, *belief*, and *observation effort*; and mathematical structures like topological spaces arise in the course of that investigation. So the modus operandi, and the overall goals, of our work are different. One should not expect insight into, say, algebraic topology from our work. What we are after is rather a kind of reconstruction of the ideas underlying topology.

Some perspective on our subject might be gleaned from comparing the ideas with those in a well-known source, Steven Vickers' book *Topology via Logic* (see Vickers, 1989). One goal of that book is to explain the non-Hausdorff topologies that arise in computer science. This goal is relevant for our chapter, more so perhaps than for other chapters of this handbook. The overall message is that (p. 3) "topology is used to explain *approximate* states of information: the points include both approximate points and more refined points, and these relate to the topology by the property that if an open set contains an approximate point, then it must contain any refinement of it." The book goes on (pp. 5–11) to discuss "finite observations," and therefore aims at a reconstruction of topology in terms of logics of observations. The algebraization of a logic of finite observations is called a *frame*; it is a complete Heyting algebra under two operations $\wedge$ (binary) and $\bigvee$ (infinitary). (Note that the opens of a topological space are a frame under the corresponding set-theoretic operations.) This notion of a frame gives us a more topological notion, a *locale*; this is based on the frame morphisms into a special two-element frame.

Now our work differs from Vickers' in the sense that we add to the notion of observation a notion of *effort*. Our discussion begins in Section 4. Our treatment of effort is in a sense fairly crude: we take the subsets of a space as the possible observations, and then more effort corresponds to a smaller set, a better approximation to being closer to the "real" point. We say that this is crude because it does not measure the amount of effort in any real sense. However, it is a refinement of treatments which do not include any modeling of effort whatsoever. We start in the next section by first discussing the classical work of Tarski and McKinsey.

## 3. The original topological interpretation of modal logic: Tarski and McKinsey's Theorem

The project of relating topology to modal logic begins with work of Alfred Tarski and J.C.C. McKinsey (see McKinsey, 1941; McKinsey, 1944). The basic idea is to study the laws of the *interior* operation on subsets of a topological space and its dual, the *closure* operator. Suppose that $\mathcal{X} = \langle X, \mathcal{O} \rangle$ is a *topological space*; this just means that $\mathcal{O}$ is a family of subsets of $X$ containing the empty set $\emptyset$ and $X$ itself, and $\mathcal{O}$ is closed under arbitrary unions and finite intersections. The family $\mathcal{O}$ is called a *topology*, and its elements are the *open sets* of the space $\mathcal{X}$. The *closed* sets are the complements of the opens. For any subset $A \subseteq X$, we define its interior $A^{\mathrm{o}}$ to be the largest open subset of $A$. Dually, the closure $\overline{A}$ is the smallest closed set including $A$. We say "dually" here to illustrate one of the properties of these operations: $A^{\mathrm{o}} = X \setminus \overline{(X \setminus A)}$. In words, the interior of $A$ is the complement of the closure of the complement of $A$. This holds for all subsets of all spaces; it is just this kind of general property that we aim to study.

In order to study these notions, we introduce a logical language $\mathcal{L}_0$. It is a modal language. We begin with an arbitrary but fixed set $At$ of atomic propositions and close under truth functions $\wedge$ and $\neg$ and the appropriate modalities; for present purposes, we use the modality $I$ for the interior operation. (As we have seen, $I$ and $C$ are interdefinable duals. So only one needs to be taken as basic in the syntax, and then the other may be regarded as a defined symbol.)

The language $\mathcal{L}_0$ is interpreted on such a space $\mathcal{X}$ together with an *interpretation map into* $\mathcal{X}$ $i : At \to \mathcal{P}(X)$. (We do not insist that each $i(p)$ is an open set.) So for atomic $p$, $i(p)$ says which points satisfy $p$. We call $\langle X, \mathcal{O}, i \rangle$ a *topological model*. Then $i$ extends to all of $\mathcal{L}_0$ by interpreting negation as complement relative to $X$, conjunction as intersection, and $C$ and $I$ as the closure and interior operators respectively. In symbols, we have

$$\begin{array}{lcl} i(\neg\phi) & = & X \setminus i(\phi) \\ i(\phi \wedge \psi) & = & i(\phi) \cap i(\psi) \\ i(I\phi) & = & (i(\phi))^{\mathrm{o}} \end{array}$$

EXAMPLE 1.1 For a very easy example, consider the usual real line $\mathcal{R}$ with $i(p) = \{1\}$. Then $i(Ip) = \emptyset$, $i(\neg p) = R \setminus \{1\}$, and $i(I\neg p) = i(\neg p)$. Moreover, one can check as well that for all $\phi$, $i(\phi)$ is always one of the sets $\emptyset$, $R$, $\{1\}$, or $R \setminus \{1\}$.

As with all attempts to study some phenomenon, the main idea here is that the basic properties of the boolean operations on sets (unions, intersection, and complement), and also the salient topological operations (interior and closure) correspond to sentences in the language, or rather to schemes of sentences. For example, consider another general fact: the interior of the intersection of two sets is the intersection of their interiors. This corresponds to the fact that for all sentences $\phi$ and $\psi$, and for all spaces $\mathcal{X}$ and all interpretations $i$ of whatever atomic sentences we have, we also have

$$i((I(\phi \wedge \psi) \leftrightarrow ((I\phi) \wedge (I\psi))) \quad = \quad X.$$

For another example, the fact that the interior operation is idempotent corresponds in the same way to the scheme

$$i((II\phi) \leftrightarrow (I\phi)) \quad = \quad X.$$

One of the natural questions to ask about this language and its semantics is: can we characterize in an enlightening way the sentences $\phi$ with the property that for all topological models $\langle X, \mathcal{O}, i\rangle$, $i(\phi) = X$? We call such sentences *topologically valid*. The reason for this is that we more generally write $x \models \phi$ for $x \in i(\phi)$. (This notation from model theory will be used throughout our chapter.) Then the topologically valid sentences are those that are true at all points in all spaces under all interpretations. Similarly, the *topologically satisfiable* sentences are those sentences $\phi$ which belong to $i(\phi)$ for some topological model. And we say that a set $S \subseteq \mathcal{L}$ is topologically satisfiable if there is a topological model with $\bigcap_{\phi \in S} i(\phi) \neq \emptyset$.

What Tarksi and McKinsey proved is that the topologically valid sentences are exactly those provable in the logical system $S4$. This is a logical system which had been proposed much earlier than their work. It is the system whose axioms are the substitution instances of the tautologies of classical propositional logic, and also the schemes below:

1 $I(\phi \rightarrow \psi) \rightarrow (I\phi \rightarrow I\psi)$

2 $I\phi \rightarrow \phi$

3 $I\phi \rightarrow II\phi$

(Note that the second of these can be read as saying that the interior of a set is a subset of it, and the last is one direction of the idempotence of the interior operation.) The rules of $S4$ are modus ponens, and also necessitation: from $\phi$, derive $I\phi$. It should be mentioned that the earliest work on $S4$ did not present this semantics, so that the Tarski-McKinsey work may be read as giving a nice semantics to an already-existing logical system.

THEOREM 1.2 (TARSKI AND MCKINSEY) *The interpretation of $\mathcal{L}_0$ in topological spaces is sound and complete in the sense that the following are equivalent:*

*1 $\phi$ is topologically valid.*

*2 $\phi$ is provable in $S4$.*

*Moreover, the set of topologically valid sentences is decidable.*

*Proof* The soundness being routine, here is a sketch of the completeness. Our work here follows a recent presentation (Aiello et al., 2003). One considers the *theories* in $S4$; these are the maximal consistent sets of sentences. We show that each theory $T$ is topologically satisfiable. This is equivalent to completeness. Actually, we shall show that all theories $T$ are satisfiable in the same *canonical topological model*.

Let $\mathcal{C}(S4)$ be the set of theories in $S4$. For any $\phi$, let

$$\widehat{\phi} \quad = \quad \{T \in \mathcal{C} : \phi \in T\}.$$

The family of sets $\{\widehat{I\phi} : \phi \in \mathcal{L}_0\}$ is the basis of a topology; this is tantamount to the fact about intersections and interiors that we noted above. In this topology, the basic open sets are those of the form $\{T : I\phi \in T\}$, for some sentence $\phi$. We use the following *canonical interpretation*

$$i(p) \quad = \quad \widehat{p} \tag{1.1}$$

for $p \in At$. This gives a topological model which we also call $\mathcal{C}(S4)$.

The main fact about $\mathcal{C}(S4)$ is the following Truth Lemma:

$$i(\phi) \quad = \quad \widehat{\phi}.$$

The proof is by induction on $\phi$. The base case for atomic sentences is by definition, and the steps for the propositional connectives use basic facts about theories. The main work is in the induction step for $I\phi$. Suppose first that $T \in \widehat{I\phi}$, so that $I\phi \in T$. Then this set $\widehat{I\phi}$ is a basic open set containing $T$. By Scheme 2 of the logic, $\widehat{I\phi} \subseteq \widehat{\phi}$. By induction hypothesis, $\widehat{\phi} = i(\phi)$. In this way, $T \in i(\phi)^{\mathrm{o}} = i(I\phi)$.

Going the other way, suppose that $T \in i(I\phi)$. Then there is some basic open set around $T$, say $\widehat{I\psi}$, included in $i(\phi)$. By induction hypothesis, $\widehat{I\psi} \subseteq \widehat{\phi}$. We claim now that $I\psi \to \phi$ is provable in $S4$. (If not, then $I\psi \wedge \neg\phi$ is consistent. So there is some theory $U \in i(I\psi)$ but not in $i(\phi)$. This contradicts $\widehat{I\psi} \subseteq \widehat{\phi}$.) Using this claim and the necessitation rule, we can prove $I(I\psi \to \phi)$. Using Scheme 1, we get $II\psi \to I\phi$. By Scheme 3, we then get $I\psi \to I\phi$. So $I\phi$ belongs to our original $T$. Hence $T \in \widehat{I\phi}$, as desired.

The final assertion of our theorem, the decidability, is a standard *effective finite model* result. The idea is that given a sentence $\phi$, one can compute a number $n = n(\phi)$ such that if $\phi$ is satisfiable on any model, then it is satisfied on a model of size at most $n$. We get such a number by estimating the size of a certain quotient of $\mathcal{C}(S4)$, a quotient which of course depends on $\phi$. We omit the details here, but see Section 8 for another decidability result. QED

There are also stronger forms of Theorem 1.2. For example, one might well wonder whether $S4$ is complete for the topological operations on specific natural spaces, such as the reals or the interval $[0, 1]$. McKinsey and Tarski showed that $S4$ is complete for every dense-in-itself separable metric space (see McKinsey, 1944). This implies the completeness of $S4$ for all of the spaces mentioned above. In recent years, there has been a series of papers simplifying the completeness arguments for these special cases; see, for example, Aiello et al., 2003.

## 3.1 The preorder semantics of the same system $S4$

We have already seen the topological interpretation of the logic $S4$. In what follows, we need not only this semantics but the more standard *relational*, or *Kripke* semantics of this and other logical systems. This section reviews this topic.

The syntax of modal logic is similar to what we have already seen. It begins with some set $At$ of *atomic sentences (or propositions)* and then considers the closure of this set under the boolean operations of $\wedge$ and $\neg$ (and others, say by abbreviation), and some modal operators. We shall continue to use $I$ as the one operator for now. (It is more standard to use symbols like $\Box$ and $K$ for the operators, with duals $\Diamond$ and $L$. We shall see these in later parts of our chapter.)

The semantics begins with a *frame*, a set whose elements are called *worlds* or *points* together with a binary relation on it. This relation is sometimes called *accessibility*, and symbols like $\to$, $\leq$, or $R$ have been used for it. To interpret $S4$, read $I$ not as interior but rather as "all points which the current point relates to". Then reading the schemes of $S4$ this way suggests that the accessibility relation be a *preorder* (transitive and reflexive): for example, $I\phi \to \phi$ says that if something is true of all points which the current point relates to, then it is true at the current point itself. To get a sound interpretation of $S4$, we should require that the current point is related to itself. We therefore define a *preorder model* to be a triple $\mathcal{X} = \langle X, \leq, i\rangle$, where $\langle X, \leq\rangle$ is a preorder and again $i : At \to \mathcal{P}(X)$.

The semantics is the same as for topological models, except that now the clause for $I$ becomes

$$i(I\phi) \qquad = \qquad \{x : \uparrow x \subseteq i(\phi)\}$$

Here $\uparrow x$ stands for $\{y : x \leq y\}$. The semantics of $C$ is given by duality, so that $C\phi \leftrightarrow \neg I\neg\phi$ is valid by definition. We again define *validity* and *satisfiability in preorder models* just as with topological models, mutatis mutandis.

THEOREM 1.3 *The interpretation of $\mathcal{L}_0$ in preorders is sound and complete in the sense that the following are equivalent:*

*1 $\phi$ is valid in preorder models.*

*2 $\phi$ is provable in $S4$.*

One can prove this result in the same manner as Theorem 1.2. We use the set $\mathcal{C}(S4)$ of theories in $S4$ and define a preorder on them by

$$T \leq U \text{ iff } \phi \in U \text{ whenever } I\phi \text{ in } T. \tag{1.2}$$

One also uses the canonical interpretation from (1.1). The rest of the completeness argument is standard, and any textbook on modal logic would contain it. Instead of giving the details, we present an alternative approach based on the connection between certain topological spaces and preorders.

Let $\mathcal{X} = \langle X, \leq\rangle$ be a preorder. Consider the *Alexandrov topology* on $\mathcal{X}$: the opens are the sets closed upwards in the order. This gives a topology which we call $\mathcal{O}_{\leq}$. This associates topological spaces to preorders. (Actually, the reflexivity is not used in verifying that we have a topology, a fact which will come into play in Section 11.2.) It also associates topological models to preorder models, just by copying the interpretation $i$. (Actually, this gives a very special kind of space: the opens are closed under arbitrary intersections.)

PROPOSITION 1.4 *For all preorder models $\langle X, \leq, i\rangle$, all $x \in X$, and all $\phi \in \mathcal{L}_0$,*

$$x \models \phi \text{ in } \langle X, \leq, i\rangle \quad \text{iff} \quad x \models \phi \text{ in } \langle X, \mathcal{O}_{\leq}, i\rangle$$

*Proof* By induction on $\phi \in \mathcal{L}_0$. The case of the atomic sentences is trivial, as are the induction steps for the boolean connectives.

Assume the lemma for $\phi$. So $i(\phi)$ is the same for both interpretations. Consider $I\phi$. First, assume that $x \in i(I\phi)$ in the preorder sense. That is, every $y \geq x$ is in $i(\phi)$. Now $\uparrow x$ is an open set, and by reflexivity it contains $x$. As we have just seen, it is included in $i(\phi)$. So $x \in i(I\phi)$ in the topological sense. Conversely, if $x \in I(\phi)$ topologically, then there is some $y$ such that $x \in \uparrow y$ and $\uparrow y \subseteq i(\phi)$. But then $y \leq x$. By transitivity, $\uparrow x \subseteq \uparrow y$. So $\uparrow x \subseteq i(\phi)$. Thus $x \in i(I\phi)$ in the preorder sense. QED

Using this easy result, Theorem 1.2 follows from Theorem 1.3. That is, every $S4$ theory is satisfied on some preorder, hence on some topological space. But with a little more work, Theorem 1.3 can be made to follow from Theorem 1.2 or rather from the related fact that sentences which are satisfiable in $S4$ have *finite* topological models. Thus they are *Alexandrov spaces*: every point has a minimal open set around it. We turn a topological model which is an Alexandrov space into a preorder model by: $x \leq y$ iff $y$ belongs to every open set around $x$: the Alexandrov property implies the transitivity. Then one notes a result just like Proposition 1.4, except now we relate the topological semantics on an Alexandrov space to the preorder semantics derived from it. The upshot is that a given topologically satisfiable sentence $\phi$ now has a finite preorder model. This implies the completeness of $S4$ in the relational semantics.

## 3.2 Adding the difference modality

One of the purposes of this chapter is to mention other work that builds on classical topics. We mention here recent work (Gabelaia, 2001; Kudinov, 2006) which adds a modal operator to the language we have been discussing. Kudinov's paper is the source for all results in this subsection and contains other material as well. Add to $\mathcal{L}_0$ the operator $[\neq]$ to get a larger language $\mathcal{L}_1$. For the semantics, we stipulate that

$$x \models [\neq]\phi \quad \text{iff} \quad \text{for all } y \text{ different from } x \text{, we have } y \models \phi.$$

So we can read $[\neq]\phi$ as "$\phi$ holds everywhere except possibly here." It is also useful to adopt an abbreviation $K\phi$ for $\phi \wedge [\neq]\phi$, (Kudinov uses $[\forall]$, but we use $K$ later for just this purpose). As usual we include for convenience a dual modality $\langle\neq\rangle$ so that $\langle\neq\rangle\phi$ abbreviates $\neg[\neq]\neg\phi$. Then as axioms, one takes $S4$ for $I$ (just as we have seen), together with the following schemes:

$$
\begin{array}{l}
[\neq](\phi \to \psi) \to ([\neq]\phi \to [\neq]\psi) \\
\phi \to [\neq]\langle\neq\rangle\phi \\
(\phi \wedge [\neq]\phi) \to [\neq][\neq]\phi \\
K\phi \to I\phi
\end{array}
$$

For rules we take the necessitation rules for both $I$ and also $[\neq]$. This axiom system is called $S4D$. It is easy to see that $S4D$ is sound for all topological interpretations.

Now in this logic we can express some interesting topological properties. The examples we have in mind are: density-in-itself, and $T_1$. The first condition means that there are no isolated points: $\{x\}$ is not open. The second means that for every $x \neq y$, there is an open set containing $x$ but not $y$.) These are examples of *correspondence phenomena*, and the formal statements involve quantification over interpretations of the atomic sentences in the model, in the following way.

Proposition 1.5 *Let $\mathcal{X} = \langle X, \mathcal{O}\rangle$ be a topological space.*

1 *The following are equivalent:*

  (a) *$\mathcal{X}$ is dense in itself.*

  (b) *For all sentences $\phi$, all interpretations $i$ into $\mathcal{X}$, and all $x \in X$, we have $x \models [\neq]\phi \to C\phi$.*

2 *The following are equivalent:*

  (a) *$\mathcal{X}$ is a $T_1$ space.*

  (b) *For all sentences $\phi$, all interpretations $i$ into $\mathcal{X}$, and all $x \in X$, we have $x \models [\neq]\phi \to [\neq]I\phi$.*

Correspondence results for the $T_0$ and $T_1$ properties were first obtained in Gabelaia, 2001. The main results about the logics in Proposition 1.5 are the following completeness theorems. For example, the second statement below means that for a sentence $\phi \in \mathcal{L}_1$, $\phi$ is provable in the system axiomatized by $S4D$ with the extra scheme $[\neq]\phi \to C\phi$ iff $\phi$ holds in every space $\mathcal{X}$ which is dense in itself, at all points, under all valuations.

THEOREM 1.6 (KUDINOV, 2006)

1 *S4D is complete for topological spaces.*

2 *S4D + "$[\neq]\phi \to C\phi$" is complete for spaces which are dense in themselves.*

3 *S4D + "$[\neq]\phi \to C\phi$" + "$[\neq]\phi \to [\neq]I\phi$" is complete for spaces which are dense in themselves and also $T_1$.*

## 4. Topologic

At this point we have seen the topological interpretation of modal logic. This topic has been pursued in many directions over the years; see Chapter **??** for a survey. It is not exactly the thrust of this chapter, however. Instead, we strike out on a different direction by considering a *bimodal* language interpreted on a larger class of models. This language was first considered in Moss and Parikh, 1992.

A *subset frame* is a pair $\mathcal{X} = \langle X, \mathcal{O} \rangle$ where $X$ is a set of *points* and $\mathcal{O}$ is a set of subsets of $X$. The elements of $\mathcal{O}$ are called *opens.* We assume that $X \in \mathcal{O}$, though this is really not necessary. $\mathcal{X}$ is an *intersection frame* if whenever $u, v \in \mathcal{O}$ and $u \cap v \neq \emptyset$, then also $u \cap v \in \mathcal{O}$. $\mathcal{X}$ is a *lattice frame* if it is an intersection frame closed under finite unions, and a *complete lattice frame* if it is closed under infinitary intersections and unions.

Note that we use the term "open" for simplicity even though it is not required that $\mathcal{O}$ be a topology. It is just that the topological case is our paradigm case, and the basis of our intuitions.

We now set up a formal language $\mathcal{L}$ which is expressive enough for simple arguments concerning subset spaces. Later we shall expand this language. The formulas of $\mathcal{L}$ are obtained from atomic propositions by closing under $\wedge$, $\neg$, $K$ and $\Box$.

A *subset space* is a triple $\mathcal{X} = \langle X, \mathcal{O}, i \rangle$, where $\langle x, \mathcal{O} \rangle$ is a subset frame, and $i : At \to \mathcal{P}(X)$. (We do not require that each $i(p)$ be an open.) If $\langle x, \mathcal{O} \rangle$ is an intersection frame, then $\mathcal{X}$ is called an *intersection space*, and similarly for lattice spaces, etc. (Often we simply speak of *models.*) For $p \in X$ and $p \in u \in \mathcal{O}$, we define the *satisfaction relation* $\models_{\mathcal{X}}$ on $(X \times \mathcal{O}) \times \mathcal{L}$ by recursion on $\phi$.

| | | |
|---|---|---|
| $p, u \models_{\mathcal{X}} A$ | iff | $p \in i(A)$ |
| $p, u \models_{\mathcal{X}} \phi \wedge \psi$ | iff | $p, u \models_{\mathcal{X}} \phi$ and $p, u \models_{\mathcal{X}} \psi$ |
| $p, u \models_{\mathcal{X}} \neg\phi$ | iff | $p, u \not\models_{\mathcal{X}} \phi$ |
| $p, u \models_{\mathcal{X}} K\phi$ | iff | $q, u \models_{\mathcal{X}} \phi$ for all $q \in u$ |
| $p, u \models_{\mathcal{X}} \Box\phi$ | iff | $p, v \models_{\mathcal{X}} \phi$ for all $v \in \mathcal{O}$ such that $p \in v \subseteq u$ |

We use $L$ as the dual of $K$ and $\Diamond$ as the dual of $\Box$. Explicitly, we have

$$\begin{array}{lll} p, u \models_{\mathcal{X}} L\phi & \text{iff} & q, u \models_{\mathcal{X}} \phi \text{ for some } q \in u \\ p, u \models_{\mathcal{X}} \Diamond\phi & \text{iff} & p, v \models_{\mathcal{X}} \phi \text{ for some } v \in \mathcal{O} \text{ such that } p \in v \subseteq u \end{array}$$

We stress that we only use the notation $p, v \models \phi$ when $p$ belongs to $v$.

As usual, we write $p, u \models \phi$ if $\mathcal{X}$ is clear from context. If $T \subseteq \mathcal{L}$, we write $T \models \phi$ if for all models $\mathcal{X}$, all $p \in X$, and all $u \in \mathcal{O}$, if $p, u \models_{\mathcal{X}} \psi$ for each $\psi \in T$, then also $p, u \models_{\mathcal{X}} \phi$. Finally, we also write, e.g., $T \models_{Int} \phi$ for the natural restriction of this notion to the class of models which are intersection spaces.

With these definitions in place, we return to a discussion of the concepts and motivation. We are considering a Kripke structure whose worlds are the pairs $(p, u)$ with $p \in u$ and $u \in \mathcal{O}$. Think of $p$ as the "real world" and $u$ as a guess as to where that world lies based on some observation. The language then uses two accessibility relations corresponding to *shrinking* an open ($\Box$) while maintaining a reference point, or to moving a reference point inside the *given open* ($K$). Another way to think of these is in terms of quantification, so we recast the semantics in English a bit: $\Box\phi$ is true at $p, u$ if for all refinements $v$ of the observation $u$, $\phi$ is still true at $p, v$. And $K\phi$ is true at $p, u$ if for all $q \in u$, if the real world happens to be $q$ rather than $p$, $\phi$ is still true at that world $q$ with the same observation $u$.

We see that the intuition behind this logic with its two modalities is that knowledge is affected not only by the *situation* we are in, but also by the amount of *effort* we have put in. For instance, recall our remarks introducing measurement in the Introduction. Suppose a policeman uses radar to determine that a car is going 51 mph in a 50 mile speed limit zone. But if the accuracy of his radar is $\pm 2$ mph, then he does not know that the car is speeding. If however, a more accurate radar with an accuracy of $\pm 1$ mph shows that the car is going 51.5 mph, then he *does know* that the car is speeding. Originally the possible speed of the car lay in the interval (49,53), which is not entirely contained in the interval $(50, \infty)$ which represents speeding. The second interval, however, is (50.5,52.5); this *is* contained in $(50, \infty)$. We can represent the policeman's *later* situation as symbolized by $K(\mathit{Speeding})$, and the earlier one as $\Diamond K(\mathit{Speeding}) \wedge L\Diamond K\neg(\mathit{Speeding})$. In the earlier case the motorist *was* speeding, and the policeman had the possibility of knowing this, but did not actually know it.

## 4.1 Preliminary examples

At this point, we present two simple spaces where we can interpret the language, and some formulas. These should help the reader to become familiar with the semantics.

First, consider the case when $X$ is the set $R$ of real numbers, and $\mathcal{O}$ is the standard topology on $R$. Suppose that there are two atomic predicates $P$ and $I$, and that $i(P) = [0, 2]$, and $i[I] = \{x \in R : x \text{ is irrational}\}$. Then $(1, (0, 3)) \models P$, Also, since $2.5 \in (0, 3)$, $(1, (0, 3)) \models L\neg P$. Moreover, $(1, (0, 3)) \models \Diamond KP$, since we can shrink $(0, 3)$ around 1 to $(.5, 1.1)$, say, and have the new neighborhood entirely inside the interpretation of $P$. On the other hand, $(2, (0, 3)) \not\models \Diamond KP$. The reason is that every open $u$ around 2 contains a point larger than 2, and so $(2, u) \models \neg KP$. We also write this last fact as $(2, (0, 3)) \models \Box L\neg P$. For a final example, $(0, R) \models K\Box LI$, since every open set containing any real is non-empty and so contains an irrational.

The next example itself will re-appear (in a slightly elaborated manner) in Section 6 as Example A. A picture of it may be found in Figure 1.3 below.

Let $X = \{a, p, q, z_1, z_2\}$, and let $\mathcal{O}$ contain $X$ and

$$\begin{array}{ll} u_1 = \{a, p, z_1\} & u_2 = \{a, q, z_2\} \\ v_1 = \{a, z_1\} & v_2 = \{a, z_2\}. \end{array}$$

Let $P$ and $Q$ be atomic sentences; we form a subset space $\langle X, \mathcal{O}, i\rangle$ via

$$i(P) = \{p\} \qquad i(Q) = \{q\}$$

Note that $a, v_1 \models K\neg P$. the reason for this is that neither $a$ nor $z_1$ satisfy $P$. Since $v_1 \subseteq u_1$, we see that $a, u_1 \models \Diamond K\neg P$. So we have

$$a, u_1 \models (LP \wedge \neg LQ) \wedge \Diamond K\neg P.$$

Similarly, we have

$$a, u_2 \models (LQ \wedge \neg LP) \wedge \Diamond K\neg Q$$

We conclude that

$$a, X \models \Diamond((LP \wedge \neg LQ) \wedge \Diamond K\neg P) \ \wedge \ \Diamond((LQ \wedge \neg LP) \wedge \Diamond K\neg Q) \quad (1.3)$$

## 4.2 Further definitions

Certain kinds of sentences will have special interest in our study. Given a model $\mathcal{X}$, and a sentence $\phi$, $\phi$ is *persistent in* $\mathcal{X}$ if for all $p, u, v$

so that $p \in v \subseteq u$, we have that if $p, u \models_{\mathcal{X}} \phi$ then $p, v \models_{\mathcal{X}} \phi$. $\phi$ is *persistent* if it is persistent in all $\mathcal{X}$. $\phi$ is *bi-persistent* if for all $\mathcal{X}$ and all $p, u, v$ so that $p \in v \subseteq u$, we have $p, u \models_{\mathcal{X}} \phi$ iff $p, v \models_{\mathcal{X}} \phi$. A sentence $\phi$ is *reliable* in a model $\mathcal{X}$ if $K\phi \to K\Box\phi$ is valid in $\mathcal{X}$. $\phi$ is *reliable* if it is reliable in every $\mathcal{X}$. In other words, once $\phi$ is known, we need not worry about its becoming false. A sentence of the form $K\Box\phi$ is itself always reliable. Reliable sentences represent reliable knowledge and have a rather intuitionistic flavor. However, our logic is classical, since we are trying to represent certain knowledge theoretic ideas in a classical setting, rather than use an intuitionistic setting where such ideas would be *presupposed*. If the topology is discrete, then the only reliable sentences will be persistent. In comparison, with the trivial topology, only *tautologies* will tend to be reliable in $\mathcal{X}$. Thus, for example, assuming that all boolean combinations of $i(A)$ and $i(B)$ are non-empty, then the only sentences involving $A$ and $B$ which are reliable will be tautologies. Note that when $v$ is a subset of $u$, then every reliable sentence known at $p, u$ is also known at $p, v$. This is in accord with the intuition that refining from $u$ to $v$ increases knowledge.

Our language allows us to express certain basic topological notions. If $\mathcal{X}$ is indeed a topology, then a set $i(A)$ will be open iff every point in $i(A)$ has an open neighborhood contained entirely in $i(A)$ iff at any $p$ in $i(A)$, the sentence $\Diamond KA$ holds. Thus $i(A)$ is *open* iff the sentence $A \to \Diamond KA$ is valid in the model. Dually, $i(A)$ is *closed* iff the sentence $\Box LA \to A$ is valid in the model. It is not hard to see that with the obvious definitions, r.e. subsets of the natural numbers will satisfy the same knowledge theoretic sentences that opens do in a topological setting, and this, we believe, is the source of our intuition that there is similarity between open sets and r.e. sets. The set $i(A)$ is *dense* iff the sentence $\Box LA$ is valid and it is *nowhere dense* if $\Diamond L\neg A$ is valid.

## 4.3 Topologic and the Tarski-McKinsey semantics

We now relate the original topological interpretation of the modal logic $\mathcal{L}_0$ from Section 3 to the bimodal logic $\mathcal{L}$ that we have been discussing. Define a map

$$^* : \mathcal{L}_0 \to \mathcal{L}$$

by recursion:

$$\begin{array}{lcl} A^* & = & A \\ (\phi \wedge \psi)^* & = & \phi^* \wedge \psi^* \\ (\neg\phi)^* & = & \neg(\phi^*) \\ (I\phi)^* & = & \Diamond K\phi^* \end{array}$$

Recall that we interpret $\mathcal{L}$ on all subset models, hence on all topological models. It is on this class that it makes sense to compare the two languages.

PROPOSITION 1.7 *For all $\phi$ in $\mathcal{L}_0$, all topological models $\mathcal{X}$, and all points $x \in X$,*

$$x \models \phi \quad \textit{iff} \quad x, X \models \phi^* \ .$$

The proof uses the fact that $\phi^*$ is always bi-persistent.

The point here is that the original language $\mathcal{L}_0$ corresponds to a *fragment* of the bimodal language $\mathcal{L}$. It should not be surprising that the larger language is strictly more expressive.

PROPOSITION 1.8 *The sentence $Lp$ is not equivalent on the class of topological spaces to $\phi^*$ for any sentence $\phi$ of $\mathcal{L}_0$. That is, there is no $\mathcal{L}_0$-sentence $\phi$ such that for all topological spaces $\mathcal{X}$, all $x \in X$, and all interpretations $i$ of atomic sentences in $X$,*

$$x \models \phi \quad \textit{iff} \quad x, X \models Lp.$$

*Proof* Here is a sketch. Consider two models, both with the same universe $R$ of reals. In $M_1$, $i(p) = \emptyset$, and in $M_2$, $i(p) = \{1\}$. ($M_1$ was presented in Example 1.1.) An induction on $\psi \in \mathcal{L}_0$ shows that the interpretation of $\psi$ in the two models is the same except possibly for the point 1. In particular, $0 \models_1 \psi$ iff $0 \models_2 \psi$. Now suppose that $\phi$ exists as in our proposition. Note that $0 \models_1 \neg Lp$ and $0 \models_2 Lp$. It follows that $0 \models_1 \neg\phi$ and $0 \models_2 \phi$; this is a contradiction. QED

## 5. A logical system: the subset space axioms

One main technical goals of this chapter is to present axiomatizations of the validities of several classes of subset space models: all spaces, intersection spaces, lattice spaces, and complete lattice spaces. The *logic of subset spaces* is described by axioms and rules of inference in Figure 1.1.

These axioms and rules say that $K$ is $S5$-like, and $\Box$ is $S4$-like. We have an axiom of *atomic permanence*: $(A \to \Box A) \wedge (\neg A \to \Box\neg A)$. This is only sound for atomic $A$. The intuition is that since an atomic sentence $A$ is true at a point irrespective of which open we are considering, shrinking the open does not alter the truth value of $A$.

Perhaps the characteristic axiom of the system is $K\Box\phi \to \Box K\phi$, called the *Cross Axiom*. It is the one axiom relating the two modalities.

| |
|---|
| Substitution instances of classical tautologies |
| $K(\phi \to \psi) \to (K\phi \to K\psi)$     $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$ |
| $K\phi \to (\phi \wedge KK\phi)$     $\Box\phi \to (\phi \wedge \Box\Box\phi)$ |
| $L\phi \to KL\phi$ |
| $(A \to \Box A) \wedge (\neg A \to \Box\neg A)$ for atomic $A$ |
| $K\Box\phi \to \Box K\phi$ (the Cross Axiom) |
| $\dfrac{\phi \to \psi,\ \phi}{\psi}$    $\dfrac{\phi}{K\phi}$    $\dfrac{\phi}{\Box\phi}$ |

*Figure 1.1.* The axioms and rules of the logic of subset spaces.

it is often used in its dual form $\Diamond L\phi \to L\Diamond\phi$. Let us check its soundness in this form. Fix a subset model $\mathcal{X}$, and assume that $p \in u \in \mathcal{O}$ has the property that $p, u \models \Diamond L\phi$. We must show that $p, u \models L\Diamond\phi$. Our assumption first gives an open $v \in \mathcal{O}$ such that $p \in v \subseteq u$ and $p, v \models L\phi$. From this, there is some $q \in v$ such that $q, v \models \phi$. Now $q \in u$ as well, and we use this point to show that indeed $p, u \models L\Diamond\phi$. That is, we claim that $q, u \models \Diamond\phi$. The reason again is that $q \in v \subseteq u$, and we have seen above that $q, v \models \phi$.

The axioms and rules of inference are sound for subset spaces. The argument is routine, and perhaps the only interesting part concerns the Cross Axiom; we saw it just above. We defer discussion of completeness to Section 7 and decidability to Section 8.

## 5.1 Axioms for spaces with closure properties: the topologic axioms

We next consider what happens in set spaces with various closure properties. We list the axioms of interest in Figure 1.2.

A *directed space* is one where for every $p, u, v$ with $p \in u$ and $p \in v$ there is a $w \in \mathcal{O}$ such that $p \in w$, $w \subseteq (u \cap v)$ An *intersection space* is one where we can take $w = u \cap v$.

We check that the axiom the Weak-Directedness Axiom (WD) from Figure 1.2. is sound for directed spaces. Suppose that $x, u \models \Diamond\Box\phi$. Let $v \subseteq u$ be such that $x, v \models \Box\phi$. To see that $x, u \models \Box\Diamond\phi$, let $u' \subseteq u$. Let $w \in \mathcal{O}$ be such that $p \in w \subseteq u' \cap v$. Then since $w \subseteq v$ we have $x, w \models \phi$. Hence $x, u' \models \Diamond\phi$. Since $u'$ is arbitrary, $x, u$ indeed satisfies $\Box\Diamond\phi$.

| | formal statement + comments |
|---|---|
| WD | $\Diamond\Box\phi \to \Box\Diamond\phi$ |
| | sound for weakly directed spaces |
| Un | $\Diamond\phi \wedge L\Diamond\psi \;\to\; \Diamond[\Diamond\phi \wedge L\Diamond\psi \wedge K\Diamond L(\phi \vee \psi)]$ |
| | sound for spaces closed under binary unions |
| topologic | This is the set space axioms + (WD) + (Un). |
| | sound for lattice spaces, complete for |
| | topological spaces – even for complete lattice spaces |
| Weak Un | $L\Diamond\phi \wedge L\Diamond\psi \;\to\; L\Diamond[L\Diamond\phi \wedge L\Diamond\psi \wedge K\Diamond L(\phi \vee \psi)]$ |
| | weaker than (Un) |
| CI | $\Box\Diamond\phi \to \Diamond\Box\phi$ |
| | sound for set spaces closed under all intersections |
| | follows from topologic axioms |
| $\mathrm{M}_n$ | $(\Box L\Diamond\phi \wedge \Diamond K\psi_1 \wedge \cdots \wedge \Diamond K\psi_n)$ |
| | $\to L(\Diamond\phi \wedge \Diamond K\psi_1 \wedge \cdots \wedge \Diamond K\psi_n)$ |
| | (WD) + all ($\mathrm{M}_n$) is complete for directed spaces |

*Figure 1.2.* Axiom schemes for set spaces with additional closure properties

As it happens, (WD) does not lead to a complete axiomatization of the valid sentences on intersection spaces. We discuss the incompleteness further in Section 6.

We next consider the Union Axioms (Un). To check soundness on spaces closed under unions, suppose that $x, u^*$ satisfies the antecedent via $u$, $y$, and $v$ such that $u \subseteq u^*$, $y \in u$, $y \in v \subseteq u^*$, $x, u \models \phi$, and $y, v \models \psi$. Let $w = u \cup v$. Then $w \subseteq u^*$. Clearly $x, w \models \Diamond\phi \wedge L\Diamond\psi$. Since each point of $w$ is either in $u$ or $v$, every point in $w$ has a neighborhood in which either $\Diamond\phi$ or $\Diamond\psi$ is satisfied. The reader might like to consider the scheme (Weak Un) and to see why it actually is weaker than (Un).

The system whose axioms are the basic axioms together with the Weak-Directedness axiom and the Union Axiom will be called topologic. The idea is that topologic should be strong enough to support elementary topological reasoning.

Theorem 1.9 (Georgatos, 1993; Georgatos, 1994a)
*The* topologic *axioms are complete for topological spaces, indeed for complete lattice spaces. Moreover, any sentence satisfiable in any topological space is satisfied in a finite topological space.*

One way to obtain the completeness result goes by considering a canonical model, building on what we have seen in Section 3 on the more standard modal logic of topology. (See Dabrowski et al., 1996.) This proof also shows that the topologic axioms give a complete axiomatization of the validities on the smaller class of complete lattice spaces. Indeed, the canonical model of topologic turns out to be a complete lattice. Further, it turns out that the (Un) cannot be replaced by (Weak Un) in this axiomatization.

This work also has some model-theoretic corollaries which might be of interest. Here are two of them, again from Dabrowski et al., 1996. In the first, we recall that $\mathcal{L}_0$ is the modal logic of the interior operation $I$, and the map $^* : \mathcal{L}_0 \to \mathcal{L}$ is the embedding of $\mathcal{L}_0$ into the bimodal logic of interest here (this map was discussed in Section 4.3). We extend the map to sets by $S^* = \{\phi^* : \phi \in S\}$.

THEOREM 1.10 *For all $S$ and $\phi$ in $\mathcal{L}_0$, the following are equivalent:*

1 $S \vdash \phi$ *in* $S4$.

2 $S^* \vdash \phi^*$ *in* topologic.

In the second result, let $\Pi$ be the smallest set of sentences containing the atomic sentences, and closed under boolean operations and the operator $\Diamond K$. This is also the image of the $^*$-translation mentioned above. It is easy to check that each $\pi \in \Pi$ is bi-persistent on all spaces closed under intersections; this can be proven formally using (WD).

THEOREM 1.11 *If $\phi$ is bi-persistent on the class of finite lattice spaces, then there is some $\pi \in \Pi$ so that $\vdash \psi \leftrightarrow \pi$ in* topologic.

Georgatos also showed that topologic has the finite model property and is therefore decidable. The logic of subset spaces and the logic of intersection spaces do not have the finite model property. However, in Section 8 we adapt filtration on certain non-standard models to show that the logic of subset spaces is decidable.

Continuing, we consider the axioms (CI). These are the converse of the (WD) axioms. We claim first that (CI) is sound for complete intersection spaces. To see this, let $x$ be a point in some space $\mathcal{X}$, and assume that $x, v \models \Box\Diamond\phi$ Let $u_x$ be the intersection of all opens containing $x$. Since $u_x$ has no open proper subsets $x, u_x \models (\Box\phi \vee \Box\neg\phi)$. Hence $x, u_x \models \Box\phi$. Since $u_x \subseteq v$, we have $x, v \models \Diamond\Box\phi$. On the other hand, (CI) is not sound for intersection spaces; this can be shown semantically using Example B of Section 6 below. Intuitively, closure under finite intersection alone is not enough to guarantee that the neighborhoods of a point eventually stabilize on $\phi$ or on $\neg\phi$. The interesting point is that this axiom *is* sound for lattice spaces. So by completeness, each instance is provable in topologic. We know of no purely syntactic argument for this.

We end with a mention of the results on intersection spaces and directed spaces. As we noted before, the (WD) axioms are sound for intersection spaces, even for directed spaces. Since these classes are of topological interest, the reader of this handbook article might find them of interest. It turns out that (WD) is not complete for either class. As Weiss has shown in an unpublished note, the two classes of spaces have the same logic, and a characterization of either class by a set of axioms also leads to a characterization of the other. That turns out to require infinitely many axiom schemes which cannot be reduced to a finite set (see Weiss and Parikh, 2002). This can be achieved using the ($\mathrm{M}_n$) schemes; we omit the proof of their soundness for directed spaces and merely state the relevant result.

THEOREM 1.12 (WEISS AND PARIKH, 2002) *The set space axioms plus the axiom (WD) and the axioms ($M_n$) are complete for directed spaces.*

The topics described in this section do not exhaust what is known in the field. For example, Georgatos, 1997 studies tree-like spaces. (Given two open sets in a tree-like topological space, they are either disjoint, or one is a subset of the other.) See also Section 9 for other results which use languages that are variants of the one presented in Section 4 and studied in this section.

## 6. Further examples

In this section, we present two further examples of set spaces and sentences. (The first of these was essentially introduced in Section 4.1.) These examples are somewhat pathological, and we present them to justify some of the claims in the last section, and also to motivate some of the work in Sections 7 and 8.

**Example A.** Let $X = \{a, p, q, z_1, z_2\}$, and let $\mathcal{O}$ contain $X$ and

$$\begin{array}{ll} u_1 = \{a, p, z_1\} & u_2 = \{a, q, z_2\} \\ v_1 = \{a, z_1\} & v_2 = \{a, z_2\}. \end{array}$$

Let $P$, $Q$, and $Z$ be atomic sentences. (Our work in Section 4.1 did not need $Z$.) We form a subset space $\langle X, \mathcal{O}, i\rangle$ via

$$i(P) = \{p\} \qquad i(Q) = \{q\} \qquad i(Z) = \{z_1, z_2\}.$$

(See Figure 1.3.)

*Figure 1.3.* Example A.

Our first observation is that the Weak Directedness axioms are validated in this model (despite the fact that the model is not actually closed under intersections). This is an instance of a very general fact. In a subset frame with only finitely many opens, every open $u$ about a point $p$ can be shrunk to a minimal open $v$ about $p$. When $v$ is minimal $(p, v)$ automatically satisfies all sentences of the form $\phi \to \Box\phi$. In this way, all finite spaces satisfy the Weak Directedness axioms.

Next, we see that the Weak Union axioms also hold (and again, the model is not actually closed under unions). Since there are a number of cases, we only present a few of them. Suppose that $p, u_1 \models \phi$ and $q, u_2 \models \psi$. Then for any $x \in X$ we have $x, X \models L\Diamond\chi$, where

$$\chi \quad \equiv \quad L\Diamond\phi \ \wedge \ L\Diamond\psi \ \wedge \ K\Diamond L(\phi \vee \psi) \ .$$

The most interesting case is where, e.g., $p, u_1 \models \phi$ and $z_2, v_2 \models \psi$. Here, $z_1, v_1$ satisfies $\psi$ as well, since $v_1$ and $v_2$ are isomorphic. Thus for any $y \in u_1$ we have $y, u_1 \models \chi$.

However, the stronger Union Axiom fails; for example

$$\begin{array}{rl} & z_2, X \models \Diamond K(\neg Q) \ \wedge \ L\Diamond(LP \wedge K\neg Q) \\ \text{but} & z_2, X \not\models \Diamond[LP \wedge K\neg Q] \end{array}$$

Note that, prefixed by an $L$, this sentence would be satisfied. This was the key idea for showing that the Weak Union Axiom is satisfied.

To summarize: this space satisfies the Weak Union Axiom but not the Union Axiom. Hence the former scheme is properly stronger. There is a second reason for introducing this example, having to do with the theories realizable in various spaces.

This example can also be used to show that the subset space logic and Weak Directedness Axioms are incomplete for the class of intersection spaces.

Concerning unions, we have seen in Section 4.1 that

$$a, X \models \Diamond((LP \wedge \neg LQ) \wedge \Diamond K\neg P) \ \wedge \ \Diamond((LQ \wedge \neg LP) \wedge \Diamond K\neg Q) \quad (1.4)$$

The witnesses are $u_1$ and $u_2$. Also

$$a, X \models K(Z \ \to \ \neg(\Diamond(LP \wedge \neg LQ) \ \wedge \ \Diamond(LQ \wedge \neg LP))) \quad (1.5)$$

Let $\psi$ be the conjunction of the sentences in (1.4) and (1.5). We claim that $\psi$ has no models which are closed under unions. For suppose $\mathcal{X}'$

*Figure 1.4.* Example B.

were such a model. Let $u_1'$ and $v_1'$ be witnesses in $\mathcal{X}'$ to (1.4); let $u_2'$ and $v_2'$ be subsets witnessing $a, u_1' \models \Diamond K\neg P$ and $a, v_1' \models \Diamond K\neg Q$ respectively. Let $w' = u_1' \cup v_2'$ (we could also use $u_2' \cup v_1'$). Let $z' \in v_2'$ be a $Z$-point. Then $z', w' \models LP \wedge \neg LQ$ while $z', u_2' \models LQ \wedge \neg LP$. So

$$z', X \models Z \ \wedge \ \Diamond(LP \wedge \neg LQ) \ \wedge \ \Diamond(LQ \wedge \neg LP)$$

This contradicts (1.5).

This fact that the model satisfies the Weak Directedness Axioms and Weak Union Axioms shows that the basic axioms of topologic together with all of these axioms cannot refute $\psi \wedge \chi$. Nevertheless $\psi \wedge \chi$ cannot hold in any lattice space (even in any directed space). So the axioms are incomplete. (Nevertheless, we do have completeness for lattice spaces using the stronger Union Axioms.)

Next, we present an example which shows a number of things about the theories realized in spaces closed under finite intersections.

**Example B.** The space $\mathcal{X}$ has points

$$a_0, a_1, a_2, \ldots, a_n, \ldots, \qquad b_0, b_1, b_2, \ldots, b_n, \ldots, \qquad \text{and } c\ .$$

There are several families of opens

$$\begin{array}{rcl} u_n & = & \{c\} \ \cup \ \{a_m : m \geq n\} \ \cup \ \{b_m : m \geq n\} \\ v_n & = & \{c\} \ \cup \ \{a_m : m > n\} \ \cup \ \{b_m : m \geq n\} \\ w_n & = & \{a_n, b_n\} \\ w_n' & = & \{b_n\} \end{array}$$

(See Figure 1.4.) We interpret three predicates $A$, $B$, and $C$ in the obvious way.

Let

$$\textsf{last-and-}B \quad \equiv \quad B \wedge \Box(KB \vee LC).$$

Informally, last-and-$B$ should hold only at a pair $(d, s)$ only when $d$ is a $b$-point, and only when $d$ is the last element of $s$ (the element with lowest subscript). Note that $b_i, v_i \models$ last-and-$B$, since every subset of $v_i$ which contains $b_i$ is either the singleton $\{b_i\}$ or contains $c$. Further, $b_i, u_i \models \neg$last-and-$B$, since $w_i \subseteq v_j$, and $b_i, w_i \models \neg(KB \vee LC)$. Finally, if $i > j$, then $b_i, v_j \models \neg$last-and-$B$ for the same reason. So in this space last-and-$B$ has the meaning that we have described.

Another fact about this example is that if $i > j$ and $i' > j'$, then $th(a_i, u_j) = th(a_{i'}, u_{j'})$, and $th(a_i, v_j) = th(a_{i'}, v_{j'})$. Similar statements hold for the $b$-points, and also $th(c, u_i) = th(c, u_{i'})$, etc. All of these facts are proved by induction. Alternatively, one may use the appropriate version of Fraisse-Ehrenfeucht games for this semantics.

This example also has important consequences for theories in the set space logic. Consider the theories $T = th(c, u_i)$ and $U = th(c, v_i)$. These theories are not equal, since $L(\mathsf{last\text{-}and\text{-}}B)$ is in the latter but not the former. Nevertheless, the two theories have the property that for every $\phi \in T$, $\Diamond\phi \in U$. In Section 7, we will introduce the notation $T \xrightarrow{\Diamond} U$ for this relation. In this example, we also see that $U \xrightarrow{\Diamond} T$. Now our intuition about the $\xrightarrow{\Diamond}$ relation is that $T \xrightarrow{\Diamond} U$ *should* mean that in every space containing $(d, s)$ with theory $T$, it is possible to shrink $s$ to $s' \subseteq s$ with $th(d, s') = U$.

A further desirable result would be that if $T \xrightarrow{\Diamond} U \xrightarrow{\Diamond} T$, then $T = U$. However, the example under consideration shows that this is not in general correct. This accounts for some of the difficulties in the completeness proof for the logic of set spaces.

A related fact is that the set space logic does not have the finite model property. To see this, consider $\phi \equiv L(\mathsf{last\text{-}and\text{-}}B)$, and note that both $T$ and $U$ contain

$$\psi \quad \equiv \quad (\Box\Diamond\phi) \wedge (\Box\Diamond\neg\phi). \tag{1.6}$$

We claim that no sentence of the form (1.6) can have a finite model. For suppose that $\mathcal{X}$ were a finite space containing $x$ and $u$ such that $x, u \models \psi$. We may assume that $u$ is a $\subseteq$-minimal open about $x$ with this property. But the minimality implies that $x, u \models \phi \wedge \neg\phi$, and this is absurd.

## 7. Completeness of the subset space axioms

The following is the main result on the subset space logic.

THEOREM 1.13 *The basic axioms are strongly complete for subset space models. That is, if $T \models \phi$, then $T \vdash \phi$.*

This section is an extended sketch of the proof, presenting many of the details and ideas but certainly leaving out a good deal of the work. The definitions at the beginning will be used in later sections, as will Propositions 1.14 and 1.15.

As in Section 3, we use *theories*; these are the maximal consistent subsets of the language $\mathcal{L}$. Let $th$ be the set of theories in $\mathcal{L}$ using the subset space axioms from Section 5. We continue to use letters like $T$,

$U$, $V$, etc., to denote theories. In order to prove that we have given a complete proof system, we need only show that for every theory $T$, there is a subset space model $\mathcal{X} = \langle X, \mathcal{O}, i\rangle$, a point $x \in X$, and a subset $u \in \mathcal{O}$ such that $p, u \models_{\mathcal{X}} T$.

To get started, we define the relations $\stackrel{L}{\longrightarrow}$ and $\stackrel{\Diamond}{\longrightarrow}$ on theories by:

$$U \stackrel{L}{\longrightarrow} V \text{ iff whenever } \phi \in V,\ L\phi \in U$$
$$U \stackrel{\Diamond}{\longrightarrow} V \text{ iff whenever } \phi \in V,\ \Diamond\phi \in U$$

Of course, the maximal consistency of theories give other characterizations. For example, $U \stackrel{L}{\longrightarrow} V$ if whenever $K\phi \in U$, $\phi \in V$.

Further, define $U \stackrel{L\Diamond}{\longrightarrow} V$ if for all $\phi \in V$, $L\Diamond\phi \in U$. And define relations such as $U \stackrel{\Diamond L}{\longrightarrow} V$ and $U \stackrel{\Box\Diamond}{\longrightarrow} V$ similarly.

Proposition 1.14 *Concerning the relations* $\stackrel{L}{\longrightarrow}$ *and* $\stackrel{\Diamond}{\longrightarrow}$ *:*

*(1)* $\stackrel{L}{\longrightarrow}$ *is an equivalence relation.*

*(2)* $\stackrel{\Diamond}{\longrightarrow}$ *is reflexive and transitive.*

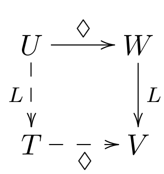
*(3) If* $L\phi \in T$, *then there is some* $U$ *so that* $\phi \in U$ *and* $T \stackrel{L}{\longrightarrow} U$.

*(4) If* $\Diamond\phi \in T$, *then there is some* $U$ *so that* $\phi \in U$ *and* $T \stackrel{\Diamond}{\longrightarrow} U$.

*Proof* These are all standard consequences of the S4-ness of $\Diamond$ and the S5-ness of $L$. QED

These facts will be used in the sequel without mention. In addition, we have the following consequence of the Cross Axiom.

Proposition 1.15 *Let* $U$ *and* $V$ *be theories, and suppose that there is a theory* $W$ *such that* $U \stackrel{\Diamond}{\longrightarrow} W \stackrel{L}{\longrightarrow} V$.




*Then there is a theory* $T$ *so that* $U \stackrel{L}{\longrightarrow} T \stackrel{\Diamond}{\longrightarrow} V$.

*Proof* Let $S = \{\Diamond\phi : \phi \in V\} \cup \{\psi : K\psi \in U\}$. We claim that $S$ is consistent; suppose towards a contradiction that it is not. Then there is a finite subset of $S$ which is inconsistent. Now the two sets whose union is $S$ are closed under conjunction, and moreover, $K$ commutes with conjunction. So there are *individual* $\phi$ and $\psi$ such that $\phi \in V$, $K\psi \in U$ and $\vdash \Diamond\phi \to \neg\psi$. Therefore $\vdash L\Diamond\phi \to L\neg\psi$; hence this sentence belongs to $U$. And also, $\phi \in V$, so $L\phi \in W$ and $\Diamond L\phi \in U$. So $L\Diamond\phi \in U$ by the Cross Axiom. It follows that $L\neg\psi \in U$. But since $K\psi \in U$, this gives the contradiction that $U$ is inconsistent. So $S$ is consistent. Let $T \supseteq S$ be maximal consistent. By construction, $U \xrightarrow{L} T \xrightarrow{\Diamond} V$. QED

Proposition 1.15 is the embodiment of the Cross Axiom in the realm of theories. The next result is a generalization which will be used in the proof of completeness.

PROPOSITION 1.16 *Let $\mathcal{L} = \langle L, \leq, \top\rangle$ be a finite, bounded, linear order, and let $T$ be an order preserving map from $\mathcal{L}$ to $\langle th, \xrightarrow{\Diamond}\rangle$. Suppose that $T_\top \xrightarrow{L} U^*$. Then there is an order preserving $U : \mathcal{L} \to th$ such that $U_\top = U^*$, and for all $m \in L$, $T_m \xrightarrow{L} U_m$.*

*Proof* $U$ is defined going down $\mathcal{L}$, using Proposition 1.15. QED

The first idea in proving Theorem 1.13 is to consider the canonical model and to show that every theory $T$ is the theory of some pair $p, u$ from that model. As we have seen, the collection $th$ of theories has relations $\xrightarrow{L}$ and $\xrightarrow{\Diamond}$ with the cross property of Proposition 1.15. We shall call such a structure a *cross axiom frame*; we can soundly interpret $\mathcal{L}$ on it using the two relations. (We shall pursue this in Section 8.) The canonical cross axiom model shows that the subset space logic is complete for interpretations of $\mathcal{L}$ in cross axiom models. However, the canonical cross axiom model is not a subset space, and there does not seem to be any straightforward way to turn it into one.

Nevertheless, we would like completeness relative to subset spaces. Again, the natural idea is to try to "spatialize" the space of canonical theories (perhaps by taking as opens the $L$-equivalence classes). However, this construction could not work for the following reasons: Call a model $\mathcal{X}$ *exact* if for every $T$ there are unique $p$ and $u$ so that $th_{\mathcal{X}}(p, u) = T$, and if $u \supseteq v$ iff $th_{\mathcal{X}}(p, u) \xrightarrow{\Diamond} th_{\mathcal{X}}(p, v)$. But in Example B we saw theories $T \xrightarrow{\Diamond} U \xrightarrow{\Diamond} T$ which are distinct. This implies that there are no exact subset space models.

For this reason, we do not approach completeness via the canonical model. The strategy is to build a space $X$ of "abstract" points. We shall also have opens given in an abstract way, via a poset $\mathsf{P}$ and an antitone (i.e., order-reversing) map $i : \mathsf{P} \to \mathcal{P}^*(X)$. The points are abstract since they are not theories. But with each $x$ and each $p$ so that $t \in i(p)$ we shall have a "target" theory $t(x, p)$. The goal of the construction is to arrange that in the overall model, $th(x, i(p)) = t(x, p)$.

We are not going to present all of the details here, only the basic ideas. We hope that the interested reader can fill in the rest, based on the following more concrete plan.

One builds

(1) A set $X$ containing a designated element $x_0$.

(2) A poset with least element $\langle \mathsf{P}, \leq, \bot \rangle$.

(3) A function $i : \mathsf{P} \to \mathcal{P}^*(X)$ such that $p \leq q$ iff $i(p) \supseteq i(q)$, and $i(\bot) = X$. (That is, a homomorphism from $\langle \mathsf{P}, \leq, \bot \rangle$ to $\langle \mathcal{P}^*(X), \supseteq, X \rangle$.)

(4) A partial function $t : X \times \mathsf{P} \to th$ with the property that $t(x,p)$ is defined iff $x \in i(p)$. Furthermore, we require the following properties for all $p \in \mathsf{P}$, $x \in i(p)$, and $\phi$:

(a.1) If $y \in i(p)$, then $t(x,p) \xrightarrow{L} t(y,p)$.

(a.2) If $L\phi \in t(x,p)$, then for some $y \in i(p)$, $\phi \in t(y,p)$.

(b.1) If $q \geq p$, then $t(x,p) \xrightarrow{\Diamond} t(x,q)$.

(b.2) If $\Diamond\phi \in t(x,p)$, then for some $q \geq p$, $\phi \in t(x,q)$.

(c) $t(x_0, \bot) = T$, where $T$ is the theory from above which we aim to model.

Suppose we have $X, \mathsf{P}, i$, and $t$ with these properties. Then we consider the subset space

$$\mathcal{X} \quad = \quad \langle X, \{i(p) : p \in \mathsf{P}\}, i \rangle.$$

where $i(P) = \{x : A \in t(x, \bot)\}$.

LEMMA 1.17 (THE TRUTH LEMMA) *Assume conditions (1) – (4) for $X$, $\mathsf{P}$, $i$, and $t$. Then for all $x \in X$ and all $p \in \mathcal{L}$ such that $x \in i(p)$,*

$$th_{\mathcal{X}}(x, i(p)) \quad = \quad t(x,p).$$

*Proof* We show induction on $\phi$ that $\phi$ belongs to the set on the left iff it belongs to the set on the right. We only give the inductive step for sentences $L\phi$.

Suppose that $x, i(p) \models L\phi$. Then there is some $y \in i(p)$ such that $y, i(p) \models \phi$. By induction hypothesis, $\phi \in t(y,p)$. By property (4a.1), $t(x,p) \xrightarrow{L} t(y,p)$. Therefore $L\phi \in t(x,p)$. On the other hand, if $L\phi \in t(x,p)$, then by property (4a.2) there is some $y \in i(p)$ such that $\phi \in t(y,p)$. By induction hypothesis, $y, i(p) \models \phi$. Therefore $x, i(p) \models L\phi$. This concludes the induction step for $L$.

The induction step for $\Diamond$ is similar and uses (3), (4b.1), and (4b.2).

QED

One builds $X$, $\mathsf{P}$, $i$, and $t$ by recursion, in a step-by-step process that goes on for countably many steps. This is not the place to enter into these details. They are fairly straightforward and can be found in Dabrowski et al., 1996.

By the Truth Lemma and property (4c) above, $theory_{\mathcal{X}}(x_0, \perp) = T$. So the theory that we started with has a model. In the usual way, this proves the Completeness Theorem.

## 8. Decidability of the subset space logic

Despite the failure of the finite model property, we prove that the logic of subset spaces is decidable. The proof here is due to Krommes, 2003, simplifying the argument from Dabrowski et al., 1996. It goes by showing that a satisfiable sentence $\phi$ has a finite *cross axiom* model. This is a kind of pseudo-model for this subject, to be defined shortly. The idea as always is that we move to a bigger class of models than the one we are primarily interested in, and this class will turn out to be better behaved.

DEFINITION 1.18 *A cross axiom frame is a tuple* $\langle J, \xrightarrow{L}, \xrightarrow{\Diamond} \rangle$ *such that* $J$ *is a set,* $\xrightarrow{L}$ *is an equivalence relation on* $J$*,* $\xrightarrow{\Diamond}$ *is a preorder on* $J$*, and the following property holds: If* $i \xrightarrow{\Diamond} j \xrightarrow{L} k$*, then there is some* $l$ *such that* $i \xrightarrow{L} l \xrightarrow{\Diamond} k$*. A cross axiom model is a cross axiom frame together with an interpretation* $i$ *of the atomic symbols of* $\mathcal{L}$*.*

Note that when we interpret the language on a cross axiom model, we have a *node* on the left side of the turnstile. That is, we write, e.g., $j \models \phi$ since there are no sets involved.

The subset space logic is complete for interpretations in cross axiom models since the latter include subset spaces.

Our main example of a cross axiom model which is not a subset space is the *canonical model* of the subset space logic:

$$\mathcal{C}(ca) \quad = \quad \langle th, \xrightarrow{L}, \xrightarrow{\Diamond} \rangle.$$

(The "ca" stands for "cross axiom".) Proposition 1.15 says that $\mathcal{C}(ca)$ actually is a cross axiom model. The standard truth lemma for this structure shows that for all $T \in th$, $th(T) = T$; that is, the set of sentences satisfied by the point $T$ in $\mathcal{C}(ca)$ is $T$ itself. We shall use a version of filtration to prove a finite model property.

Let $\phi$ be any sentence which is consistent in the logic of set spaces, and consider the following sets:

$\Sigma^{\neg}$ = all subformulas of $\phi$ and their negations.

$\Sigma'$ = all conjunctions and disjunctions of (finite) subsets of $\Sigma^{\neg}$.

$\Sigma''$ = all conjunctions and disjunctions of (finite) subsets of $\Sigma'$.

$\Sigma^{KL} = \{L\psi : \psi \in \Sigma''\} \cup \{K\psi : \psi \in \Sigma''\}$.

$\Sigma = \Sigma'' \cup \Sigma^{KL}$.

Note that $\Sigma$ is finite. Note also that modulo propositional logic, $\Sigma''$ is closed under the boolean connectives; in effect we are taking disjunctive normal forms. Finally, $\Gamma$ depends on the original $\phi$.

Write $U \equiv V$ if $U \cap \Gamma = V \cap \Gamma$, and let $[U]$ be the equivalence class of $U$ under this relation. (Note that $\equiv$ also depends on $\phi$, but we save a little notation by suppressing this.) We define relations $\xrightarrow{L}$ and $\xrightarrow{\Diamond}$ on $[\mathcal{C}(ca)]$ as follows:

$$[S] \xrightarrow{L} [T] \quad \text{iff} \quad \text{there exist } S' \in [S] \text{ and } T' \in [T] \text{ such that } S' \xrightarrow{L} T'.$$

This is the definition used in minimal filtrations. In contrast, we want $\xrightarrow{\Diamond}$ to be transitive, and so we define

$$[S] \xrightarrow{\Diamond} [T] \quad \text{iff} \quad \begin{array}{l}\text{there exist } n \geq 0 \text{ and } S', S_1, \ldots, S_n, S'_n, T' \text{ such that} \\ S \equiv S' \xrightarrow{\Diamond} S_1 \equiv S'_1 \xrightarrow{\Diamond} \cdots \xrightarrow{\Diamond} S_n \equiv S'_n \xrightarrow{\Diamond} T' \equiv T\end{array}$$

We call the tuple

$$[\mathcal{C}(ca)] = \langle [\mathcal{C}(ca)], \xrightarrow{L}, \xrightarrow{\Diamond} \rangle$$

the *quotient of* $\mathcal{C}(ca)$ *by* $\equiv$. We shall show that this quotient is a cross axiom frame; the main points are the transitivity of $\xrightarrow{L}$ and the cross axiom property. We turn the structure into a model in the evident way, via the interpretation $i(A) = \{[T] : A \in T \cap \Sigma\}$.

Standard reasoning shows that $[\mathcal{C}(ca)]$ is a filtration of $\mathcal{C}(ca)$. It follows that since the sentence $\phi$ with which we began is consistent, it has a *finite* cross axiom model. Thus the decidability reduces to the verification of the properties of $[\mathcal{C}(ca)]$ that we mentioned in the last paragraph.

We need several results, starting with an important lemma. For each theory $T$, let

$$\gamma_T \quad = \quad \bigwedge (T \cap \Gamma).$$

That is, we take the conjunction of the set $T \cap \Gamma$. This is an analog of *atoms*, as we find them in the completeness proof of PDL and other places (see Kozen and Parikh, 1981).

LEMMA 1.19 *Let $T$ and $U$ be theories. If $L\gamma_T \in U$, then whenever $V \equiv U$, we also have $L\gamma_T \in V$.*

*Proof* Fix $U$ containing $L\gamma_T$, and let $V \equiv U$. We split $\gamma_T$ into two conjuncts

$$\gamma''_T = \bigwedge(T \cap \Gamma'') \qquad \gamma^{KL}_T = \bigwedge(T \cap \Gamma^{KL})$$

Again, we have $\gamma_T \equiv \gamma''_T \wedge \gamma^{KL}_T$. Our goal is to show that $L\gamma''_T \in V$.

Let $\psi \in T \cap \Gamma^{KL}$. (Notice at this point that $\psi$ belongs to $V$.) This $\psi$ is either $L\chi$ or $K\chi$ for some $\chi \in \Gamma''$. The S5 laws of $K$ in subset spaces tell us that $K\chi \leftrightarrow KK\chi$ and $L\chi \leftrightarrow KL\chi$. So $\psi \equiv K\psi$. We therefore see that $K\psi \in V$. This for all $\psi \in T \cap \Gamma^{KL}$ shows that $K\gamma^{KL}_T \in V$.

Note second that since $U$ contains $L\gamma_T$, it also contains $L\gamma''_T$. But this last sentence belongs to $\Gamma^{KL} \subseteq \Gamma$. Because $V \equiv U$, we see that $L\gamma''_T \in V$.

We conclude that $V$ contains $K\gamma^{KL}_T$ and $L\gamma''_T$. Thus it contains $L(\gamma^{KL}_T \wedge \gamma''_T)$. But this is equivalent to $L\gamma_T$. So $L\gamma_T \in V$. QED

LEMMA 1.20 *Suppose that $[S] \xrightarrow{L} [T]$. Then for all $S' \in [S]$ there is some $T' \in [T]$ such that $S' \xrightarrow{L} T'$.*

*Proof* Fix $S' \in [S]$. Let $S'' \in [S]$ and $T'' \in [T]$ be such that $S'' \xrightarrow{L} T''$. Then $L\gamma_{T''} \in S''$. By Lemma 1.19, $L\gamma_{T''} \in S'$. And thus there is some $T'$ such that $S' \xrightarrow{L} T'$ and $\gamma_{T''} \in T'$. Since $\gamma_{T''} = \gamma_T$, we see that $T' \equiv T$.
QED

Lemma 1.20 easily implies the transitivity of $\xrightarrow{L}$ in $[\mathcal{C}(ca)]$. We are thus left with the verification of the cross axiom property. Suppose that $[S] \xrightarrow{\Diamond} [T] \xrightarrow{L} [U]$. We need to find $W$ so that $[S] \xrightarrow{L} [W] \xrightarrow{\Diamond} [U]$, and for this we argue by induction on $n$ in the definition of the relation $[S] \xrightarrow{\Diamond} [T]$. When $n = 0$, we have $S \equiv T$. In this trivial case we have $[S] = [T]$ and may take $[W] = [S]$. Assume our result for $n$, and suppose that $[S] \xrightarrow{\Diamond} [T]$ via a chain of length $n+1$, say

$$S \equiv S' \xrightarrow{\Diamond} S_1 \equiv S'_1 \xrightarrow{\Diamond} \cdots \xrightarrow{\Diamond} S_n \equiv S'_n \xrightarrow{\Diamond} S_{n+1} \equiv S_{n'+1} \xrightarrow{\Diamond} T' \equiv T.$$

Then $[S_1] \xrightarrow{\Diamond} [T]$ via a chain of length $n$, and so by induction hypothesis we have some $W$ such that $[S_1] \xrightarrow{L} [W] \xrightarrow{\Diamond} [U]$. By Lemma 1.20 there is some $W' \equiv W$ such that $S_1 \xrightarrow{L} W'$. So as theories, $S' \xrightarrow{\Diamond} S_1 \xrightarrow{L} W'$. Since $\mathcal{C}(ca)$ is a cross axiom frame, there is some $X$ such that $S' \xrightarrow{L} X \xrightarrow{\Diamond} W'$. We have $[S] \xrightarrow{L} [X]$ in the quotient. We also have $[X] \xrightarrow{\Diamond} [W] \xrightarrow{\Diamond} [U]$. By transitivity we also have $[X] \xrightarrow{\Diamond} [U]$. This completes the proof of decidability: from a consistent sentence, we have effectively produced a finite model.

## 9. Heinemann's extensions to topologic

This section discusses two of the many extensions of topologic due to Bernhard Heinemann. One of his papers (Heinemann, 1998) studies spaces which satisfy *chain conditions* of various sorts. We begin with the relevant definitions; let $\mathcal{X} = \langle X, \mathcal{O} \rangle$ be a subset frame.

1 $\mathcal{X}$ satisfies the *weak bounded chain condition* (*wbcc*) if for each point $x \in X$, every descending sequence of opens around $x$ is finite.

2 $\mathcal{X}$ satisfies the *finite chain condition* (*fcc*) if every descending sequence of opens is finite.

3 $\mathcal{X}$ satisfies the *bounded chain condition* (*bcc*) if for some $n$, every descending sequence of opens is of length at most $n$.

In order to axiomatize the logics of these classes of spaces, it is necessary to alter the basic semantics of topologic. Up until now we have

$$p, u \models \Box\phi \qquad \text{iff} \qquad p, v \models \phi \text{ for all } v \in \mathcal{O} \text{ such that } v \subseteq u$$

In the study of chain conditions, we alter this to

$$p, u \models \Box\phi \qquad \text{iff} \qquad p, v \models \phi \text{ for all } v \in \mathcal{O} \text{ such that } v \subset u$$

(So $\Box$ now quantifies over *proper* subsets.) Turning to the logics themselves, first note that the topologic axiom $\Box\phi \to \phi$ is no longer valid. So this axiom is dropped, and the rest of the system is retained.

To axiomatize the validities in the wbcc spaces, one then adds the scheme

$$\Box(\Box\phi \to \phi) \to \Box\phi \tag{1.7}$$

This scheme is familiar from provability logic (also called Löb logic). The resulting logic lacks the finite model property, and indeed the decidability remains open (see Heinemann, 1998).

Turning to bcc spaces, we would add (1.7) (actually a weaker scheme suffices) and also

$$K\Box(K\Box\phi \to \phi) \to K\Box\phi)$$

It turns out that the resulting system is sound for fcc spaces and complete even for the smaller class of bcc spaces, and that it is decidable.

Around the same time as we wrote this chapter, Heinemann discovered a modality which is basic for our enterprise. This is *overlap operator* $\mathsf{O}$, (see Heinemann, 2006). We *add* it to the basic language of topologic and then the semantics is extended to

$$p, u \models \mathsf{O}\phi \qquad \text{iff} \qquad p, v \models \phi \text{ for all } v \in \mathcal{O} \text{ such that } p \in v$$

Note that the difference between this and the semantics of $\Box\phi$ is that we do not require that $v \subseteq u$. As a result, we see that the axiom $\mathsf{O}\phi \to \Box\phi$ is valid. Also, the S5 axioms are valid for $\mathsf{O}$:

$$\begin{array}{l} \mathsf{O}(\phi \to \psi) \to (\mathsf{O}\phi \to \mathsf{O}\psi) \\ \mathsf{O}\phi \to \phi \wedge \mathsf{OO}\phi \\ \phi \to \mathsf{OP}\phi \end{array}$$

In the last of these, the operator $\mathsf{P}$ is the dual of $\mathsf{O}$, so $\mathsf{P}\phi$ is defined by $\neg\mathsf{O}\neg\phi$. Turning from axioms to rules of inference, we see easily that the necessitation for $\mathsf{O}$ preserves validity. So we get a logical system called $\mathsf{ET}$ by adding the axioms and rule to the system of topologic. Heinemann proves completeness and decidability of this system (see Heinemann, 2006). The details are elaborations of the arguments which we have already seen for the basic system of topologic, the subset space axioms.

Recall that the class of directed spaces is not finitely axiomatizable (see Weiss and Parikh, 2002). This is remedied in the larger language: consider

$$\mathsf{P}\Box\phi \to \Diamond\phi.$$

Further, the *weak connectedness* of the space may be captured. This condition says that given any two overlapping sets, one is included in the other. The relevant axiom is

$$(\phi \to \Diamond\psi) \vee \mathsf{O}(\psi \to \Diamond\phi)$$

But this is not so topologically natural. It would be more in the spirit of things to capture the condition that two overlapping sets have a common superset. The natural move here is to add *nominals* $I$, $J$, etc. whose values are taken to be opens of the underlying space. Then one defines the semantics as in hybrid logic, using interpretations on the new nominals and the obvious satisfaction relation. The sentence

$$I \wedge \mathsf{P}J \to \mathsf{P}(\Diamond I \wedge \Diamond J)$$

Then captures the condition we mentioned. One can also add nominals for points, again following the lead of hybrid logic . There is an outline of the completeness and decidability of the resulting logical system in Heinemann, 2006.

We close with a mention of an avenue for further work. Most of the results in this area concern completeness and decidability of various systems, and also results on expressive power. Results on *complexity* are scarce. One exception is (Heinemann, 1999a), where an NP–completeness result is proved for a satisfiability problem related to *topological nexttime logic*. But again, there are many open complexity problems.

A complete list of Bernhard Heinemann's papers on topologic can be found in the author's homepage:
http://www.informatik.fernuni-hagen.de/thi1/bernhard.heinemann/

## 10. Common knowledge in topological settings

The interaction of the topological semantics of modal logic with the topic of *common knowledge* is the topic of a recent paper (van Benthem and Sarenac, 2004). Like those discussed in Section 9, we expect this paper to inspire others. Our purpose here is to give a high-level overview of it that connects it with the theme of our chapter.

The most standard setting of modal logic is that given by the semantics on Kripke models. We shall go into more detail on this in Section 11 just below. In the standard setting, one adds *transitive closure* operators $\Box^*$ to modal logic. For the semantics, one takes a model $M$ which lives on a relation $R$, considers the reflexive-transitive closure $R^*$ of $R$, and finally defines

$$x \models \Box^* \phi \qquad \text{iff} \qquad y \models \phi \text{ for all } y \in M \text{ such that } xR^*y$$

So $\Box^*\phi$ is semantically equivalent to the infinite conjunction $\phi \wedge \Box\phi \wedge \Box^2\phi \wedge \cdots$. We sometimes read $\Box^*\phi$ as *fully-aware* knowledge; $\phi$ is true, the agent knows this, s/he knows that s/he knows *this*, etc. But note that this reading hides a great deal. Indeed, it would be better to note that there is an intuitively important idea of fully-aware knowledge and then to take $\Box^*$ as a *proposal* for modeling it. A second proposal is to take the set of states satisfying $\Box^*\phi$ to be the largest fixed point of the following operator on the power set of the set of states in our model:

$$X \mapsto \{w \in W : w \models \phi, \text{ and all } R \text{ successors of } w \text{ belong to } X\}.$$

Yet another proposal might be to have some very explicit evidence, and of course this is not capturable in the standard semantics in the first place. (In the same way, we should always remember that the standard semantics of $\Box$ is a proposal for the modeling of intuitive knowledge in the first place.)

To capture this new operator $\Box^*$ in a sound, complete, and decidable logic, one adds necessitation for the new operator, an axiom called Mix:

$$\Box^*\phi \to (\phi \wedge \Box\Box^*\phi)$$

and also an induction rule: from $\psi \to \phi \wedge \Box\psi$, infer $\psi \to \Box^*\phi$.

So far in our chapter we have only dealt with epistemic logic with one agent, and this is how the subject of epistemic logic germinated. But the whole subject blossoms in the *multi-agent* setting. For simplicity, we shall deal with two agents, call them $A$ and $B$. The intuitive idea of common knowledge is the two-agent version of the fully-aware knowledge that we saw above. So we add an operator $\Box^*$ (sometimes it is decorated with the names of the agents $A$ and $B$) to the basic modal language. Then the question arises as to the semantics. One way to begin is to start with a model $M$ living on two different relations, say $R_A$ and $R_B$. The standard proposal for the semantics of $\Box^*\phi$ is

$$x \models \Box^*\phi \qquad \text{iff} \qquad y \models \phi \text{ for all } y \in M \text{ such that } x(R_A \cup R_B)^* y$$

So we take the reflexive-transitive closure of the union of the two relations. In plainer terms, this proposal amounts to saying that $\phi$ is common knowledge to $A$ and $B$ if

(0) $\phi$ is true;

(1) $A$ and $B$ both know (0);

(2) $A$ and $B$ both know (1);

and so on. To summarize, this standard proposal is to identify common knowledge of $\phi$ with a certain infinite conjunction of iterated knowledge statements of $\phi$.

We have taken great care in this discussion to keep separate the intuitive concepts and the standard formulation of them. These are usually conflated, and the main point of van Benthem and Sarenac's paper is that a two-agent version of the topological semantics of epistemic logic allows one to distinguish between different formalizations that are always identified in the relational semantics. The need to do this goes back to an influential paper on common knowledge (Barwise, 1988). In that paper, Barwise wishes to question the basic modeling of common knowledge as an infinite iteration. More precisely, he wishes to distinguish between the infinite iteration and the related fixed point; he also is concerned with the notion of "agents having a shared situation".

Recall the topological semantics of epistemic logic, and especially the translation of the interior operator $I$ from Section 4.3. So the analog of $\Box^*\phi$ is now

$$\phi \wedge K\Diamond\phi \wedge K\Diamond K\Diamond\phi \wedge \cdots$$

And the natural operator on the power set of the state set is

$$X \mapsto \{w \in X^{\mathrm{O}} : w \models \phi\}$$

(Note that we are using the interior operator here.) For a sentence in the logic at hand, the two definitions agree. And one of the points of the paper of van Benthem and Sarenac is to show a separation of the two formalizations on one particular model for the two-agent version of the logic. This model is

$$Q \times Q \quad = \quad \langle Q \times Q, \mathcal{O}_1, \mathcal{O}_2 \rangle.$$

Thus the states are pairs of rational numbers. A set $X$ is open in $\mathcal{O}_1$ if for all $(r, s) \in X$ there is some rational $\epsilon$ such that

$$\{r\} \times ((s - \epsilon, s + \epsilon) \cap Q) \subseteq X.$$

And $\mathcal{O}_2$ is defined similarly. The analog of the infinite iteration now reflects two agents:

$$\phi \wedge \bigwedge_i \Box_i \phi \wedge \bigwedge_i \bigwedge_i \Box_i \phi \wedge \cdots$$

(so that $i$ ranges over $\{1, 2\}$). The fixed point now is for the operator which takes a set $X$ to the intersection of the interiors of $X$ in the two topologies. Again, the paper proves that the two notions differ. See also van Benthem et al., 200 and Sections 3.2 and 3.4 of Chapter **??** for this and related material, such as the completeness of the two-agent version of S4 on the rational square $Q \times Q$.

## 11. The topology of belief

We have already seen in Section 3 that $S4$, a logic of knowledge, has a topological semantics. For the purposes of this chapter, the important conclusion is that the interior operator acts like *knowledge*. In more detail, the properties of this operator as rendered in modal terms correspond to the axioms of a logic $S4$ that may be considered a *logic of knowledge*. The purpose of this section of our chapter is to show that there is a topological operator which acts like *belief* in this same sense. Our work in this section is based on the recent dissertation by one of the authors (Steinsvold, 2006).

Most commonly, the difference between knowledge and belief is taken to be that belief should not imply truth; this corresponds to the assertion that analogues of statements like $I\phi \to \phi$ should *not* be valid. To emphasize that we are dealing with belief, we change the modality to $B$ in this section. So again, we do not want to work with a logic that includes the $T$-scheme $B\phi \to \phi$.

The most standard logic of belief is the logic $KD45$. $KD45$ is axiomatized using the schemes listed in Figure 1.5, together with the rules of

Modus Ponens and Necessitation which we have seen in Section 3. The $D$ scheme corresponds to the assertion that beliefs are consistent; an agent who believes $\phi$ should not also believe $\neg\phi$. The 4 and 5 schemes correspond to assertions of *introspection*: if an agent believes $\phi$, then they believe the assertion that they believe $\phi$. This is the content of 4. As for 5, it says that if an agent does not believe $\phi$, then again they believe the assertion of that disbelief. Despite obvious problems, $KD45$ is a standard logic of belief in the sense that the provable sentences may be taken as a first approximation to the properties of belief. Here is the plan of this section of our chapter. We first recall and compare the relational and topological semantics for $KD45$. We then discuss the *derived set* operation from topology (defined in Section 11.2 below). and provide an interpretation of $KD45$ in it. Finally, a topological completeness proof is presented for $KD45$ with respect to a class of spaces that we call DSO spaces.

We should note in passing that other authors have explored the properties of this derived set operation using modal logic; see, e.g., Esakia, 1981; Shehtman, 1990; Bezhanishvili et al., 2005. However, none of these works offered epistemic connections. This is one of our goals. Actually, we are more interested in *doxastic* connections, that is connections to the notion of *belief*.

Before plunging into the details, we should mention why we chose this topic for a chapter on Topology and Epistemic Logic. We have already seen various logics in the paper, including a standard logic of knowledge ($S4$) and a standard logic with common knowledge (the logic of $\Box^*$). One certainly can view what we are doing here as being work in the same general direction: find a logic corresponding to a semantics that is already of interest. But this section can also be read in the other direction: given a logic that we believe to be sensible or at least worthy of study, construct a semantics for it. The work of this section suggests a topological development along these lines. In a nutshell, the proposal is to read belief as the dual of the derived set operator on a topological space satisfying some conditions. For a complete discussion of the philosophical aspects of this proposal, see Steinsvold, 2006.

## 11.1 Relational semantics of $KD45$

We have already discussed the notion of a *frame* in Section 3.1. We quickly moved from frames in general to preorders there, but here we need the more general notion. Again, a *frame*, $F = \langle W, R \rangle$, is a set $W$ with a relation $R$ on $W$. We'll refer to the members of $W$ as points or worlds, interchangeably. If $x$ bears the relation $R$ to $y$, we'll write either

| Ax | formal statement | relational correspondent |
|---|---|---|
| $K$ | $B(\phi \to \psi) \to (B\phi \to B\psi)$ | (none) |
| $D$ | $B\phi \to \neg B \neg \phi$ | every point has an $R$-successor |
| 4 | $B\phi \to BB\phi$ | $R$ is transitive |
| 5 | $\neg B\phi \to B\neg B\phi$ | $R$ is Eucldiean |

*Figure 1.5.* Axiom schemes of $KD45$ with their relational correspondents.

$xRy$ or $(x, y) \in R$. A *model* is a frame together with an interpretation $i$ of atomic sentences in it. We say that the model is *based* on the frame.

The semantics on models is given in the usual way, with the clause for $B$ that

$$w \models B\phi \quad \text{iff} \quad (\forall z)(wRz \text{ implies } z \models \phi).$$

So $B$ is defined using a universal quantifier, like $I$ was in our earlier work.

We read $w \models \phi$ as saying that $\phi$ *is true at* $w$ or that $w$ *satisfies* $\phi$. A sentence $\phi$ is *valid in a model* iff $\phi$ is true at every point in the model. A sentence $\phi$ is *valid in a frame* iff $\phi$ is valid in every model based on the frame.

The correspondences of Figure 1.5 are well known. The precise nature of this correspondence is that a frame $F$ satisfies each instance of $D$ (say) iff $F$ meets the condition that every point in it has a successor. A model which meets the condition corresponding to $D$ (that every point have a successor) is called *serial*. The Euclidean condition corresponding to the 5 axioms is

$$(\forall x)(\forall y)(\forall z)((xRy \wedge xRz) \to yRz).$$

$KD45$ is complete with respect to models which have the properties listed above. That is, if $T \cup \{\phi\}$ is a set of sentences, then $T \vdash \phi$ in $KD45$ iff for all serial, transitive, and Eucldiean models $M$ and all points $x \in M$, if $x \models T$, then $x \models \phi$.

This completeness is a parallel to facts we saw in Section 3 for $S4$: One considers the set theories $T$ in $KD45$ and gets the rest of a "canonical model" structure of a model using (1.2) and (1.1). $KD45$ also is decidable via the finite model property.

## 11.2 The derivative operation on topological spaces

We are now going to be proposing a *different* semantics for modal logic, one based on an operation from topology called the *derived set* operation. We collect in this section some topological preliminaries about this operation. Let $\mathcal{X} = \langle X, \mathcal{O}\rangle$ be a topological space. For a set $A \subseteq X$, we define the *derived set* of $A$, $d(A)$,

$$w \in d(A) \text{ iff } (\forall U \in \mathcal{O})(\text{if } w \in U \text{ then } (\exists x \in U \setminus \{w\})(x \in A)).$$

In words, $w$ belongs to the derived set of $A$ iff every open set $U$ around $A$ contains some point of $A$ different from $w$. It also might be useful to write out the complement:

$$w \notin d(A) \text{ iff } (\exists U \in \mathcal{O}))\ (w \in U \text{ but } (\forall x \in U \setminus \{w\})(x \notin A)).$$

This *derived set* $d(A)$ has many other names in the literature on point-set topology: *derivative*, *Cantor-Bendixson derivative*, *set of limit points*, *set of accumulation points*, and *set of cluster points* (of $A$). It is usually written as $A'$.

Example 1.21 Let $\mathcal{R}$ be the reals with the usual topology, and let

$$A \quad = \quad \left\{\frac{1}{n} : n = 1, 2, \ldots\right\}$$

Then $d(A) = \{0\}$. That is, every open set containing 0 contains some number $1/n$. But for all $r \neq 0$, there is an open neighborhood of $r$ which contains no number in $A$. Next, consider

$$B \quad = \quad \left\{\frac{1}{n} + \frac{1}{n+1+m} : n, m = 1, 2, \ldots\right\}.$$

This is like $A$, but with a copy of $A$ itself next to each of its points. The copies are disjoint. Then $d(B) = A \cup \{0\}$. And so $d(d(B)) = \{0\}$. Further, $d(d(d(B))) = \emptyset$. More generally, the derived set of any singleton in any topological space is always empty.

Example 1.22 Here is another example of how $d$ works, one which will be elaborated in Section 11.4 below. Let $W$ be an infinite set, let $D$ be an infinite subset of $W$, and let $\mathcal{O}$ be the family of subsets $U \subseteq W$ such that $D \setminus U$ is finite, together with the empty set. It is easy to see that $\mathcal{O}$ is closed under finite intersections, since

$$D \setminus (U \cap V) \quad = \quad (D \setminus U) \cup (D \setminus V),$$

and arbitrary unions. (In fact, every superset of a nonempty open set is itself open.) In this way, $\mathcal{W} = \langle W, \mathcal{O}\rangle$ is a topological space.

We compute $d(A)$ for all $A \subseteq W$:

$$d(A) \quad = \quad \begin{cases} \emptyset & \text{if } D \cap A \text{ is finite} \\ A & \text{if } D \cap A \text{ is infinite} \end{cases}$$

In the first case, $D \setminus (D \setminus A) = D \cap A$ is finite. For $w \in W$, let $U_w = \{w\} \cup (D \setminus A)$. Then $U_w$ is an open set containing $w$, but for all $x \in U_w \setminus \{w\}$, $x \notin A$. So $w \notin d(A)$. Thuse $d(A) = \emptyset$. We turn to the second case. Let $w \in W$ and let $U$ be an open set containing $w$. We claim that $U \cap A \cap D$ is infinite. For suppose not. Then $D \cap A$ is infinite but $U \cap (A \cap D)$ is finite. So $(D \cap A) \setminus U$ is infinite. So its superset $D \setminus U$ is also infinite. But by definition of the topology, $D \setminus U$ is finite. This contradiction shows that indeed $U \cap D \cap A$ is infinite. In particular, $U$ contains an element of $A$ different from the original $w$.

Example 1.23 Here is a final example, one suggestive of the connection between the relational semantics and the derived set operation. Let $M = \langle W, R\rangle$ be a set with a transitive relation on it. (We do not require that $R$ be reflexive.) Recall the Alexandrov topology from Section 3.1: the opens are the $R$-closed sets. We again compute $d$ on sets, and this time

$$d(A) \quad = \quad \{x \in W : (\exists y \in A)\ (y \neq x \text{ and } xRy)\}.$$

Here is the reasoning: suppose first that $x \in d(A)$. Consider the open set $U_x = \{x\} \cup \{y : xRy\}$. $A$ must contain some point of $U_x$ other than $x$. In the other direction, assume that $y \neq x$, $xRy$, and $y \in A$. Then every open set around $x$ contains $y$.

A suggestive observation: the standard Kripke semantics of modal logic is *almost* related to the derived set by

$$i(\Diamond\phi) \quad = \quad d(i(\phi)).$$

Indeed, if we modified the definition of $x \in d(A)$ to not require witnesses different from $x$ in the open neighborhoods, we would have the equation above. Or, if $R$ were irreflexive, then the equation would hold. See Esakia, 2001 for more on this.

Proposition 1.24 below contains standard results in point-set topology about the operation $d$. In the first part, we recall that a topological space satisfies the $T_1$ separation property iff for all distinct points $x$ and $y$, there is an open set containing $x$ but not $y$. Another formulation: all singletons are closed sets. This property can also be neatly expressed using $d$: $d(\{x\}) = \emptyset$ for all $x$. (Recall also that $x \notin d(\{x\})$.) In the second part, we mention the $T_d$ separation property. This holds if every singleton is the intersection of an open and a closed set. Equivalently, every set of the form $d(A)$ is closed. This alternate formulation is more useful in this chapter.

PROPOSITION 1.24 *Concerning the derivative operation* $d$ *on a topological space* $\mathcal{X}$*:*

1 *If every set of the form* $d(\{z\})$ *is open, then* $\mathcal{X}$ *is a* $T_1$ *space.*

2 *If* $\mathcal{X}$ *is a* $T_1$ *space, then* $\mathcal{X}$ *is a* $T_d$ *space.*

3 *For all topological spaces,* $d(d(A)) \subseteq A \cup d(A)$*.*

*Proof* For the first assertion, assume that each $d(\{x\})$ is open. Suppose towards a contradiction that $y \in d(\{x\})$. As we know, $x \notin d(\{x\})$. But since $d(\{x\})$ is open, contains $y$, and is disjoint from $\{x\}$, we see that $y \notin d(\{x\})$ after all. This is a contradiction.

For the second assertion, consider a derived set $d(A)$. Let $x \notin d(A)$. Then there is an open set $U$ containing $x$ such that $(U \setminus \{x\}) \cap A = \emptyset$. We claim that $U \cap d(A) = \emptyset$. Since we already know $x \notin d(A)$, we only need to consider some $y \neq x$ in $U$. By the $T_1$ property, let $V$ be an open set with $y \in V$ but $x \notin V$. Then $(U \cap V) \cap A = \emptyset$. Since $U \cap V$ is open and contains $y$, we see that $y \notin d(A)$. But $y$ is arbitrary, and this establishes the claim that $U \cap d(A) = \emptyset$. That is, $U$ is an open set, $x \in U$, and $U \subseteq X \setminus d(A)$. Since $x$ is arbitrary, $X \setminus d(A)$ is open.

For the last part, suppose that $x \in d(d(A))$ but $x \notin A$. Let $U$ be an open set containing $x$; we show that $U \cap (A \setminus \{x\}) \neq \emptyset$. Let $y \in d(A) \cap U$. Since $y \in d(A) \cap U$, let $z \in U \cap (A \setminus \{y\})$. Then $z \neq x$, since $x \notin A$. QED

The assertions in Proposition 1.24 suggest a definition: we say that a space $\mathcal{X}$ is a *DSO space* if every derived set $d(A)$ is open but the space is dense in itself. (This last condition means that there are no open singletons. We add this because the conjunction of these two properties will be important in our later work: the DSO spaces turn out to play the role of serial, transitive, Euclidean relations in the Kripke semantics of $KD45$.)

EXAMPLE 1.25 Every space built as in Example 1.22 is a DSO space. To see this, recall that we have computed the derivative operation on all subsets of the space. The upshot is that the only derived sets are the empty set and the whole space. So every derived set is open. Further, no singletons are open. So we have a DSO space.

Example 1.26 On the other hand, consider a space $\langle W, \mathcal{O}\rangle$ built from a transitive relation $\langle W, R\rangle$ as in Example 1.23. An open singleton corresponds to a point $x$ with no $R$-successors different from $x$ itself. We do not usually have a DSO space: usually there are derived sets which are not open. (Incidentally, there are no finite DSO spaces: such a space would be $T_1$, and hence all singletons would be open.) In fact, there does not seem to be an elegant frame-theoretic correspondent to the DSO condition.

## 11.3 Derived set semantics of $KD45$

Recall that we are aiming towards a completeness theorem for the doxastic logic $KD45$ with respect to a semantics concerned with the derived set operation. We have seen the axioms of $KD45$ in the opening part of Section 11. We now give the semantics. Let $\mathcal{X} = \langle X, \mathcal{O}, i\rangle$ be a *topological model*, a topological space with an interpretation of some fixed background set of atomic sentences.

We interpret the basic modal language in $\mathcal{X}$ using the classical interpretation of the connectives, and most critically,

$$w \models B\phi \quad \text{iff} \quad (\exists U \in \mathcal{O})(w \in U \wedge (\forall x \in U \setminus \{w\})(x \models \phi)).$$

In words, $B\phi$ is true at $w$ when there is an open set $U$ containing $w$ such that every point in $U$ *except possibly for* $w$ satisfies $\phi$. So we have

$$w \models \neg B\neg\phi \text{ iff } (\forall U \in \mathcal{O})(\text{ if } w \in U \text{ then } (\exists x \in U \setminus \{w\})(x \models \phi)).$$

And therefore

$$i(\neg B\neg\phi) \quad = \quad d(i(\phi)).$$

At this point we have a hint as to why the semantics of $B$ was taken with the existential condition rather than the universal one. In Example 1.23 we saw that the derivative operation corresponds to the existential modality $\Diamond$. And since the $KD45$ axioms for $B$ are universal, we take the semantics using the dual of $d$ rather than $d$ itself.

We have the standard semantic definitions: A sentence $\phi$ is *valid in a topological model* $\mathcal{X}$ iff $\phi$ is true at every $x \in X$. A sentence $\phi$ is *valid in a topological space* $\langle X, \mathcal{O}\rangle$ iff $\phi$ is valid in every topological model based on $\langle X, \mathcal{O}\rangle$.

We first study some correspondence phenomena and then use these to motivate a completeness theorem. The correspondence for $D$ may be found in Shehtman, 1990, and the one for 4 in Esakia, 2001).

Theorem 1.27 *The correspondences of Figure 1.6 hold. That is, each scheme is valid on a topological space $\mathcal{X} = \langle X, \mathcal{O}\rangle$ iff $\mathcal{X}$ has the specified topological property.*

| Ax | formal statement | topological correspondent |
|---|---|---|
| $K$ | $B(\phi \to \psi) \to (B\phi \to B\psi)$ | (none) |
| $D$ | $B\phi \to \neg B \neg \phi$ | dense in itself |
| 4 | $B\phi \to BB\phi$ | $T_d$ separation property |
| 5 | $\neg B\phi \to B\neg B\phi$ | all derived sets are open |

*Figure 1.6.* Axiom schemes of $KD45$ again, this time with their topological correspondents.

*Proof* We fix a space $\mathcal{X} = \langle X, \mathcal{O} \rangle$ in this proof. The $K$ axioms are easily seen to be valid on $\mathcal{X}$; this amounts to the closure of topologies under intersection.

We turn to the correspondence for the $D$ axioms. If $\{w\}$ is open, consider the model obtained by $i(p) = \{w\}$. In it, $w \models Bp \wedge B\neg p$, counter to $D$. In the other direction, fix an interpretation $i$ and assume that $w \models B\phi \wedge B\neg\phi$. Then there are opens $U$ and $V$ containing $w$ such that at all points of $U$ besides $w$, $\phi$ holds, and all points of $V$ besides $w$, $\phi$ fails. Then the open set $U \cap V$ must be $\{w\}$.

Next, suppose that the space is $T_d$, so every derived set $d(A)$ is closed. Fix $i$ and $\phi$, and suppose that $w \models B\phi$. Let $A = i(\neg\phi)$, so that $w \in X \setminus d(A)$. Let $U$ be such that $w \in U \subseteq X \setminus d(A) = i(B\phi)$. $U$ shows that $w \in i(BB\phi)$. Going the other way, suppose that every 4 axiom holds under every interpretation. Let $A \subseteq X$. We check that $X \setminus d(A)$ is open. Let $i(p) = X \setminus A$, so that $i(Bp) = X \setminus d(A)$. Let $x \in X \setminus d(A)$. So

$$x \in i(Bp) \quad \subseteq \quad i(BBp) \quad = \quad X \setminus (d(X \setminus i(Bp)).$$

Thus there is an open set $U$ containing $x$ with the property that all points in $U$ are in $i(Bp)$, except possibly $x$. But $i(Bp) = X \setminus d(A)$. And as we know, $x$ itself belongs to $X \setminus d(A)$. So $U$ shows that $X \setminus d(A)$ is indeed open.

For the correspondence result for the 5 axioms, suppose that $d(A)$ is always open in $\mathcal{X}$. Fix $i$ and $\phi$, and suppose that $w \models \neg B\phi$. Let $A = i(\neg\phi)$, so that $i(B\phi) = X \setminus d(A)$ and $w \in d(A)$. Then $d(A)$ is open, contains $w$, and every point of it is outside of $i(B\phi)$. So $w \in i(B\neg B\phi)$. Going the other way, suppose that every 5 axiom holds under every interpretation. Let $A \subseteq X$. We check that $d(A)$ is open. As in the last part, let $i(p) = X \setminus A$. Let $x \in d(A) = i(\neg Bp)$. Then $x \in i(B\neg Bp)$. So there is an open set $U$ containing $x$ such that $U \setminus \{x\} \subseteq i(\neg Bp) = d(A)$. Hence $U \subseteq d(A)$. QED

There are a number of other general facts concerning this semantics which might be of interest. For one, no topological space validates $B\phi \to \phi$. (This is the scheme $T$ which we frowned on before.) But since our main goal is a completeness theorem, we shall not digress.

Recall the notion of a DSO space from Section 11.2, and also Proposition 1.24. We see from the correspondence results that every DSO space satisfies all of the $KD45$ axioms under every interpretation. In fact, it is not hard to check that the logic $KD45$ is sound for DSO spaces in the following sense. Let $T \cup \{\phi\}$ be any set of modal sentences (using $B$ as a modality). We say that $T \vdash \phi$ is *provable in* $KD45$ if there is a finite subset $\{\psi_1, \ldots, \psi_n\} \subseteq T$ such that $(\psi_1 \wedge \cdots \wedge \psi_n) \to \phi$ in $KD45$. It is easy to check that $KD45$ is sound for DSO spaces: if $T \vdash \phi$, then $T \models \phi$.

**The derived set logic as a fragment of a richer topologic-like system.** Much of our chapter has been concerned with the two-sorted ontology of points and sets, and with the resulting bimodal logical systems used in its study. Our work in this section appears to be on a different track. However, it is possible to unify the two. Consider the language $\mathcal{L}$ for points and sets from Section 4; this is the language with modalities $\Box$ and $K$. Instead of $K$, take as primitive an "everywhere but here" operator $[\neq]$ as in Section 3.2, but this time with semantics

$$p, u \models [\neq]\phi \quad \text{iff} \quad \text{for all } q \neq p \text{ in } u, \text{ we have } q, u \models \phi$$

Then the $K$ of topologic is recovered as an abbreviation. Call the resulting language $\mathcal{L}_2$. It is easy to check that the correspondence results of Proposition 1.5 still hold. We can translate the language of this section into $\mathcal{L}_2$ in the manner of Section 4.3 via

$$(B\phi)^* \quad \text{iff} \quad \Diamond[\neq]\phi^*$$

The logic $S4D$ must be modified a bit: the axioms $K\phi \to \Box\phi$ are no longer sound. This reflects the fact that the sentences in $\mathcal{L}_2$ are not always persistent.

Here is a sample of the use of this logic. We saw in Proposition 1.24 the general fact about derived sets $d(d(A)) \subseteq A \cup d(A)$. Here we show that this can be proved in the logic of set spaces together with the 'weak 4' scheme $(\phi \wedge [\neq]\phi) \to [\neq][\neq]\phi$. That is, we can prove

$$(\neg p \wedge \Box\langle\neq\rangle\Box\langle\neq\rangle p) \to \Box\langle\neq\rangle p$$

for atomic $p$. The first conjunct on the left and atomic permanence gives $\Box\neg p$. The second conjunct on the left easily gives $\Box\langle\neq\rangle\langle\neq\rangle p$. Using weak 4 and modal reasoning, we have $\Box\langle\neq\rangle p$.

But we must mention that $\mathcal{L}_2$ has not been studied previously. So all of the natural questions about the logic are open.

## 11.4 Completeness of $KD45$ for DSO spaces

The final result of our chapter is the following completeness theorem.

THEOREM 1.28 *The logical system $KD45$ is sound and strongly complete for DSO topological models. That is, if $T \models \phi$, then $T \vdash \phi$.*

*Proof* As the reader expects, the method of proof is to show that a set $S$ of modal sentences which is consistent in $KD45$ has a DSO topological model. So fix such a set $S$. By the relational completeness noted in Section 11.1, we have a model

$$M \quad = \quad \langle W, \to, i \rangle,$$

which is serial, transitive, and Euclidean; and $w^* \in W$ so that $w^* \models S$.

Let $N$ be the set of natural numbers. We build a topological space $\mathcal{M}$ whose set of points is $N \times W$. Here we meld the constructions in Examples 1.22 and 1.23. For the topology, we consider the following family $\mathcal{O}$ of subsets of $N \times W$:

$$U \in \mathcal{O} \quad \text{iff} \quad \begin{array}{l} \text{for all } (n,x) \in U, \text{ if } x \to y, \\ \text{then for all but finitely many } m,\ (m,y) \in U. \end{array}$$

It is easy to see that $\mathcal{O}$ is closed under arbitrary unions. So to check that $\mathcal{O}$ really is a topology we only need to consider binary intersections. This amounts to the fact that the intersection of two cofinite sets is again cofinite. As an example, take any $(n,x) \in N \times W$ and consider

$$U_{(n,x)} \quad = \quad \{(n,x)\} \cup \{(m,y) : m \in N \text{ and } x \to y\} \tag{1.8}$$

This is open by the transitivity of $\to$ in $M$.

We check that this space $\mathcal{M} = \langle N \times W, \mathcal{O} \rangle$ is a DSO space. No singletons are open, using the fact that the original $M$ is serial. To check that derived sets are open, and also for future use, we again compute derived sets:

$$d(A) \quad = \quad \begin{array}{l} \{(n,x) : n \in N \text{ and there is some } y \text{ such that } x \to y \\ \text{and infinitely many } m \text{ such that } (m,y) \in A\} \end{array}$$

(We leave the verification of this to the reader: see Examples 1.22 and 1.23 for parallel work.) We check that such a set $d(A)$ is open. Suppose that $(n,x) \in d(A)$. Fix some $y$ such that $x \to y$ and for which there are infinitely many $m$ such that $(m,y) \in A$. Let $U_{(n,x)}$ be as in

(1.8). We claim that $U_{(n,x)} \subseteq d(A)$; let $x \to z$, and consider $(p,z)$. Let $V$ be any open set containing $(p,z)$. The key here is that since $x \to y$ and $x \to z$, we have $z \to y$ by the Euclidean property. Almost all $j$ have the property that $(j,y) \in V$, and infinitely many of them have the property that $(j,y) \in A$. So there is one (indeed infinitely many) $j$ so that $(j,y) \in V \cap A$. Since $V$ is arbitrary, $(p,z) \in d(A)$, as desired.

So far we have converted our relational model $M$ into a DSO space $\mathcal{M}$. We consider it as a model via $i^{\mathcal{M}}(p) = N \times i^M(p)$.

LEMMA 1.29 (TRUTH LEMMA) *For all $(n,x) \in N \times W$, $x \models \phi$ in $M$ iff $(n,x) \models \phi$ in $\mathcal{M}$.*

*Proof* By induction on $\phi$. Only the inductive step for $B$ needs an argument. Suppose $x \models B\phi$ in $M$, so that for all $y$ such that $x \to y$, $y \models \phi$. By induction hypothesis, we see that for all $m$, $(m,y) \models \phi$ in $\mathcal{M}$. So each point of $U_{(n,x)}$ from (1.8) above satisfies $\phi$ in $\mathcal{M}$, except possibly for $(n,x)$. Thus our topological semantics of $B\phi$ tells us that $(n,x) \models B\phi$ in $\mathcal{M}$. Conversely, assume that $(n,x) \models B\phi$ in $\mathcal{M}$. Let $U$ be an open set containing $(n,x)$ with the property that all points in $U$ satisfy $\phi$ in $\mathcal{M}$, save possibly for $(n,x)$ itself. Let $x \to y$ in $M$. There is some $m$ such that $(m,y) \in U$. By induction hypothesis, $y \models \phi$ in $M$. And since $y$ is arbitrary, we see that in $M$, $x \models B\phi$. QED

This Truth Lemma completes the proof of Theorem 1.28. We started with a set $S$ and a world $w^* \in W$ such that $w^* \models S$. Then $\mathcal{M}$ is a DSO space, and in it $(0, w^*) \models S$. QED

Incidentally, the same proof shows the following results: $K4$ is complete for spaces in which every derived set is closed (due to Esakia, 2001), and $KD4$ is complete for spaces which are dense in themselves and in which every derived set is closed. For this last result, see Shehtman, 1990 and also Section 3 of Chapter **??** of this Handbook. The point is that the construction in the proof of Lemma 1.29 only used the Euclidean property in verifying that the space $\mathcal{M}$ has the property that all derived sets are open. We can drop this point, and instead verify directly that $\mathcal{M}$ is $T_1$. The rest of the arguments are the same. Perhaps the progenitor of all of these results is Esakia's completeness theorem for (finite) topological spaces using weak $K4$, the modal logic $K$ supplemented with the scheme $(\phi \wedge B\phi) \to BB\phi)$ (Esakia, 2001). See also Theorem 1.62 of Chapter **??**.

## 12. Other work connected to this chapter

Chapters **??** and **??** contain much material that is relevant to the concerns of this chapter.

We mention a few other papers which also are relevant, but which we did not discuss in the chapter. Davoren, 1999 and Davoren and Gore, 2002 study a propositional bimodal logic consisting of two $S4$ modalities $\Box$ and $[a]$, together with the interaction axiom scheme $\langle a \rangle \Box \phi \rightarrow \Box \langle a \rangle \phi$. In the intended semantics, the plain $\Box$ is given the McKinsey-Tarski interpretation of interior, while the labeled $[a]$ is given the standard Kripke semantics using a preorder $R_a$. The interaction axiom has the flavor of the Cross-Axiom, and here it expresses the property that the $R_a$ relation is lower semi-continuous with respect to the topology.

Pacuit and Parikh, 2005 study a non-spatial application of topologic, thereby showing that the area of application may indeed be wide. They consider a set of agents connected in a communication graph, and such that agent $i$ may receive information from agent $j$ only if there is an edge from $i$ to $j$. The logic which arises uses a language very similar to *topologic*, and it is shown that for each graph, the logic is decidable, and completely characterizes the graph. An application to the Valerie Plame affair, a notorious political affair from the early years of this century, is also described.

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