LOGICS BEYOND THE SYLLOGISTIC BOUNDARY

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- logics with variables, going beyond syllogistic logic
- logics for the sizes of sets

Tomorrow's topic:

- the monotonicity calculus

Let's say we're talking about a big table full of fruit.

Every sweet fruit is bigger than every ripe fruit
Every pineapple is bigger than every kumquat
Every non-pineapple is bigger than every unripe (fruit)

Every fruit bigger than some sweet fruit is bigger than every kumquat

If we assume (or believe, or know) all the sentences above the line, then we should do the same for the sentence below the line.

A computer can do this ... or can it?

Every sweet fruit is bigger than every ripe fruit
Every pineapple is bigger than every kumquat
Every non-pineapple is bigger than every unripe fruit

Every fruit bigger than some sweet fruit is bigger than every kumquat

---

all x all y (sweet(x) & ripe(y) $\rightarrow$ bigger(x,y)).
all x all y (pineapple(x) & kumquat(y) $\rightarrow$ bigger(x,y)).
all x all y (-pineapple(x) & -ripe(y) $\rightarrow$ bigger(x,y)).

---

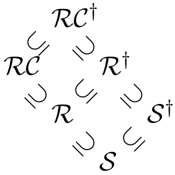
all x (exists y (sweet(y) & bigger(x,y))
  $\rightarrow$ (all y (kumquat(y) $\rightarrow$ bigger(x,y)))).

```
 % number = 1
% seconds = 0
% Interpretation of size 2

 c1 : 0

 c2 : 1

 c3 : 0

bigger :
       | 0 1
    ---+----
     0 | 0 1
     1 | 1 1

 kumquat :
        0 1
    --------
        1 0

 pineapple :
        0 1
    --------
        0 0

 ripe :
        0 1
    --------
        1 0
```

Output from Prover9

This is after adding the enthymeme about transitivity of "bigger than".

```
============================== prooftrans ============================
Prover9 (32) version Dec-2007, Dec 2007.
Process 49557 was started by larry on 129-79-94-213.dhcp-bl.indiana.edu,
Wed Sep 23 10:46:06 2009
The command was "/Users/larry/Desktop/Prover9-Mace4-v05B.app/Contents/Resources/bin-mac-
============================== end of head ===========================

============================== end of input ==========================

============================== PROOF =================================

% -------- Comments from original proof --------
% Proof 1 at 0.00 (+ 0.00) seconds.
% Length of proof is 19.
% Level of proof is 6.
% Maximum clause weight is 9.
% Given clauses 3.

1 (all x all y (sweet(x) & ripe(y) -> bigger(x,y))) # label(non_clause).  [assumption].
2 (all x all y (pineapple(x) & kumquat(y) -> bigger(x,y))) # label(non_clause).  [assump
3 (all x all y (-pineapple(x) & -ripe(y) -> bigger(x,y))) # label(non_clause).  [assumpt
4 (all x all y all z (bigger(x,y) & bigger(y,z) -> bigger(x,z))) # label(non_clause).  [
5 (all x ((exists y (sweet(y) & bigger(x,y))) -> (all y (kumquat(y) -> bigger(x,y))))) #
6 sweet(c2).  [deny(5)].
7 -sweet(x) | -ripe(y) | bigger(x,y).  [clausify(1)].
8 pineapple(x) | ripe(y) | bigger(x,y).  [clausify(3)].
9 -pineapple(x) | -kumquat(y) | bigger(x,y).  [clausify(2)].
10 ripe(x) | bigger(y,x) | -kumquat(z) | bigger(y,z).  [resolve(8,a,9,a)].
11 kumquat(c3).  [deny(5)].
12 ripe(x) | bigger(y,x) | bigger(y,c3).  [resolve(10,c,11,a)].
13 -ripe(x) | bigger(c2,x).  [resolve(6,a,7,a)].
14 -bigger(x,y) | -bigger(y,z) | bigger(x,z).  [clausify(4)].
15 bigger(c1,c2).  [deny(5)].
16 -bigger(c1,c3).  [deny(5)].
17 bigger(x,y) | bigger(x,c3) | bigger(c2,y).  [resolve(12,a,13,a)].
20 bigger(c2,c3).  [factor(17,b,c),merge(b)].
23 $F.  [ur(14,a,15,a,c,16,a),unit_del(a,20)].

============================== end of proof ==========================
}
```



$\mathcal{S}$	classical syllogistic
$\mathcal{S}^\dagger$	syllogistic with noun-level negation
$\mathcal{R}$	relational syllogistic
$\mathcal{R}^\dagger$	relational syllogistic with noun-level negations
$\mathcal{RC}$	relational syllogistic allowing subject NPs to be relative clauses
$\mathcal{RC}^\dagger$	relational syllogistic allowing subject NPs to be relative clauses and full noun-level negation

All p are q	*All p aren't q*
Some p are q	*Some p aren't q*

The interpretation is the natural one.

This language is called $\mathcal{S}$.

The language $\mathcal{S}^\dagger$ has complemented variables $\overline{p}$ on top of $\mathcal{S}$, and the interpretation is via set complement.

The (complete) logical system $\mathsf{S}^\dagger$ for $\mathcal{S}^\dagger$

$$\frac{}{\textit{All } p \textit{ are } p} \qquad \frac{\textit{Some } p \textit{ are } q}{\textit{Some } p \textit{ are } p} \qquad \frac{\textit{Some } p \textit{ are } q}{\textit{Some } q \textit{ are } p}$$

$$\frac{\textit{All } p \textit{ are } r \quad \textit{All } r \textit{ are } q}{\textit{All } p \textit{ are } q} \qquad \frac{\textit{Some } r \textit{ are } s \quad \textit{All } s \textit{ are } q}{\textit{Some } p \textit{ are } q}$$

$$\frac{\textit{All } q \textit{ are } \overline{q}}{\textit{All } q \textit{ are } p} \qquad \frac{\textit{All } \overline{q} \textit{ are } q}{\textit{All } p \textit{ are } q}$$

$$\frac{\textit{All } p \textit{ are } \overline{q}}{\textit{All } q \textit{ are } \overline{p}} \qquad \frac{\textit{Some } p \textit{ are } \overline{p}}{\varphi} \textit{ Ex falso quodlibet}$$

$$\begin{tikzcd}[column sep=small, row sep=small]
 & \mathcal{RC}^\dagger & & \\
\mathcal{RC} \arrow[ur, phantom, sloped, "\subseteq"] & & \mathcal{R}^\dagger \arrow[ul, phantom, sloped, "\subseteq"] & \\
 & \mathcal{R} \arrow[ul, phantom, sloped, "\subseteq"] \arrow[ur, phantom, sloped, "\subseteq"] & & \mathcal{S}^\dagger \arrow[ul, phantom, sloped, "\subseteq"] \\
 & & \mathcal{S} \arrow[ul, phantom, sloped, "\subseteq"] \arrow[ur, phantom, sloped, "\subseteq"] &
\end{tikzcd}$$

$\mathcal{S}$	classical syllogistic
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The language uses transitive verbs such as "see":

All p are q	*All p aren't q* $\equiv$ *No p are q*
Some p are q	*Some p aren't q*
All p see all q	*All p don't see all q* $\equiv$ *No p sees any q*
All p see some q	*All p don't see some q* $\equiv$ *No p sees all q*
Some p see all q	*Some p don't see any q*
Some p see some q	*Some p don't see some q*

The interpretation is the natural one, using the subject wide scope readings in the ambiguous cases.

This is $\mathcal{R}$.

The language $\mathcal{R}^{\dagger}$ has complemented variables $\overline{p}$ on top of $\mathcal{R}$.

Theorem

There are no purely syllogistic logical systems complete for $\mathcal{R}$. However, there is a logical system **R** which uses reductio ad absurdum

$$\dfrac{\begin{array}{c}\cdots[\bar{\varphi}] \quad \cdots[\varphi]\cdots \\ \vdots \\ \bot\end{array}}{\psi}\ \mathsf{RAA}$$

and which is complete.

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$$\dfrac{\begin{matrix}\cdots[\bar{\varphi}] \quad \cdots[\varphi]\cdots \\ \vdots \\ \bot\end{matrix}}{\psi}\ \mathsf{RAA}$$

and which is complete.

Theorem

There are no finite, complete syllogistic logical systems for $\mathcal{R}^\dagger$, even ones which allow *RAA*.

A complete system $\mathbf{R}$ for $\mathcal{R}$

On top of the system $\mathcal{S}$, one rule is missing, and so is *RAA*

$$\begin{gathered}
\textit{All } X^{\color{red}\downarrow} \textit{ (don't) see all } Y^{\color{red}\downarrow}\\
\textit{Some } X^{\color{red}\uparrow} \textit{ (don't) see all } Y^{\color{red}\downarrow}\\
\textit{All } X^{\color{red}\downarrow} \textit{ (don't) see some } Y^{\color{red}\uparrow}\\
\textit{Some } X^{\color{red}\uparrow} \textit{ (don't) see some } Y^{\color{red}\uparrow}
\end{gathered}$$

$$\frac{\textit{All } X \textit{ aren't } X}{\textit{All } X \textit{ see all } Y}$$

$$\frac{\textit{All } X \textit{ (don't) see all } Z \quad \textit{Some } Y \textit{ are } Z}{\textit{All } X \textit{ (don't) see some } Y}$$

$$\frac{\textit{All } Z \textit{ (don't) see all } Y \quad \textit{Some } X \textit{ are } Z}{\textit{Some } X \textit{ (don't) see all } Y}$$

$$\frac{\textit{Some } X \textit{ don't see some } Y \quad \textit{All } X \textit{ see all } Y}{\textit{No } X \textit{ are } X}$$

$$\frac{\textit{Some } X \textit{ (don't) see some } Y}{\textit{Some } Y \textit{ is a } Y}$$

$$\begin{tikzcd}[column sep=small, row sep=small]
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\mathcal{RC} \arrow[ur, phantom, "\subseteq" sloped] & & \mathcal{R}^\dagger \arrow[ul, phantom, "\supseteq" sloped] & \\
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$\mathcal{RC}$ allows the subject noun phrases to contain relative clauses of the form

who see all p	*who see some p*
who don't see all p	*who don't see some p*

$\mathcal{RC}^{\dagger}$ has full negation on nouns.

$$\frac{\textit{All p are q}}{\textit{All (see all q) (see all p)}}$$

$$\frac{\textit{All p are q}}{\textit{All (see some p) (see some q)}}$$

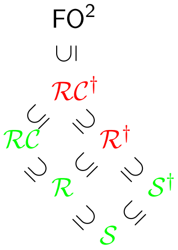
$$\frac{\textit{Some p are q}}{\textit{All (see all p) (see some q)}}$$

$$\frac{\textit{Some p see some q}}{\exists q}$$

$$\frac{\textit{All p aren't p}}{\textit{All (see all q) see all p}}$$

The Aristotle boundary

direct = syllogistic system, indirect = syllogistic system using RAA, complexity results are for validity



$\mathcal{S}$	direct	NLOGSPACE
$\mathcal{S}^\dagger$	direct	NLOGSPACE
$\mathcal{R}$	direct	NLOGSPACE
$\mathcal{R}^\dagger$	not even indirect	EXPTIME
$\mathcal{RC}$	indirect	Co-NPTIME [McA-G]
$\mathcal{RC}^\dagger$	not even indirect	EXPTIME
FO^2		Co-NEXPTIME [GKV]

I'll give a complete logical system for $\mathcal{RC}^{\dagger}$,
or rather for a closely related system with a different syntax.

The logical system will not be of the syllogistic type,
since this is impossible,
and instead it will use (something like) variables.

I'll then go on to add comparative adjectives to that system.

Expression	Variables	Syntax
unary atom	p, q	
binary atom	r, s	
constant	j, k	
unary literal	l	$p \mid \bar{p}$
binary literal	r	$s \mid \bar{s}$
set term	c, d, b	$l \mid \exists(c, r) \mid \forall(c, r)$
sentence	φ, ψ	$\forall(c, d) \mid \exists(c, d) \mid c(j) \mid r(j, k)$

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Think of the constants as proper names: John, Mary, etc.
the unary atoms as predicates like boys or boys,
the binary atoms by transitive verbs such as likes or sees.

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We form unary and binary literals using the bar notation.

We think of this as expressing classical negation.
So we take it to be involutive: $\bar{\bar{p}} = p$ and $\bar{\bar{s}} = s$.

The set terms in this language are the only recursive construct.
If b is read as boys and s as sees,
then one should read $\forall(b, s)$ as sees all boys,
and $\exists(b, s)$ as sees some boys.
Hence these set terms correspond to simple verb phrases.

We also allow negation on the atoms, so we have $\forall(b, \overline{s})$;
this can be read as fails to see all boys,
or (better) sees no boys or doesn't see any boys.

We also have $\exists(b, \overline{s})$, fails to see some boys.

But the recursion allows us to embed set terms, and so we have set terms like

$$\exists(\forall(\forall(b, \overline{s}), h), a)$$

which may be taken to symbolize
a verb phrase such as
admires someone who hates everyone who does not see any boy.

We should note that the relative clauses which can be obtained in this way are all "subject relatives", never "object relatives".

The language is too poor to express predicates like
λx.all boys see x.

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unary atom	p, q	
binary atom	s	
constant	j, k	
unary literal	l	$p \mid \bar{p}$
binary literal	r	$s \mid \bar{s}$
set term	b, c, d	$l \mid \exists(c, r) \mid \forall(c, r)$
sentence	φ, ψ	$\forall(c, d) \mid \exists(c, d) \mid c(j) \mid r(j, k)$

The sentences $\forall(b, c)$ and $\exists(b, c)$ can be read as *all b are c* and *some b are c*

We also have sentences using the constants, such as $\forall(g, s)(m)$, corresponding to Mary sees all girls.

But we are not able to say all girls see Mary; the syntax again is too weak.

Unary atoms appear to be one-place relation symbols, especially because we shall form sentences of the form $p(j)$.

However, we do not have sentences $p(x)$, since we have no variables at this point in the first place.

Similar remarks apply to binary atoms and two-place relation symbols.
So we chose to speak of atoms and not relation symbols

A structure (for this language $\mathcal{L}$) is a pair $\mathfrak{M} = \langle M, [\![\ ]\!]\rangle$, where M is a non-empty set, $[\![p]\!] \subseteq M$ for all unary atoms p, $[\![s]\!] \subseteq M^2$ for all binary atoms s and $[\![j]\!] \in M$ for all constants j.

Given a model $\mathfrak{M}$, we extend the interpretation function $[\![\ ]\!]$ to the rest of the language by setting

$$\begin{array}{lcl} [\![\overline{p}]\!] & = & M \setminus [\![p]\!] \\ [\![\overline{s}]\!] & = & M^2 \setminus [\![s]\!] \\ [\![\exists(l,r)]\!] & = & \{x \in M : \text{ for some } y \text{ such that } [\![l]\!](y), [\![r]\!](x,y)\} \\ [\![\forall(l,r)]\!] & = & \{x \in M : \text{ for all } y \text{ such that } [\![l]\!](y), [\![r]\!](x,y)\} \end{array}$$

We define the truth relation $\models$ between models and sentences by:

$$\begin{array}{lcl} \mathcal{M} \models \forall(c,d) & \text{iff} & [\![c]\!] \subseteq [\![d]\!] \\ \mathcal{M} \models \exists(b,c) & \text{iff} & [\![c]\!] \cap [\![d]\!] \neq \emptyset \\ \mathcal{M} \models c(j) & \text{iff} & [\![c]\!]([\![j]\!]) \\ \mathcal{M} \models r(j,k) & \text{iff} & [\![r]\!]([\![j]\!],[\![k]\!]) \end{array}$$

If Γ is a set of formulas, we write $\mathcal{M} \models \Gamma$ if for all $\varphi \in \Gamma$, $\mathcal{M} \models \varphi$.

We consider a simple case, with one unary atom p, one binary atom s, and two constants j and k.

Consider the following model.
We set $M = \{w, x, y, z\}$, and $[\![p]\!] = \{w, x, y\}$.
For the relation symbol, s, we take the arrows below:

$$\begin{tikzcd}
w \arrow[r] \arrow[dr] & x \arrow[d] \arrow[dl] \\
y \arrow[u] \arrow[r, leftrightarrow] & z \arrow[loop right]
\end{tikzcd}$$

For example, $[\![\overline{p}]\!] = \{z\}$, $[\![\forall(p, s)]\!] = \emptyset$, $[\![\exists(\overline{p}, s)]\!] = M$, and $[\![\exists(\forall(p, \overline{s}), s)]\!] = \emptyset$.
Here are two $\mathcal{L}$-sentences true in $\mathcal{M}$: $\forall(p, \exists(\overline{p}, s))$ and $\forall(\exists(\forall(p, \overline{s}), s), \overline{p})$.

Example

We consider a simple case, with one unary atom p, one binary atom s, and two constants j and k.

Consider the following model.
We set $M = \{w, x, y, z\}$, and $[\![p]\!] = \{w, x, y\}$.
For the relation symbol, s, we take the arrows below:

$$
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$$

For example, $[\![\overline{p}]\!] = \{z\}$, $[\![\forall(p, s)]\!] = \emptyset$, $[\![\exists(\overline{p}, s)]\!] = M$, and $[\![\exists(\forall(p, \overline{s}), s)]\!] = \emptyset$.
Now set $[\![j]\!] = w$ and $[\![k]\!] = x$.
We get additional sentences true in $\mathcal{M}$ such as $s(j, k)$, $\overline{s}(k, j)$, and $\exists(\overline{p}, s)(k)$.

General sentences in this fragment are what usually are called formulas.
We prefer to change the standard terminology to make the point that here, sentences are not built from formulas by quantification.
Sentences in our sense do not have variable occurrences.
But general sentences do allow variables.

Expression	Variables	Syntax
individual variable	x, y	
individual term	t, u	$x \mid j$
general sentence	α	$\varphi \mid c(t) \mid r(t,u) \mid \bot$

It will turn out that for this fragment, only two variables are needed.

$$\frac{c(t) \quad \forall(c,d)}{d(t)}\ \forall E \qquad\qquad \frac{c(u) \quad \forall(c,r)(t)}{r(t,u)}\ \forall E$$

$$\frac{c(t) \quad d(t)}{\exists(c,d)}\ \exists I \qquad\qquad \frac{r(t,u) \quad c(u)}{\exists(c,r)(t)}\ \exists I$$

Proof system: the second half of the rules

$$\dfrac{\begin{array}{c}[c(x)]\\ \vdots \\ d(x)\end{array}}{\forall(c,d)}\ \forall I \qquad \dfrac{\begin{array}{c}[c(x)]\\ \vdots \\ r(t,x)\end{array}}{\forall(c,r)(t)}\ \forall I$$

$$\dfrac{\exists(c,d) \qquad \begin{array}{c}[c(x)] \quad [d(x)]\\ \vdots \\ \alpha\end{array}}{\alpha}\ \exists E \qquad \dfrac{\exists(c,r)(t) \qquad \begin{array}{c}[c(x)] \quad [r(t,x)]\\ \vdots \\ \alpha\end{array}}{\alpha}\ \exists E$$

$$\dfrac{\alpha \quad \overline{\alpha}}{\bot}\ \bot I \qquad \dfrac{\begin{array}{c}[\overline{\varphi}]\\ \vdots \\ \bot\end{array}}{\varphi}\ RAA$$

Proof system: side conditions

$$
\dfrac{\begin{array}{c}[c(x)]\\ \vdots \\ d(x)\end{array}}{\forall(c,d)}\ \forall I
\qquad\qquad
\dfrac{\begin{array}{c}[c(x)]\\ \vdots \\ r(t,x)\end{array}}{\forall(c,r)(t)}\ \forall I
$$

$$
\dfrac{\exists(c,d) \qquad \begin{array}{c}[c(x)] \quad [d(x)]\\ \vdots \\ \alpha\end{array}}{\alpha}\ \exists E
\qquad\qquad
\dfrac{\exists(c,r)(t) \qquad \begin{array}{c}[c(x)] \quad [r(t,x)]\\ \vdots \\ \alpha\end{array}}{\alpha}\ \exists E
$$

In $(\forall I)$, x must not occur free in any uncanceled hypothesis.

In $(\exists E)$, the variable x must not occur free in the conclusion α or in any uncanceled hypothesis in the subderivation of α.

In contrast to usual first-order natural deduction systems, there are no side conditions on the rules $(\forall E)$ and $(\exists I)$.

Example 1: the classical syllogism Darii: $\forall(b,d), \exists(c,b) \vdash \exists(c,d)$

$$
\begin{prooftree}
\AxiomC{$\exists(c,b)$}
\AxiomC{$[b(x)]^1$}
\AxiomC{$\forall(b,d)$}
\RightLabel{$\forall E$}
\BinaryInfC{$d(x)$}
\AxiomC{$[c(x)]^1$}
\RightLabel{$\exists I$}
\BinaryInfC{$\exists(c,d)$}
\RightLabel{$\exists E^1$}
\BinaryInfC{$\exists(c,d)$}
\end{prooftree}
$$

Example 2: $\forall(c, d) \vdash \forall(\exists(c, r), \exists(d, r))$

IF ALL WATCHES ARE EXPENSIVE ITEMS, THEN EVERYONE WHO OWNS A WATCH OWNS AN EXPENSIVE ITEM

$$
\dfrac{[\exists(c,r)(x)]^2 \quad \dfrac{[r(x,y)]^1 \quad \dfrac{[c(y)]^1 \quad \forall(c,d)}{d(y)}\ \forall E}{\exists(d,r)(x)}\ \exists I}{\dfrac{\exists(d,r)(x)}{\forall(\exists(c,r),\exists(d,r))}\ \forall I^2}\ \exists E^1
$$

EXAMPLE 2: $\forall(c,d) \vdash \forall(\exists(c,r),\exists(d,r))$

IF ALL WATCHES ARE EXPENSIVE ITEMS, THEN EVERYONE WHO OWNS A WATCH OWNS AN EXPENSIVE ITEM

Here is the same derivation, rendered in Fitch-Jaskowski style:

1		$\forall(c,d)$	hyp
2	x	$\exists(c,r)(x)$	hyp
3		$c(y)$	$\exists E$, 2
4		$r(x,y)$	$\exists E$, 2
5		$d(y)$	$\forall E$, 1, 3
6		$\exists(d,r)(x)$	$\exists I$, 4, 5
7		$\forall(\exists(c,r),\exists(d,r))$	$\forall I$, 1–6

One could write this out in plain English, as Fitch 1973 does, and then this and syllogistic logic would be a didactic alternative to FOL.

Example 3: $\forall(c, \overline{c}) \vdash \forall(d, \forall(c, r))$

If there are no watches, then Everyone owns all watches

$$\dfrac{[d(x)]^2 \qquad \dfrac{\dfrac{\dfrac{\dfrac{[c(y)]^1 \quad \forall(c,\overline{c})}{\overline{c}(y)}\,\forall E \qquad [c(y)]^1}{\bot}\,\bot I}{r(x,y)}\,RAA}{\forall(c,r)(x)}\,\forall I^1}{\forall(d,\forall(c,r))}\,\forall I^2$$

1		Every sweet fruit is bigger than every ripe fruit	hyp
2		Every pineapple is bigger than every kumquat	hyp
3		Every $\overline{\text{pineapple}}$ is bigger than every $\overline{\text{ripe fruit}}$	hyp
4	x	x is a sweet fruit	hyp
5		x is bigger than every ripe fruit	$\forall E$, 1, 4
6		x is a pineapple	hyp
7		x is bigger than every kumquat	$\forall E$, 2, 6
8		x is a $\overline{\text{pineapple}}$	hyp
9		x is bigger than every $\overline{\text{ripe fruit}}$	$\forall E$, 3, 8
10	y	y is a kumquat	hyp
11		y is a ripe fruit	hyp
12		x is bigger than y	$\forall E$, 5, 11
13		y is a $\overline{\text{ripe fruit}}$	hyp
14		x is bigger than y	$\forall E$, 9, 13
15		x is bigger than y	cases, 13–14, 11–12
16		x is bigger than every kumquat	$\forall I$, 10–15
17		x is bigger than every kumquat	cases, 6–7, 8–16
18		Every sweet fruit is bigger than every kumquat	$\forall I$, 4–17

Theorem

If $\Gamma \models \varphi$, then $\Gamma \vdash \varphi$.

If Γ is finite and consistent,
then Γ has a model of size at most 2^{2n},
where n is the number of set terms in Γ.

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If $\Gamma \models \varphi$, then $\Gamma \vdash \varphi$.

If Γ is finite and consistent,
then Γ has a model of size at most 2^{2n},
where n is the number of set terms in Γ.

The completeness result can be proved with a standard Henkin-style argument.

The easiest proof of the finite model property is to just translate to FO^2.
But the better bound of 2^{2n} requires more.
One can get it by adapting modal filtration,
a kind of quotient with finite index.

The best result is the EXPTIME-completeness of the validity problem.
This takes a lot more work, and two different proofs are known.

We extend our language $\mathcal{L}$ to a language $\mathcal{L}(adj)$ by taking a basic set of comparative adjective phrases in the base.

In this talk the only example will be bigger than.

We use a as a variable to range over the comparative adjectives.
For the syntax, we take adjectives to be binary atoms,
just as transitive verbs are.

Binary literals are expressions of the form
s, $\overline{s}$, a, or $\overline{a}$.

We require that (in every model $\mathcal{M}$)
for an adjective a, $[\![a]\!]$ must be a transitive relation.

(We could also require $[\![a]\!]$ to be irreflexive, but I won't do that in this talk.)

We add a rule:

$$\frac{a(t_1, t_2) \quad a(t_2, t_3)}{a(t_1, t_3)}\ \text{trans}$$

This rule is added for all comparatives a.

$$
\dfrac{[\exists(\mathsf{sweet},\mathsf{bigger})(x)]^3 \qquad \dfrac{\dfrac{[\mathsf{bigger}(x,y)]^2 \qquad \dfrac{[\mathsf{kumquat}(z)]^1 \qquad \dfrac{[\mathsf{sweet}(y)]^2 \qquad \begin{array}{c}\vdots\\ \forall(\mathsf{sweet},\forall(\mathsf{kumquat},\mathsf{bigger}))\end{array}}{\forall(\mathsf{kumquat},\mathsf{bigger})(y)}\ \forall E}{\mathsf{bigger}(y,z)}\ \forall E}{\mathsf{bigger}(x,z)}\ \textcolor{red}{\mathsf{trans}}}{\forall(\mathsf{kumquat},\mathsf{bigger})(x)}\ \forall I^1}{\dfrac{\forall(\mathsf{kumquat},\mathsf{bigger})(x)}{\forall(\exists(\mathsf{sweet},\mathsf{bigger}),\forall(\mathsf{kumquat},\mathsf{bigger}))}\ \forall I^3}\ \exists E^2
$$

Every sweet fruit is bigger than every ripe fruit
Every pineapple is bigger than every kumquat
Every non-pineapple is bigger than every unripe fruit

Every sweet fruit is bigger than every kumquat

Every fruit bigger than some sweet fruit is bigger than every kumquat

THEOREM

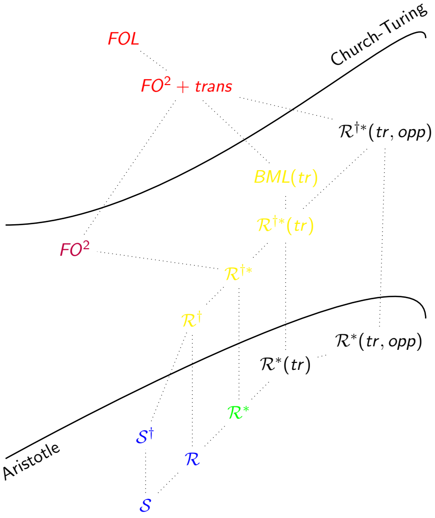
If $\Gamma \models \varphi$, then $\Gamma \vdash \varphi$.

The validity problem for this logic is also in EXPTIME.

The complexity result follows from a result of Lutz and Sattler (2001) on the complexity of boolean modal logic on transitive frames, but it can also be obtained using filtration.

Complexity

(mostly) best possible results on the validity problem



undecidable
Church 1936
Grädel, Otto, Rosen 1999

in co-NEXPTIME
EXPTIME
Lutz & Sattler 2001

Co-NEXPTIME
Grädel, Kolaitis, Vardi '97
EXPTIME
Pratt-Hartmann 2004

lower bounds also open

Co-NP
McAllester & Givan 1992

NLOGSPACE

One can also add relational converses to the syntax.

In this way, one could render sentences like all girls see Mary:

$$\forall(\mathsf{girl}, \mathsf{see}^{-1})(\mathsf{m})$$

and also sentences with object relative clauses.

It is easy to get a complete proof system,
based on what we've seen.

And the filtration-style work goes through to show that the satisfiability problem is in NExpTime.

But it is not clear that this satisfiability problem is in ExpTime
(it should be).

$$\frac{c(t) \quad \forall(c,d)}{d(t)}\ \forall E \qquad\qquad \frac{c(u) \quad \forall(c,r)(t)}{r(t,u)}\ \forall E$$

$$\frac{c(t) \quad d(t)}{\exists(c,d)}\ \exists I \qquad\qquad \frac{r(t,u) \quad c(u)}{\exists(c,r)(t)}\ \exists I$$

$$\dfrac{\begin{array}{c}[c(x)] \\ \vdots \\ d(x)\end{array}}{\forall(c,d)}\ \forall I \qquad\qquad \dfrac{\begin{array}{c}[c(x)] \\ \vdots \\ r(t,x)\end{array}}{\forall(c,r)(t)}\ \forall I$$

$$\dfrac{\exists(c,d) \quad \begin{array}{c}[c(x)]\quad[d(x)] \\ \vdots \\ \alpha\end{array}}{\alpha}\ \exists E \qquad\qquad \dfrac{\exists(c,r)(t) \quad \begin{array}{c}[c(x)]\quad[r(t,x)] \\ \vdots \\ \alpha\end{array}}{\alpha}\ \exists E$$

$$\begin{array}{c} r(j,k) \quad s(j,k) \\ \hline\hline (r \wedge s)(j,k) \end{array}\ \wedge \qquad\qquad \begin{array}{c} r^{-1}(k,j) \\ \hline\hline r(j,k) \end{array}\ \text{inv}$$

$$\frac{\alpha \quad \overline{\alpha}}{\perp}\ \perp I \qquad\qquad \dfrac{\begin{array}{c}[\overline{\varphi}] \\ \vdots \\ \perp\end{array}}{\varphi}\ RAA$$

$$\frac{\textit{Bao is seen and heard by every student} \quad \textit{Amina is a student}}{\textit{Amina sees Bao}}$$

$$\frac{\textit{Bao is seen and heard by every student} \quad \textit{Amina is a student}}{\textit{Amina sees Bao}}$$

$$\cfrac{\cfrac{\cfrac{\forall(\mathsf{student}, \mathsf{see}^{-1} \wedge \mathsf{hear}^{-1})(\mathsf{Bao}) \quad \mathsf{student}(\mathsf{Amina})}{(\mathsf{see} \wedge \mathsf{hear})^{-1}(\mathsf{Bao}, \mathsf{Amina})}\ \forall E}{(\mathsf{see} \wedge \mathsf{hear})(\mathsf{Amina}, \mathsf{Bao})}\ \mathsf{inv}}{\mathsf{see}(\mathsf{Amina}, \mathsf{Bao})}\ \wedge$$

Every wolf is a animal. Every fox is a animal.
Every bird is a animal. Every caterpillar is a animal.
Every snail is a animal. Some wolf exists.
Some fox exists. Some bird exists. Some caterpillar exists. Some snail exists. Every grain is a plant. Some grain exists.
Every caterpillar is smaller than every bird.
Every snail is smaller than every bird.
Every bird is smaller than every fox.
Every fox is smaller than every wolf.
It is not true that some wolf eats some fox. It is not true that some wolf eats some grain.
Every bird eats every caterpillar.
It is not true that some bird eats some snail. Every caterpillar eats some plant. Every snail eats some plant.
Every animal eats every plant or every animal that is smaller than itself and eats some plant.
Show that Every animal eats some animal that eats some grain.

Transitivity should not be treated as a meaning postulate, since this can't be expressed!

Anyways, it takes 3 variables, and 3 variable FOL is undecidable.

A final observation

Transitivity should not be treated as a meaning postulate, since this can't be expressed!

Anyways, it takes 3 variables, and 3 variable FOL is undecidable.

Instead, it is a proof rule, much like the proof rules yesterday.

Speaking of transitivity, let's take a musical break.