

Q520: Linear Algebra Homework, On Eigenvectors, etc.
Due: Thursday, April 3

It looks like this homework is very long, but all of the problems are short.
Be sure to note that this homework set continues on the back side of this page.

1. When dealing with vector space representations of concepts, one often is interested in a similarity measure between vectors. If v and w are vectors, we define the similarity to be the angle θ so that

$$\cos \theta = \frac{v \cdot w}{\|v\| \cdot \|w\|}$$

(On the bottom, we have the product of the lengths of v and w .)

Explain why the following is true: if we multiply one of the two vectors v and w by a positive real number, the angle between is not changed. That is, the angle between v and w is the same as the angle between cv and w , whenever $c > 0$.

2. An *orthogonal matrix* is a square matrix M with the property that each column vector x of M has length 1, and if x and y are different column vectors, then $x \cdot y = 0$. Show that if M is orthogonal, then $M^{-1} = M^{tr}$. [To do this, show that $M \times M^{tr} = I$.]

Is the converse of this true? That is, if someone tells you that M is an $n \times n$ matrix and $M \times M^{tr} = I$, does it have to be the case that M is orthogonal?

3. Let v be a column vector of length n , and let M be an orthogonal $n \times n$ matrix. Show that $\|v\| = \|Mv\|$. [Hint: express the matrix product Mv in terms of the dot product. Then use results we have seen. The verification here should be short.]

4. Is the following true or false (and why?): The vector $\begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$ is an eigenvector of the

$$\text{matrix } \begin{pmatrix} 2 & -1 & -1 \\ 6 & 3 & 1 \\ 2 & 1 & 5 \end{pmatrix}.$$

5. What are the eigenvalues of the matrix below?

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

6. What is the definition of the rank of a matrix? Give an example of a 4×4 matrix with rank 3.

7. Let M be an $n \times n$ symmetric matrix, and let the eigenvalues of M be $\lambda_1, \lambda_2, \dots, \lambda_n$. Let v^1, \dots, v^n be a basis for R^n consisting of eigenvectors, with v^1 and eigenvector of λ_1, \dots, v^i and eigenvector of λ_i , etc. Note that the list of λ 's can have repeats, but the list of eigenvectors cannot have repeats because it is supposed to make a basis.

Each v^i is a vector, and so let's write it as a_1^i, \dots, a_n^i . Note that these a 's are all numbers.

By the Spectral Theorem,

$$M = \begin{pmatrix} a_1^1 & a_1^2 & \cdots & a_1^n \\ a_2^1 & a_2^2 & \cdots & a_2^n \\ \vdots & \vdots & \ddots & \vdots \\ a_n^1 & a_n^2 & \cdots & a_n^n \end{pmatrix} \times \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \times \begin{pmatrix} a_1^1 & a_2^1 & \cdots & a_n^1 \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^n & a_2^n & \cdots & a_n^n \end{pmatrix}$$

Write out in full the following matrix M_1 .

$$M_1 = \begin{pmatrix} a_1^1 & a_1^2 & \cdots & a_1^n \\ a_2^1 & a_2^2 & \cdots & a_2^n \\ \vdots & \vdots & \ddots & \vdots \\ a_n^1 & a_n^2 & \cdots & a_n^n \end{pmatrix} \times \begin{pmatrix} \lambda_1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} \times \begin{pmatrix} a_1^1 & a_2^1 & \cdots & a_n^1 \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^n & a_2^n & \cdots & a_n^n \end{pmatrix}$$