Monotonicity and Polarity

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Why this topic?

There are several reasons why people interested in the topic of natural logic are also interested in our last topic:

- Most of the rules of inference which we have seen in the course are monotonicity principles.
- In trying for the logic of bigger fragments, monotonicity gives a first shot
  at a set of sound principles.
  This might well be the best one could hope to do.
- The study of negative polarity items is interesting and important.
- Even when one doesn't have a formal semantics (or even a formal grammar!),
  it seems that similar phenomena are present and useful to study.
  For example, factives.
- There are also reasons from cognitive science, where one sees proposals that use monotonicity.

Sources

The course notes mention papers which cover most of the work in the area:

van Benthem 1986, 1991, 2008.

Sánchez Valencia, 1991

Fyodorov, Winter, and Francez 2000

Bernardi 2002

Zamansky, Francez, and Winter 2006

van Eijck 2007

My presentation is a “subset of the intersection” of these papers, reworked in terms of opposites of posets.

One motivation, based on Review from Wednesday: relational syllogistic logic

Most rules are monotonicty principles

$$
\begin{array}{ll}
\exists(p^{\uparrow}, q^{\uparrow}) & \forall(p^{\downarrow}, q^{\uparrow}) \\
\exists(p^{\uparrow}, \forall(q^{\downarrow}, t)) & \exists(p^{\uparrow}, \exists(q^{\uparrow}, t)) \\
\forall(p^{\downarrow}, \forall(q^{\downarrow}, t)) & \forall(p^{\downarrow}, \exists(q^{\uparrow}, t))
\end{array}
$$

So we want a clear explanation of the $^{\uparrow}$ and $^{\downarrow}$ notation.
Another primary goal is to put all of these principles under one general roof in as simple a way as possible.

Another motivation, based on the word any

- If anyone here can can understand John's book, then anyone here can understand John's book.

One might want to say that the meaning of any depends on the "polarity" of the context.

But what exactly does this mean?

What we'll do

I am mainly concerned in this lecture in presenting the foundational aspects of this topic.

These have not been exposed before, I think, except for the pioneering work of Johan van Benthem
and the 1991 dissertation of Victor Sánchez.

And so I am interested in finding a workable presentation of the topic.

Posets

A poset is a pair $\mathbb{P} = (P, \leq)$ consisting of a set P together with a relation $\leq$ which is
reflexive, transitive, and anti-symmetric.

This means that the following hold:

1. $p \leq p$ for all $p \in P$.
2. If $p \leq q$ and $q \leq r$, then $p \leq r$.
3. If $p \leq q$ and $q \leq p$, then $p = q$.

Actually, the requirement of anti-symmetry plays no role, and we could just as easily drop it and generalize to the larger class of preorders.

Examples of Posets

For any set X, we have a poset $\mathbb{X} = (X, \leq)$, where $x \leq y$ iff $x = y$. This is called the flat poset on X.

More interestingly, for any set X we have a power set poset $\mathcal{P}(X)$ whose set part is the set of subsets of X, and where
$p \leq q$ iff p is a subset of q.

Another poset is $\mathbb{2} = \{\mathsf{false}, \mathsf{true}\}$ with false $\leq$ true.
(In fact, $\mathbb{2}$ is (isomorphic to) the power set poset of a one-point set.)

Maps between posets

The natural class of maps between posets $\mathbb{P}$ and $\mathbb{Q}$
is the set of monotone functions:
the functions $f : P \rightarrow Q$ with the property that

$$\text{if } p \leq q \text{ in } \mathbb{P}, \text{ then } f(p) \leq f(q) \text{ in } \mathbb{Q}.$$

We write $[\mathbb{P}, \mathbb{Q}]$ for the set of monotone functions from $\mathbb{P}$ to $\mathbb{Q}$.
And we always take this to be a poset with the pointwise order:

$$f \leq g \text{ in } [\mathbb{P}, \mathbb{Q}] \quad \text{iff} \quad \text{for all } p \in P,\ f(p) \leq g(p) \text{ in } \mathbb{Q}$$

Monotone and Antitone functions

Monotone $f : \mathbb{P} \to \mathbb{Q}$

If $p \leq q$ in $\mathbb{P}$, then $f(p) \leq f(q)$ in $\mathbb{Q}$.

Antitone $f : \mathbb{P} \to \mathbb{Q}$

If $p \leq q$ in $\mathbb{P}$, then $f(q) \leq f(p)$ in $\mathbb{Q}$.

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If $p \leq q$ in $\mathbb{P}$, then $f(q) \leq f(p)$ in $\mathbb{Q}$.

For example:
The identity is a monotone function on any poset.
The negation operation on truth values is antitone: $\neg : \mathbb{2} \to \mathbb{2}$.

Opposites of posets

We can express things in a more elegant way using the concept of the opposite poset $\mathbb{P}^{op}$ of a given poset $\mathbb{P}$. This is the poset with the same set part, and with

$$p \leq q \text{ in } \mathbb{P}^{op} \quad \text{iff} \quad q \leq p \text{ in } \mathbb{P}$$

Monotone and antitone, revisited

$$\begin{array}{ll} & f : \mathbb{P} \to \mathbb{Q} \text{ is antitone} \\ \textit{iff} & f : \mathbb{P} \to \mathbb{Q}^{op} \text{ is monotone} \end{array}$$

Henceforth, when we write $f : \mathbb{P} \to \mathbb{Q}$, it will always mean that f is monotone.

Facts

Proposition

For all posets $\mathbb{P}$, $\mathbb{Q}$, and $\mathbb{R}$:

1. $(\mathbb{P}^{op})^{op} = \mathbb{P}$.
2. $[\mathbb{P}, \mathbb{Q}^{op}] = [\mathbb{P}^{op}, \mathbb{Q}]^{op}$.
3. If $f : \mathbb{P} \to \mathbb{Q}$ and $g : \mathbb{Q} \to \mathbb{R}$, then $g \cdot f : \mathbb{P} \to \mathbb{R}$.
4. If $f : \mathbb{P} \to \mathbb{Q}^{op}$ and $g : \mathbb{Q} \to \mathbb{R}^{op}$, then $g \cdot f : \mathbb{P} \to \mathbb{R}$.

For part (1), note that $\mathbb{P}$ and its opposite have the same points. So $\mathbb{P}$ and its double opposite again have the same points.

We also check that the orders on $\mathbb{P}$ and $(\mathbb{P}^{op})^{op}$ are the same. If $x \leq y$ in $\mathbb{P}$ iff $y \leq x$ in $\mathbb{P}^{op}$ iff $x \leq y$ in $(\mathbb{P}^{op})^{op}$.

Proof that $[\mathbb{P}, \mathbb{Q}^{op}] = [\mathbb{P}^{op}, \mathbb{Q}]^{op}$.

Suppose that $f : \mathbb{P} \rightarrow \mathbb{Q}^{op}$.
Then f is a set function from P to Q.
To see that f is a monotone function from $\mathbb{P}^{op}$ to $\mathbb{Q}$,
let $x \leq y$ in $\mathbb{P}^{op}$.
Then $y \leq x$ in $\mathbb{P}$. So $f(y) \leq f(x)$ in $\mathbb{Q}^{op}$.
And thus $f(x) \leq f(y)$ in $\mathbb{Q}$, as desired.

This verifies that as sets, $[\mathbb{P}, \mathbb{Q}^{op}] = [\mathbb{P}^{op}, \mathbb{Q}]$.

To show that as posets $[\mathbb{P}, \mathbb{Q}^{op}] = [\mathbb{P}^{op}, \mathbb{Q}]^{op}$,
we must check that if $f \leq g$ in $[\mathbb{P}, \mathbb{Q}^{op}]$, then $g \leq f$ in $[\mathbb{P}^{op}, \mathbb{Q}]$.

For this, let $p \in \mathbb{P}^{op}$; so $p \in P$. Then $f(p) \leq g(p)$ in $\mathbb{Q}^{op}$.
Thus $g(p) \leq f(p)$ in $\mathbb{Q}$.
This for all $p \in P$ shows that $g \leq f$ in $[\mathbb{P}^{op}, \mathbb{Q}]$.

Proof, continued

Proposition

For all posets $\mathbb{P}$, $\mathbb{Q}$, and $\mathbb{R}$:

1. $(\mathbb{P}^{op})^{op} = \mathbb{P}$.
2. $[\mathbb{P}, \mathbb{Q}^{op}] = [\mathbb{P}^{op}, \mathbb{Q}]^{op}$.
3. If $f : \mathbb{P} \to \mathbb{Q}$ and $g : \mathbb{Q} \to \mathbb{R}$, then $g \cdot f : \mathbb{P} \to \mathbb{R}$.
4. If $f : \mathbb{P} \to \mathbb{Q}^{op}$ and $g : \mathbb{Q} \to \mathbb{R}^{op}$, then $g \cdot f : \mathbb{P} \to \mathbb{R}$.

For the third part, assume $x \leq y$.
Then $f(x) \leq f(y)$, and so $g(f(x)) \leq g(f(y))$.

Here is the reasoning in the last part:
By part (2), $g : \mathbb{Q}^{op} \to \mathbb{R}$.
So by part (3), $g \cdot f : \mathbb{P} \to \mathbb{R}$.

Notation later in this talk

We'll sometimes write

$\mathbb{P}^{-}$	for the opposite of $\mathbb{P}$
$\mathbb{P}^{+}$	for the poset $\mathbb{P}$ itself

Examples

Let X be any set.
We use letters like p and q to denote elements of $[\mathbb{X}, \mathbb{2}]$.
The set of (montone) functions from $\mathbb{X}$ to $\mathbb{2}$ is in one-to-one correspondence with the set of subsets of X.
Define

$$\begin{aligned} \text{every} : & \quad [\mathbb{X}, \mathbb{2}] \to [[\mathbb{X}, \mathbb{2}], \mathbb{2}]^{op} \\ \text{some} : & \quad [\mathbb{X}, \mathbb{2}] \to [[\mathbb{X}, \mathbb{2}], \mathbb{2}] \\ \text{no} : & \quad [\mathbb{X}, \mathbb{2}] \to [[\mathbb{X}, \mathbb{2}], \mathbb{2}^{op}] \end{aligned}$$

as follows:

$$\text{every}(p)(q) = \begin{cases} \text{true} & \text{if } p \leq q \\ \text{false} & \text{otherwise} \end{cases}$$

$$\text{some}(p)(q) = \begin{cases} \text{true} & \text{if for some } x \in X,\ p(x) = \text{true} = q(x) \\ \text{false} & \text{otherwise} \end{cases}$$

$$\text{no}(p)(q) = \begin{cases} \text{true} & \text{if some}(p)(q) = \text{false} \\ \text{false} & \text{if some}(p)(q) = \text{true} \end{cases}$$

Verification that no : $[\mathbb{X}, \mathbb{2}] \to [[\mathbb{X}, \mathbb{2}], \mathbb{2}^{op}]$

Let $p \le p'$ in $[\mathbb{X}, \mathbb{2}]$.
We show that $\mathsf{no}(p) \le \mathsf{no}(p')$ in $[[\mathbb{X}, \mathbb{2}], \mathbb{2}^{op}]$.

Let $q \in [\mathbb{X}, \mathbb{2}]$.
By monotonicity of some, $\mathsf{some}(p)(q) \le \mathsf{some}(p')(q)$ in $\mathbb{2}$.
Thus in $\mathbb{2}$,

$$\mathsf{no}(p)(q) \quad = \quad \neg\mathsf{some}(p)(q) \quad \ge \quad \neg\mathsf{some}(p')(q) \quad = \quad \mathsf{no}(p')(q).$$

And so in $\mathbb{2}^{op}$,

$$\mathsf{no}(p)(q) \quad \le \quad \mathsf{no}(p')(q).$$

This for all q shows that $\mathsf{no}(p) \le \mathsf{no}(p')$ in $[[\mathbb{X}, \mathbb{2}], \mathbb{2}^{op}]$.

Something to think about

Why is it true that for any set X and any poset $\mathbb{P}$

$$[\mathbb{X}, \mathbb{P}] \quad = \quad [\mathbb{X}, \mathbb{P}]^{op} \text{ ?}$$

Exercise

For any set X, show that

$$[\mathbb{X}, \mathbb{2}] \cong \mathcal{P}(X).$$

In more detail, consider the function f that takes $p \in [\mathbb{X}, \mathbb{2}]$ to $\{x \in X : p(x) = \text{true}\}$.

Show that f is a one-to-one correspondence with the property that $p \leq q$ iff $f(p) \leq f(q)$.

Question

Which of the following is true, and which is false?

1. $[\mathbb{P}^{op}, \mathbb{Q}^{op}] = [\mathbb{P}, \mathbb{Q}]$.
2. $[\mathbb{P}^{op}, \mathbb{Q}^{op}] = [\mathbb{P}, \mathbb{Q}]^{op}$.

A problem

Let $\mathbb{P}$ and $\mathbb{Q}$ be any posets.

1. Let $f, g : \mathbb{P} \longrightarrow \mathbb{Q}$, let $f \leq g$, and let $p \leq q$ in $\mathbb{P}$. Show that $f(p) \leq g(q)$ in $\mathbb{Q}$.
2. Let $f, g : \mathbb{P} \longrightarrow \mathbb{Q}^{op}$, let $f \leq g$, and let $p \leq q$ in $\mathbb{P}$. Show that $g(q) \leq f(p)$ in $\mathbb{Q}$. [Use part (1)]

Higher-order Terms over Posets

Fix a set $\mathcal{T}_0$ of basic types.
We form the full set $\mathcal{T}$ of types as follows:

1. $\mathcal{T}_0 \subseteq \mathcal{T}$.
2. If $\sigma, \tau \in \mathcal{T}$, then also $\sigma \xrightarrow{+} \tau$ and $\sigma \xrightarrow{-} \tau$ belong to $\mathcal{T}$.

In order not to state two copies of many definitions,
we sometimes write $\sigma \xrightarrow{*} \tau$ to mean either $\sigma \xrightarrow{+} \tau$ or $\sigma \xrightarrow{-} \tau$.

Higher-order Terms over Posets

Definition

A typed language is a set $\mathcal{T}_0$ of basic types,
together with a collection of typed variables $v : \sigma$
and typed constants $c : \sigma$.

We generally assume that the set of typed variables includes infinitely many of each type.
But there might be no constants whatsoever.
We use $\mathcal{L}$ to denote a typed language.

Higher-order Terms over Posets

Let $\mathcal{L}$ be a typed language.
We form typed terms $t : \sigma$ as follows:

1. If $v : \sigma$ (as a typed variable), then $v : \sigma$ (as a typed term).
2. If $c : \sigma$ (as a typed constant), then $c : \sigma$ (as a typed term).
3. If $t : \sigma \xrightarrow{*} \tau$ and $u : \sigma$, then $t(u) : \tau$.

(The cognoscente will add in lambda terms.)

Every typed term has a topmost polarity $P(t)$. It is given by

$$\begin{array}{lcll} P(t) & = & + & \text{iff } t \text{ is of basic type} \\ P(t(u)) & = & + & \text{iff } t \text{ is of the form } \sigma \xrightarrow{+} \tau \\ P(t(u)) & = & - & \text{iff } t \text{ is of the form } \sigma \xrightarrow{-} \tau \end{array}$$

Semantics

For the semantics of our higher-order language $\mathcal{L}$ we use models $\mathcal{M}$ of a specific form.

$\mathcal{M}$ consists of an assignment of posets $\mathbb{P}_\sigma$ to each basic type $\sigma \in \mathcal{T}_0$,
together with an assignment $[\![c]\!] \in P_\sigma$ for each constant $c : \sigma$, and also a typed map f;
this is just a map which to a typed variable $v : \sigma$ gives some $f(v) \in P_\sigma$.

Then we set

$$
\begin{aligned}
\mathbb{P}_{\sigma \xrightarrow{+} \tau} &= [\mathbb{P}_\sigma, \mathbb{P}_\tau] \\
\mathbb{P}_{\sigma \xrightarrow{-} \tau} &= [\mathbb{P}_\sigma, \mathbb{P}_\tau^{op}]
\end{aligned}
$$

Ground Terms

A ground term is a term with no free variables.
Each ground term $t : \sigma$ has a denotation $[\![t]\!] \in P_\sigma$ defined in the obvious way:

$$\begin{array}{lcl} [\![c]\!] & = & \text{is given at the outset for } \sigma \text{ basic and } c : \sigma \\ [\![t(u)]\!] & = & [\![t]\!]([\![u]\!]) \end{array}$$

Extended Example: Syntax

In this example (and for future developments of it), we fix a vocabulary of unary and binary atoms as earlier in the course.

First, we describe a language $\mathcal{L}$ corresponding to this vocabulary.
Let's take our basic types to be t and pr.
(These stand for truth value and property.)

Note that complex types includes ones like $pr \xrightarrow{+} t$, $pr \xrightarrow{-} t$, $(pr \xrightarrow{+} t) \xrightarrow{+} t$, etc.

Extended Example: Syntax

In this example (and for future developments of it), we fix a vocabulary of unary and binary atoms as earlier in the course.

First, we describe a language $\mathcal{L}$ corresponding to this vocabulary. The rest of the language consists of three sets of symbols:

1. Each unary atom p gives a typed constant $\mathsf{p} : pr$.
2. We also have typed constants

$$\begin{aligned} \mathsf{every} : &\quad pr \xrightarrow{-} (pr \xrightarrow{+} t) \\ \mathsf{some} : &\quad pr \xrightarrow{+} (pr \xrightarrow{+} t) \\ \mathsf{no} : &\quad pr \xrightarrow{+} (pr \xrightarrow{-} t) \end{aligned}$$

3. Finally, every binary atom r gives two type constants:

$$\begin{aligned} \mathsf{r}_{+} : &\quad (pr \xrightarrow{+} t) \xrightarrow{+} pr \\ \mathsf{r}_{-} : &\quad (pr \xrightarrow{-} t) \xrightarrow{-} pr \end{aligned}$$

EXTENDED EXAMPLE: SEMANTICS

Up until now, we have only given the language $\mathcal{L}$.
Now we describe a model of it (actually, a family of models).
Fix a model $\mathcal{M}$ in our earlier sense, consisting of a set M and interpretations $[\![p]\!] \subseteq M$ and $[\![r]\!] \subseteq M^2$.
We wish to turn $\mathcal{M}$ into a model for our typed language which we shall also call $\mathcal{M}$.

The underlying universe M gives a flat poset $\mathbb{M}$.
We take $\mathbb{P}_{pr} = [\mathbb{M}, \mathbb{2}] \cong \mathcal{P}(M)$.
We also take $\mathbb{P}_t = \mathbb{2}$.
We interpret the typed constants $\mathsf{p} : pr$ corresponding to unary atoms by

$$[\![\mathsf{p}]\!](m) = \text{true} \quad \text{iff} \quad m \in [\![p]\!].$$

(On the right we use the interpretation of p in the model $\mathcal{M}$.)
Usually we write p instead of $[\![\mathsf{p}]\!]$.

Extended Example: Semantics of the constants

The interpretations of every, some, and no are as before (taking the set X to be the universe M of $\mathcal{M}$).

Recall that we have two typed constants r_+ and r_- corresponding to binary atom r:

$$\begin{array}{rl} \mathsf{r}_+ : & (pr \xrightarrow{+} t) \xrightarrow{+} pr \\ \mathsf{r}_- : & (pr \xrightarrow{-} t) \xrightarrow{-} pr \end{array}$$

Both are interpreted in model in the same way, by

$$\mathsf{r}_*(q)(m) \quad = \quad q(\{m' \in M : [\![r]\!](m, m')\})$$

It is clear that for all $q \in \mathbb{P}_{pr \xrightarrow{+} t} = [[\mathbb{M}, 2], 2]$,
$\mathsf{r}_*(q)$ is a function from M to $\{\text{true}, \text{false}\}$.
The monotonicity of this function is trivial, since $\mathbb{M}$ is flat.
It is good to check that r_+ and r_- have the right types.

Extended Example, Continued

Note that we have terms like $\mathsf{r}_+(\mathsf{some}(\mathsf{p}))$, but not $\mathsf{r}_+(\mathsf{no}(\mathsf{p}))$. For the latter, we would have to use $\mathsf{r}_-(\mathsf{no}(\mathsf{p}))$, since $\mathsf{no}(\mathsf{p}) : pr \xrightarrow{-} t$.

It should also be noted that we have more complex ground terms, such as

$$\mathsf{r}_-(\mathsf{no}(\mathsf{s}_+(\mathsf{some}(\mathsf{r}_-(\mathsf{no}(\mathsf{p}))))))$$

Here s_+ is a typed constant corresponding to a binary atom s.

Important fact

The semantics so far agrees with our previous semantics.
That is:

1. $\mathcal{M} \models \forall(p, q)$ iff $[\![\text{every}(\text{p})(\text{q})]\!] = \text{true}$.
2. $\mathcal{M} \models \exists(p, q)$ iff $[\![\text{some}(\text{p})(\text{q})]\!] = \text{true}$.
3. $\mathcal{M} \models \forall(p, \overline{q})$ iff $[\![\text{no}(\text{p})(\text{q})]\!] = \text{true}$.
4. $[\![\forall(p, r)]\!] = \text{r}_{+}(\text{every}(\text{p}))$.
5. $[\![\exists(p, r)]\!] = \text{r}_{+}(\text{some}(\text{p}))$.
6. $[\![\forall(p, \overline{r})]\!] = \text{r}_{-}(\text{no}(\text{p}))$.

Question

Suppose that r represents the relation of respecting and s the relation of seeing.

What subset of M corresponds to the term below?

$$r_{-}(no(s_{+}(some(r_{-}(no(p))))))$$

Exercise

Suppose that $[\![r]\!] \subseteq [\![s]\!]$, and consider r_+, r_-, s_+ and s_-.

1. Show that $r_+ \leq s_+$ in $[[[\mathbb{M}, 2], 2], [\mathbb{M}, 2]]$.
2. Show that $r_- \leq s_-$ in $[[[\mathbb{M}, 2], 2^{op}], [\mathbb{M}, 2]^{op}]$.

Solution

Here is the solution to (2).
Fix $q \in [[\mathbb{M}, \mathbb{2}], \mathbb{2}^{op}]$.
We must show that $\mathsf{r}_{-}(q) \leq \mathsf{s}_{-}(q)$ in $[\mathbb{M}, \mathbb{2}]^{op}$; that is $\mathsf{s}_{-}(q) \leq \mathsf{r}_{-}(q)$ in $[\mathbb{M}, \mathbb{2}]$.

For this, take $m \in M$. We show that $\mathsf{s}_{-}(q)(m) \leq \mathsf{r}_{-}(q)(m)$ in $\mathbb{2}$.
Let $p_r = \{m' \in M : [\![r]\!](m, m')\}$, and let p_s be defined similarly.
Then $p_r \leq p_s$ by our assumption that $[\![r]\!] \subseteq [\![s]\!]$.

Also, $\mathsf{r}_{-}(q)(m) = q(p_r)$ and $\mathsf{s}_{-}(q)(m) = q(p_s)$.
The type of q tells us that $q(p_r) \leq q(p_s)$ in $\mathbb{2}^{op}$; that is,
$q(p_s) \leq q(p_r)$ in $\mathbb{2}$.
And therefore $\mathsf{s}_{-}(q)(m) \leq \mathsf{r}_{-}(q)(m)$, just as desired.

Contexts

A context is a typed term with exactly one variable, say x.
(This variable may be of any type.)
All the rest of the subterms are constants.
We write t for a context,

We'll be interested in contexts of the form $t(u)$.
Note that if $t(u)$ is a context and if x appears in u,
then t is a ground term; and vice-versa.

Contexts

In the definition below, we remind you that subterms are not necessarily proper.
That is, a variable x is a subterm of itself.

Definition

The polarity of a context t is given as follows.

1. $\mathrm{pol}(x) = +$.
2. $\mathrm{pol}(x(u)) = P(x)$, the topmost polarity of x.
3. If x is a proper subterm of t, then $\mathrm{pol}(t(u)) = \mathrm{pol}(t)$.
4. If x is a subterm of u, then $\mathrm{pol}(t(u)) = P(t) \cdot \mathrm{pol}(u)$, where $P(t)$ is the topmost polarity of t.

$$(+)\cdot(+) = (+) \quad (+)\cdot(-) = (-) \quad (-)\cdot(+) = (-) \quad (-)\cdot(-) = (+)$$

Every context gives a function

Definition

Let $x : \rho$, and let $t : \sigma$ be a context. We associate to t a set function

$$f_t : P_\rho \longrightarrow P_\sigma$$

in the following way:

1. If $t = x$, so that $\sigma = \rho$, then $f_x : P_\sigma \longrightarrow P_\sigma$ is the identity.
2. If x is a subterm of t then $f_{t(u)}$ is

$$a \in P_\rho \mapsto f_t(a)([\![u]\!]) \ .$$

3. If x is a subterm of u, then $f_{t(u)}$ is $[\![t]\!] \cdot f_u$.
   That is, $f_{t(u)}$ is

$$a \in P_\rho \mapsto [\![t]\!] f_u(a) \ .$$

Context Lemma (van Benthem)

Lemma

Let t be a context, where $t : \sigma$ and $x : \rho$.

Then f_t may be regarded as element of $[\mathbb{P}_\rho, \mathbb{P}_\sigma^{\mathrm{pol}(t)}]$.
That is,

1. If $\mathrm{pol}(t) = +$, then $f_t : \mathbb{P}_\rho \rightarrow \mathbb{P}_\sigma$.
2. If $\mathrm{pol}(t) = -$, then $f_t : \mathbb{P}_\rho \rightarrow \mathbb{P}_\sigma^{op}$.

The proof is by induction on t.

When t is a type variable $x : \sigma$,
f_x is the identity on $\mathbb{P}_\sigma$, the polarity of x is $+$,
and indeed the identity is a monotone function on any poset.

Context Lemma (van Benthem)

Lemma

Let t be a context, where $t : \sigma$ and $x : \rho$.

Then f_t may be regarded as element of $[\mathbb{P}_\rho, \mathbb{P}_\sigma^{\mathsf{pol}(t)}]$.
That is,

1. If $\mathsf{pol}(t) = +$, then $f_t : \mathbb{P}_\rho \to \mathbb{P}_\sigma$.
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Context Lemma (van Benthem)

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That is,

1. If $\mathsf{pol}(t) = +$, then $f_t : \mathbb{P}_\rho \to \mathbb{P}_\sigma$.
2. If $\mathsf{pol}(t) = -$, then $f_t : \mathbb{P}_\rho \to \mathbb{P}_\sigma^{op}$.

Finally, assume that x appears in u.
Let τ be such that $t : \tau \xrightarrow{*} \sigma$ and $u : \tau$.
Recall that $f_{t(u)}$ is $[\![t]\!] \cdot f_u$, and the polarity of $t(u)$ is $P(t) \cdot \mathsf{pol}(u)$.

Let $i = P(t)$ and $j = \mathsf{pol}(u)$.
By induction hypothesis, $f_u : \mathbb{P}_\rho \to \mathbb{P}_\tau^j$.
And since t is a ground term in this step, $[\![t]\!] : \mathbb{P}_\sigma \to \mathbb{P}_\tau^i$.
Using an earlier result, $[\![t]\!] : \mathbb{P}_\sigma^j \to \mathbb{P}_\tau^{i \cdot j}$.
And thus

$$[\![t]\!] \cdot f_u : \mathbb{P}_\rho \to \mathbb{P}_\tau^{i \cdot j} \ .$$

This completes the proof.

The polarity of a context

Definition

Let u be a subterm occurrence in t.
Then there is a unique context $t'(x)$ such that $t = t'[u \leftarrow x]$.
We say that u has positive polarity if $\mathrm{pol}(t') = +$,
and that u has negative polarity if $\mathrm{pol}(t') = -$.

The proof of the Context Lemma gives an algorithm for telling whether a given subterm occurrence is positive or negative.
There are several variants of this algorithm in the literature.
In effect, these take a term and find the polarities of all of the subterms in one or two steps.

Monotonicity Principles

From what we have done so far, we may read off the following

Monotonicity Principles

$$\frac{u \leq v}{t(u) \leq t(v)}\ \mathsf{pol}(t) = + \qquad \frac{u \leq v}{t(v) \leq t(u)}\ \mathsf{pol}(t) = -$$

Example

Suppose that

$$[\![\text{captivate}]\!] \subseteq [\![\text{fascinate}]\!] \subseteq [\![\text{interest}]\!]$$

and also that

$$[\![\text{baby}]\!] \subseteq [\![\text{child}]\!] \subseteq [\![\text{person}]\!]$$

Consider the following three terms of type *pr*:

1. $\text{fascinate}_{+}(\text{every}(\text{child}))$
2. $\text{fascinate}_{+}(\text{some}(\text{child}))$
3. $\text{fascinate}_{-}(\text{no}(\text{child}))$

We calculate the polarity of the verb and noun terms in all of these, using the recursive definition.

term	polarity of verb	polarity of noun
$\text{fascinate}_{+}(\text{every}(\text{child}))$	$+$	$-$
$\text{fascinate}_{+}(\text{some}(\text{child}))$	$+$	$+$
$\text{fascinate}_{-}(\text{no}(\text{child}))$	$-$	$-$

Example

We have baby $\leq$ child $\leq$ person in $\mathbb{P}_{pr}$.
In addition, by a result which we have seen before,

$$\begin{array}{ll} \text{captivate}_+ \leq \text{fascinate}_+ \leq \text{interest}_+ & \text{in } [[[\mathbb{M}, 2], 2], [\mathbb{M}, 2]] \\ \text{captivate}_- \leq \text{fascinate}_- \leq \text{interest}_- & \text{in } [[[\mathbb{M}, 2], 2^{op}], [\mathbb{M}, 2]^{op}] \end{array}$$

By the monotonicity principles, the following all hold in $\mathbb{P}_{pr}$:

$$\begin{array}{lcl} \text{fascinate}_+(\text{every}(\text{child})) & \leq & \text{fascinate}_+(\text{every}(\text{baby})) \\ \text{fascinate}_+(\text{every}(\text{child})) & \leq & \text{interest}_+(\text{every}(\text{child})) \\ \text{fascinate}_+(\text{some}(\text{child})) & \leq & \text{fascinate}_+(\text{some}(\text{person})) \\ \text{fascinate}_+(\text{some}(\text{child})) & \leq & \text{interest}_+(\text{some}(\text{child})) \\ \text{fascinate}_-(\text{no}(\text{child})) & \leq & \text{fascinate}_-(\text{no}(\text{baby})) \\ \text{fascinate}_-(\text{no}(\text{child})) & \leq & \text{captivate}_-(\text{no}(\text{child})) \end{array}$$

How these get used

We want to use

captivate ⇒ fascinate ⇒ interest

as background facts.
That is, they are not stated directly anywhere.
(This contrasts with

baby ⇒ child ⇒ person

which can indeed be stated, even in our tiny language.)
Instead, the background facts feed into the Monotonicity Principle to give facts like

every(fascinate_(no(child)))(captivate_(no(child)))

These facts are valid on all interpretations satisfying the background facts.

Exercise

Find the polarity of the second verb in

see_(no(fascinate_(no(child))))

Using that, find the relation in $\leq$ of the term above to

see_(no(captivate_(no(child))))

What we might want

$S \rightarrow NP\ VP$

$NP \rightarrow N_{prop}$

$N_{prop} \rightarrow John$, $N_{prop} \rightarrow Mary$, $N_{prop} \rightarrow Susan$, ...

$NP \rightarrow Det\ N$

$Det \rightarrow every$, $Det \rightarrow some$, $Det \rightarrow most$, $Det \rightarrow one$, ...

$N \rightarrow woman$, $N \rightarrow man$, $N \rightarrow boy$, ...

$N \rightarrow girl$, ...

$VP \rightarrow V_{intrans}$

$V_{intrans} \rightarrow walks$, $V_{intrans} \rightarrow talks$, ...

$VP \rightarrow V_{trans}\ NP$

$V_{trans} \rightarrow sees$, $V_{trans} \rightarrow knows$, $V_{trans} \rightarrow likes$, ...

$N \rightarrow N\ who\ SRC$

$SRC \rightarrow VP$

$N \rightarrow N\ whom\ ORC$

$ORC \rightarrow NP\ V_{trans}$

Convet to Chomsky normal form, then to a CG, etc.

$S \to NP$ *walks*	$S \to NP$ *talks*	...
$S \to NP$ *sees* NP	$S \to NP$ *knows* NP	$S \to NP$ *likes* NP
$NP \to$ *John*	$NP \to$ *Mary*	...
$NP \to$ *every* N	$NP \to$ *some* N	$NP \to$ *the* N
$VP \to$ *walks*	$VP \to$ *talks*	...
$VP \to$ *sees* NP	$VP \to$ *knows* NP	$VP \to$ *likes* NP
$SRC \to$ *sees* NP	$SRC \to$ *knows* NP	$SRC \to$ *likes* NP
$ORC \to NP$ *sees*	$ORC \to NP$ *knows*	$ORC \to NP$ *likes*

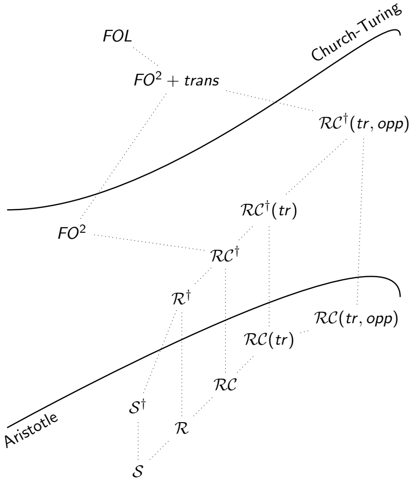
Then this converts immediately to a simple categorial grammar.
The extension to an "ordered categorial grammar" is straightforward.
And then the monotonicity principles give sound and useful logical principles.

We can go further and use the same ideas to do logic beyond semantics or even logic beyond grammar.

Last words: summary and open areas

What we did: logical systems

Open: do more! And, find the "biggest" logical system for NL

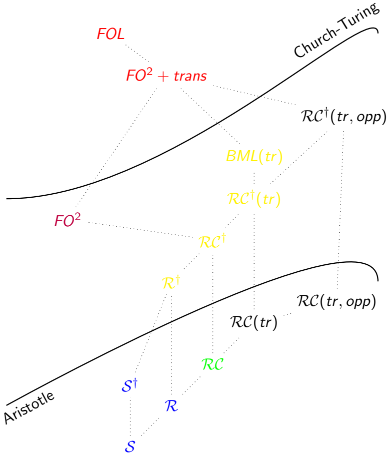


† adds full *N*-negation
* adds relative clauses

opp adds opposites
of comparative adjectives

Complexity

(mostly) best possible results on the validity problem
Open: do the average case analysis



undecidable
Church 1936
Grädel, Otto, Rosen 1999

in co-NEXPTIME
EXPTIME
Lutz & Sattler 2001

Co-NEXPTIME
Grädel, Kolaitis, Vardi '97
EXPTIME
Pratt-Hartmann 2004

lower bounds also open

Co-NP
McAllester & Givan 1992

NLOGSPACE

More open stuff

- Look at combinations of simple linguistic reasoning with default logic, or logics of presupposition.
- For computational linguistics: how complex are the expressions of general-purpose background knowledge?
- For semantics: is a complete logic a semantics?
- Proof-theoretic semantics (cf. Francez)
- Connect this work to more active areas, such as textual entailment and distributional models of word meaning.