Q520 Notes on Natural Logic

Larry Moss

We have seen examples of what are traditionally called *syllogisms* already:

$$\frac{\begin{array}{c}\textit{All men are mortal.}\\ \textit{Socrates is a man.}\end{array}}{\textit{Socrates is mortal.}}$$

The idea again is that the sentences above the line should *semantically entail* the one below the line. Specifically, in every context[1] in which *All men are mortal* and *Socrates is a man* are true, it must be the case that *Socrates is mortal* is also true.

What we want to do here is to turn this *semantic* entailment into the a rule in a *formal proof system.*

1 The fragment with sentences *All X are Y.*

It will be easier to postpone the introduction of names a bit, so to begin we'll only deal with sentences *All X are Y.* More formally, we have *variables* $X_1, X_2, \ldots, X_n, \ldots$ ranging over common nouns. The X's denote the v variables. A *sentence* is an expression S of the form *All X are Y*, where X and Y are one of the v variables. For example, *All X_{12} are X_{7132}* is a sentence in this first fragment.

Next, we need a notion of semantics for this fragment. In this discussion, a *model* (or *interpretation*, or *context*), will be a pair $(U, [\![\ ]\!])$, where U is a set, and $[\![X_i]\!] \subseteq U$ for all i. We then use this to define the truth or falsity of sentences. The definition is

$$[\![\text{All } X \text{ are } Y]\!] \quad = \quad \begin{cases} \mathsf{true} & \text{if } [\![X]\!] \subseteq [\![Y]\!] \\ \mathsf{false} & \text{otherwise} \end{cases}$$

If Γ is a set of sentences and S be a sentence, we say that Γ *semantically implies* S and we write $\Gamma \models S$ if: for all contexts $(U, [\![\ ]\!])$, if $[\![A]\!] = \mathsf{true}$ for all $A \in \Gamma$, then also $[\![S]\!] = \mathsf{true}$. For example,

$$\{\textit{All } X_1 \textit{ are } X_2, \textit{All } X_2 \textit{ are } X_3\} \models \textit{All } X_1 \textit{ are } X_3.$$

On the other hand,

$$\{\textit{All } X_1 \textit{ are } X_2, \textit{All } X_3 \textit{ are } X_2\} \not\models \textit{All } X_1 \textit{ are } X_3.$$

What one wants to do in a formal system of logic is to give a purely syntactic counterpart $\vdash$ to $\models$. We read $\vdash$ as "proves" or "derives". The goal is to give a definition of a relation $\Gamma \vdash S$ and then to check that it agrees with the earlier notion $\Gamma \models S$.

A *proof tree over* Γ for this fragment is a tree with the following properties:

[1] In these notes, we use "context" and "model" synonymously.

1. The leaves are either labeled with sentences in Γ, or with sentences of the form *All X are X*.
2. The interior leaves have two children (drawn above them); if the label of the parent (on the bottom) is *All X are Y*, then the label of the left child is *All X are Z*, and the label of the right child is *All Z are Y*.

We draw these trees with the root at the bottom and the leaves at the top.

We summarize this schematically by indicating the *inference rules* as follows:

$$\frac{\textit{All } X \textit{ are } Z \quad \textit{All } Z \textit{ are } Y}{\textit{All } X \textit{ are } Y} \qquad \frac{}{\textit{All } X \textit{ are } X}$$

If there is a proof tree over Γ whose root is labeled S, we say $\Gamma \vdash S$. This is how our proof system works.

Here is an example: Let Γ be

$$\{\textit{All } A \textit{ are } B, \textit{All } Q \textit{ are } A, \textit{All } B \textit{ are } D, \textit{All } C \textit{ are } D, \textit{All } A \textit{ are } Q\}$$

Let S be *All Q are D*. Here is a proof tree showing that $\Gamma \vdash S$:

$$\dfrac{\textit{All } Q \textit{ are } A \qquad \dfrac{\dfrac{\textit{All } A \textit{ are } B \quad \textit{All } B \textit{ are } B}{\textit{All } A \textit{ are } B} \quad \textit{All } B \textit{ are } D}{\textit{All } A \textit{ are } D}}{\textit{All } Q \textit{ are } D}$$

Note that all of the leaves belong to Γ except for one that is *All B are B*. Note also that some elements of Γ are not used as leaves. This is permitted according to our definition. The proof tree above shows that $\Gamma \vdash S$. Also, there is a smaller proof tree that does this, since the use of *All B are B* is not really needed. (The reason why we allow leaves to be labeled like this is so that that we can have one-element trees labeled with sentences of the form *All A are A*.

Lemma 1.1 (Soundness) *If* $\Gamma \vdash S$, *then* $\Gamma \models S$.

Proof To prove this formally, you need the notion of *induction on proof trees.* We didn't really cover this, so our proof will be a bit new. Suppose $\mathcal{T}$ is a tree that shows $\Gamma \vdash S$. If $\mathcal{T}$ is a one-element tree, let S be the label on the one node. Either S belongs to Γ, or else S is of the form *All A are A*. In the first case, every model satisfying every sentence in Γ clearly satisfies S, as S belongs to Γ. And in the second case, every model whatsoever satisfies S.

So we know our result for one-element trees $\mathcal{T}$. In a more general setting, let's suppose that we have a tree $\mathcal{T}$ that has as immediate subtrees $\mathcal{T}_1$ and $\mathcal{T}_2$. What do the labels look like? Well, the label on $\mathcal{T}$ must be a sentence *All X are Z*, and then for some Y the label of the left child is *All X are Y*, and the label of the right child is *All X are Y*. Now $\mathcal{T}_1$ and $\mathcal{T}_2$ are smaller than our $\mathcal{T}$. So *we assume the result we are trying to establish about* $\mathcal{T}$ *for the smaller trees* $\mathcal{T}_1$ *and* $\mathcal{T}_2$. (This is where induction comes in!) This tells us that $\Gamma \models$ *All X are Y*, and also $\Gamma \models$ *All Y are Z*. We claim that $\Gamma \models$ *All X are Y*. Take any content in which all sentences in Γ are true. Then $[\![X]\!] \subseteq [\![Y]\!]$ by our first point above. And $[\![Y]\!] \subseteq [\![Z]\!]$ by second. So $[\![X]\!] \subseteq [\![Z]\!]$ by transitivity of subset. Since the context here is arbitrary, we conclude that $\Gamma \models$ *All X are Z*. $\dashv$

Theorem 1.2 (Completeness) *In the fragment with* All, *if* $\Gamma \models S$, *then* $\Gamma \vdash S$.

Proof Let $Z_1, \ldots, Z_k$ be all the variables that occur in Γ or in S. Let S be *All X are Y*.

Define a model by $U = \{*\}$,[2] and

$$[\![Z_i]\!] \quad = \quad \begin{cases} U & \text{if } \Gamma \vdash \textit{All } X \textit{ are } Z_i \\ \emptyset & \text{otherwise} \end{cases} \tag{1}$$

We claim that if Γ contains *All* V *are* W, then $[\![V]\!] \subseteq [\![W]\!]$. For this, we may assume that $[\![V]\!] \neq \emptyset$ (otherwise the result is trivial). So $[\![V]\!] = U$. Thus $\Gamma \vdash$ *All* X *are* V. So we have a proof tree as on the left below:

$$\dfrac{\begin{matrix}\vdots \\ \textit{All } X \textit{ are } V\end{matrix} \quad \textit{All } V \textit{ are } W}{\textit{All } X \textit{ are } W}$$

The tree overall has as leaves *All* V *are* W plus the leaves of the tree above *All* X *are* V. Overall, we see that all leaves are labelled by sentences in Γ. This tree shows that $\Gamma \vdash$ *All* X *are* W. From this we conclude that $[\![W]\!] = U$. In particular, $[\![V]\!] \subseteq [\![W]\!]$.

Now our claim implies that the context we have defined makes all sentences in Γ true. So it must make the conclusion true. Therefore $[\![X]\!] \subseteq [\![Y]\!]$. And $[\![X]\!] = U$, since we have a one-point tree for *All* X *are* X. Hence $[\![Y]\!] = U$ as well. But this means that $\Gamma \vdash$ *All* X *are* Y, just as desired. $\dashv$

A Stronger Result Theorem 1.2 proves the completeness of the logical system. But it doesn't give us all the information we would need to have an efficient proceedure to decide whether or not $\Gamma \vdash S$ in this fragment. For that, we need a little more. Define a relation $\preceq$ on the variables in question by: $V \preceq W$ if there is a sequence

$$V = V_0, V_1, \ldots, V_k = Z$$

such that for $i = 0, \ldots, k-1$, $\Gamma \vdash$ *All* V_i *are* V_{i+1}. We allow the sequence to just have $V = V_0 = V$, so that we have $V \preceq V$ for all V.

Lemma 1.3 *Let* Γ *be a set of sentence in our fragment, and define* $\leq$ *by*

$$U \leq V \quad \textit{iff} \quad \Gamma \vdash \textit{All } U \textit{ are } V \tag{2}$$

Then

1. *For all* U, $U \leq U$.
2. *If* $U \leq V$ *and* $V \leq W$, *then* $U \leq W$.

Theorem 1.4 *Let* Γ *be any set of sentences in this fragment, let* $\preceq$ *be as above. Let* X *and* Y *be any variables. Then the following are equivalent:*

1. $\Gamma \vdash$ All X are Y.

[2] This just means that U is some one-element set. It doesn't matter which one-element set. Actually, it doesn't even matter that U has just one element: *any* non-empty set U would work.

2. $\Gamma \models \text{All } X \text{ are } Y$*.*

3. $X \preceq Y$*.*

The original definition of the entailment relation $\Gamma \models S$ involves looking at *all* models of the language. Theorem 1.4 is important because part (3) gives a criterion the entailment relation that is algorithmically sensible. To see whether $\Gamma \models$ *All X are Y* or not, we only need to compute $\leq$ from Γ. This amounts to constructing $\preceq$. This is the reflexive-transitive closure of a relation, so it is computationally very manageable. (In graph theoretic terms, one can make a graph of the variables in question using as the edge relation the relation $\rightarrow$ given by $Y \rightarrow Z$ iff $\Gamma \vdash$ *All Y are Z*. Then $\preceq$ just means that there is a path in this graph from Y to Z.)

Exercise 1 Prove Theorem 1.4. [Hint: show (1)$\Longrightarrow$(2), (2)$\Longrightarrow$(3), and (3)$\Longrightarrow$(1). The hard step is (2)$\Longrightarrow$(3). For this, assume that $X \not\preceq Y$. Build a context which makes all the sentences in Γ true and *All X are Y* false. You'll need to modify (1).]]

2 *Some X are Y*

We want to now enrich our language by adding assertions *Some X are Y*. We call these sentences *existentials*, since formalizing them in logic would use the existential quantifier $\exists$. We extend our semantics by

$$[\![\text{Some } X \text{ are } Y]\!] \quad = \quad \begin{cases} \mathsf{true} & \text{if } [\![X]\!] \cap [\![Y]\!] \neq \emptyset \\ \mathsf{false} & \text{otherwise} \end{cases}$$

We again write $\Gamma \models S$ for the semantic entailment relation defined the same way as before.

Exercise 2 Check that

$$\{\textit{Some X are Y}, \textit{Some Y are Z}\} \not\models \textit{Some X are Z}$$

by building a model in which $[\![X]\!] \cap [\![Y]\!] \neq \emptyset$ and $[\![Y]\!] \cap [\![Z]\!] \neq \emptyset$, but $[\![X]\!] \cap [\![Z]\!] = \emptyset$.

We add the following three proof rules to our system:

$$\frac{\textit{Some X are Y}}{\textit{Some Y are X}} \qquad \frac{\textit{Some X are Y}}{\textit{Some X are X}}$$

$$\frac{\textit{All Y are Z} \quad \textit{Some X are Y}}{\textit{Some X are Z}}$$

At this point we have a new proof system, but we keep the notation $\Gamma \vdash S$. If we need to differentiate the proof system here from the previous one, we would need to indicate this somehow in our notation.

Theorem 2.1 (Completeness) *In the fragment with* All *and* Some*, if* $\Gamma \models S$*, then* $\Gamma \vdash S$*.*

Proof Suppose that $\Gamma \models S$. There are two cases, depending on whether S is of the form

All X *are* Y or of the form *Some* X *are* Y. The cases are handled differently. Since the first is easier, we leave it to you as an Exercise. So we fix X and Y as suppose that $\Gamma \models$ *Some* X *are* Y. We need a proof of this fact in our system.

List all of the existential sentences in Γ in a list:

$$\textit{Some } V_1 \textit{ are } W_1, \textit{Some } V_2 \textit{ are } W_2, \ldots, \textit{Some } V_n \textit{ are } W_n \tag{3}$$

Note that we might have repeats among the V's and Ws, and that some of these might well coincide with the X and Y that we are dealing with in this proof. For the universe U we take $\{1, \ldots, n\}$, where n is the number in (3). For each variable Z, we define

$$[\![Z]\!] \quad = \quad \{i : \text{either } V_i \leq Z \text{ or } W_i \leq Z\}. \tag{4}$$

(The relation $\leq$ is defined in (2).) This defines an model of the language.

Consider a sentence *All* P *are* Q in Γ. Then $P \leq Q$. It follows from (3) and Lemma 1.3 that $[\![P]\!] \subseteq [\![Q]\!]$. Second, take an existential sentence *Some* V_i *are* W_i on our list in (3) above. Then i itself belongs to $[\![V_i]\!] \cap [\![W_i]\!]$, so this intersection is not empty.

These facts imply that all sentences in Γ are made true in our model. Recall that our sentence S is *Some* X *are* Y. By our assumption that $\Gamma \models$ *Some* X *are* Y, $[\![X]\!] \cap [\![Y]\!] \neq \emptyset$. Let i belong to this set. We have four cases, depending on whether $V_i \leq X$ or $V_i \leq Y$, and whether $W_i \leq X$ or $W_i \leq Y$. One case is when $V_i \leq X$ and $W_i \leq Y$. Recalling that *Some* V_i *are* W_i belongs to Γ, we have a proof tree as follows:

$$\dfrac{\textit{All } W_i \textit{ are } Y \qquad \dfrac{\dfrac{\textit{All } V_i \textit{ are } X \qquad \dfrac{\textit{Some } V_i \textit{ are } W_i}{\textit{Some } W_i \textit{ are } V_i}}{\textit{Some } W_i \textit{ are } X}}{\textit{Some } X \textit{ are } W_i}}{\textit{Some } X \textit{ are } Y}$$

The other cases are similar. ⊣

Exercise 3 Complete the proof of Theorem 2.1 by showing that if Γ is a set of sentences in the fragment of this section and $\Gamma \models$ *All* X *are* Y, then also $\Gamma \vdash$ *All* X *are* Y. You will need to modify the proof of Theorem 1.2 a little bit, since Γ now may have existential sentences.

Exercise 4 Adapt the proof of Theorem 2.1 to show that if

$$\Gamma \not\vdash \textit{Some } X \textit{ are } Y.$$

then there is a model making all sentences in Γ true and also making *Some* X *are* Y false. By directly following the construction, the size of the model will be the number of existential sentences in Γ. Can we do better?

1. Show by modifying (4) that we can shrink our model down to one of size at most 2.

2. Show that 2 is the smallest we can get by showing that if we only look at one-element models

$$\{\textit{Some } X \textit{ are } Y, \textit{Some } Y \textit{ are } Z\} \models \textit{Some } X \textit{ are } Z$$

Exercise 5 Give an algorithm which takes finite sets Γ in the fragment of this section and also single sentences S and tells whether $\Gamma \models S$ or not. [You may be sketchy, as we were in our discussion of this matter at the end of Section 1.]

Exercise 6 Suppose that one wants to say that *All X are Y* is true when $[\![X]\!] \subseteq [\![Y]\!]$ *and also* $[\![X]\!] \neq \emptyset$. Then the following rule becomes sound:

$$\frac{\textit{All } X \textit{ are } Y}{\textit{Some } X \textit{ are } Y}$$

Show that if we add this rule to our proof system, then we get a complete system for the modified semantics. [Hint: Given Γ, let $\overline{\Gamma}$ be Γ with all sentences *Some X are Y* such that *All X are Y* belongs to Γ. Show that $\Gamma \vdash S$ in the modified system iff $\overline{\Gamma} \vdash S$ in the old system.]

Exercise 7 What would you do to the system to add sentences of the form *Some X exists*?

3 Adding names

We continue by adding names so that we can deal with sentences like *John is a secretary*. To our formal language we add *individual variables* $X_1, \ldots, X_n, \ldots$; we abbreviate these $J, M, J_1, \ldots, J_n, \ldots$, etc. The sentences we add to the fragment are *J is an X* and *J is M*, where J and M are individual variables and X is a common noun variable. We assume that in a model U, $[\![J]\!] \in U$. We have to say when sentences with names are true and when they are false. The natural definition is:

$$[\![J \text{ is an } X]\!] \quad = \quad \begin{cases} \mathsf{true} & \text{if } [\![J]\!] \in [\![X]\!] \\ \mathsf{false} & \text{otherwise} \end{cases}$$

$$[\![J \text{ is } M]\!] \quad = \quad \begin{cases} \mathsf{true} & \text{if } [\![J]\!] = [\![M]\!] \\ \mathsf{false} & \text{otherwise} \end{cases}$$

To get a proof system, we add the remaining rules in Figure 1.

We intend to show the completeness of the logic in Figure 1. For this, we need a definition. Fix a set Γ of sentences in this fragment. Let $\equiv$ be the relation on names defined by

$$J \equiv M \quad \text{iff} \quad \Gamma \vdash J \textit{ is } M. \tag{5}$$

Lemma 3.1 *$\equiv$ is an equivalence relation.*

Theorem 3.2 (Soundness and Completeness) $\Gamma \vdash S$ *iff* $\Gamma \models S$.

Proof [Sketch] The soundness half of this result is routine. We omit some of the details which are similar to the completeness proof we have already seen. Assume that $\Gamma \models S$. We must show that $\Gamma \vdash S$. Again, we have cases as to the syntactic form of S. Perhaps the most interesting is when S is *Some X are Y*.

As before, we define $\leq$ to be from (2). We also have the equivalence relation $\equiv$ from (5). Let the existential sentences in Γ be listed as in (3). Let the set of equivalence classes of $\equiv$ be $[J_1], \ldots, [J_m]$.

$$\frac{\textit{All } X \textit{ are } Z \quad \textit{All } Z \textit{ are } Y}{\textit{All } X \textit{ are } Y} \qquad\qquad \frac{}{\textit{All } X \textit{ are } X}$$

$$\frac{\textit{Some } X \textit{ are } Y}{\textit{Some } Y \textit{ are } X} \qquad\qquad \frac{\textit{Some } X \textit{ are } Y}{\textit{Some } X \textit{ are } X}$$

$$\frac{\textit{All } Y \textit{ are } Z \quad \textit{Some } X \textit{ are } Y}{\textit{Some } X \textit{ are } Z}$$

$$\frac{\textit{All } X \textit{ are } Y \quad J \textit{ is an } X}{J \textit{ is a } Y} \qquad\qquad \frac{J \textit{ is an } X \quad J \textit{ is a } Y}{\textit{Some } X \textit{ are } Y}$$

$$\frac{}{J \textit{ is } J} \qquad\qquad \frac{J \textit{ is } M \quad M \textit{ is } F}{J \textit{ is } F}$$

$$\frac{M \textit{ is } J}{J \textit{ is } M} \qquad\qquad \frac{M \textit{ is an } X \quad J \textit{ is } M}{J \textit{ is an } X}$$

Figure 1: The rules of our formal logic for the sentences in this section.

We take U to be $\{1, \ldots, n\} \cup \{[J_1], \ldots, [J_m]\}$. We assume these sets are disjoint. We define

$$[\![Z]\!] \quad = \quad \{i : \text{either } V_i \leq Z \text{ or } W_i \leq Z\} \cup \{[J] : \text{ for some } M \in [J],\ \Gamma \vdash M \text{ is a } Z\} \tag{6}$$

To finish defining our context, we take $[\![J]\!] = [J]$. That is, the semantics of J is the equivalence class $[J]$.

It is easy to see that the semantics is monotone in the sense that if $V \leq W$, then $[\![V]\!] \subseteq [\![W]\!]$. This implies that all of the universal assertions of Γ are true in our model. The existential assertions in Γ are *Some* V_i *is* W_i for $i \leq n$, and for each i, the number i belongs to $[\![V_i]\!] \cap [\![W_i]\!]$. Finally, consider a sentence J *is a* Z in Γ. Then $\Gamma \vdash J$ *is a* Z. So $[\![J]\!] = [J] \in [\![Z]\!]$. This means that our sentence J *is a* Z is true in our context.

We conclude that *Some* X *are* Y also is true in this context. If there is some number i in $[\![X]\!] \cap [\![Y]\!]$, then the proof of Theorem 2.1 shows that $\Gamma \vdash$ *Some* X *are* Y. The only alternative is when for some J, $[J] \in [\![X]\!] \cap [\![Y]\!]$. By the definition in (6), there are $M \in [J]$ and $N \in [J]$ such that $\Gamma \vdash M$ *is an* X and $\Gamma \vdash N$ *is a* Y. Since $M \in [J]$ and $N \in [J]$ we also have $\Gamma \vdash M$ *is* N. We exhibit a proof tree over Γ:

$$\dfrac{\begin{array}{c}\vdots \\ M \textit{ is an } X\end{array} \qquad \dfrac{\begin{array}{c}\vdots \\ N \textit{ is a } Y\end{array} \quad \begin{array}{c}\vdots \\ M \textit{ is } N\end{array}}{M \textit{ is a } Y}}{\textit{Some } X \textit{ are } Y}$$

The vertical dots $\vdots$ mean that there is some tree over Γ establishing the sentence at the bottom of the dots. So $\Gamma \vdash$ *Some* X *are* Y, as desired. ⊣

Exercise 8 Read the statement of the Completeness Theorem above. The proof that I gave only covers the cases when S is of the form *Some* X *are* Y. The other cases are easier. Give the arguments in those cases.

4 No

In this section, we consider the fragment with *No* X *are* Y on top of *All* X *are* Y. In addition to the rules of Section 1, we take the following:

$$\frac{All\ X\ are\ Z \quad No\ Z\ are\ Y}{No\ X\ are\ Y} \qquad \frac{No\ X\ are\ Y}{No\ Y\ are\ X} \qquad \frac{No\ X\ are\ X}{No\ X\ are\ Y} \qquad \frac{No\ X\ are\ X}{All\ X\ are\ Y}$$

The soundness of this system is routine.

Theorem 4.1 (Completeness) *In the fragment with* All *and* No*, if* $\Gamma \models S$*, then* $\Gamma \vdash S$.

Proof Fix a set Γ. We construct a model $U_\Gamma = (U, [\![\ ]\!])$ and then show that S is true in U_Γ iff $\Gamma \vdash S$. We take for U the set of nonempty sets s of variables satisfying the following conditions:

1. If $V \in s$ and $V \leq W$, then $W \in s$.
2. If $V, W \in s$, then $\Gamma \not\vdash$ *No* V *are* W.

(Note as a special case of the last condition that if $V \in s$, then $\Gamma \not\vdash$ *No* V *are* V.) We set

$$[\![V]\!] \quad = \quad \{s \in U : V \in s\}. \tag{7}$$

We claim that each sentence in Γ is true in U_Γ. Condition (1) implies that if *All* V *are* W belongs to Γ, then $[\![V]\!] \subseteq [\![W]\!]$. Suppose that *No* V *are* W belongs to Γ. Let $s \in [\![V]\!]$. Then $V \in s$. By condition (2), $W \notin s$. So $s \notin [\![W]\!]$. This argument shows that $[\![V]\!] \cap [\![W]\!] = \emptyset$.

We show that if S is true in U_Γ, then $\Gamma \vdash S$. We first deal with the case that S is the of the form *All* X *are* Y. Let

$$s \quad = \quad \{Z : X \leq Z\}.$$

Case I: $s \notin U$. Then there are $V, W \in s$ such that $\Gamma \vdash$ *No* V *are* W. In this case,

$$\cfrac{All\ X\ are\ W \qquad \cfrac{\cfrac{All\ X\ are\ V \quad No\ V\ are\ W}{No\ X\ are\ W}}{No\ W\ are\ X}}{\cfrac{No\ X\ are\ X}{All\ X\ are\ Y}} \tag{8}$$

Case II: $s \in U$. Then since $s \in [\![X]\!]$, we have $s \in [\![Y]\!]$. (7) tells us that $Y \in s$, and so $\Gamma \vdash$ *All* X *are* Y, as desired.

This concludes our work when S is *All* X *are* Y. Suppose that S is *No* X *are* Y. Let

$$s \quad = \quad \{Z : X \leq Z \text{ or } Y \leq Z\}.$$

Note that $X, Y \in s$. Then $s \notin U$, lest $s \in [\![X]\!] \cap [\![Y]\!]$. So there are $V, W \in s$ such that $\Gamma \vdash$ *No* V *are* W.

Case I: $\Gamma \vdash$ *All* X *are* V, and $\Gamma \vdash$ *All* Y *are* W. We have the tree:

$$\cfrac{All\ Y\ are\ W \qquad \cfrac{All\ X\ are\ V \quad No\ V\ are\ W}{No\ X\ are\ W}}{No\ X\ are\ Y} \tag{9}$$

Case II: $\Gamma \vdash$ *All* X *are* V, and $\Gamma \vdash$ *All* X *are* W. In this case, take the proof tree in (8) and change the root to *No* X *are* Y. ⊣