

Q520: Last Linear Algebra Homework, On PageRank
Due: Thursday, April 10

To do this homework, you will need to read two things:

First, the web site

<http://www.ams.org/featurecolumn/archive/pagerank.html>

You actually only need to read the first 1/3 or so of this. You may stop at the point “Computing I ”.

Second, you’ll need to also read the handout I distributed called “The \$25,000,000 eigenvector”. (I only passed out the first six pages, and if anyone wants the remaining five pages I can certainly copy it for them. You can also find this article on the web at

<http://www.rose-hulman.edu/~bryan/google.html>

1. Exercise 1 on page 5 of “The \$25,000,000 eigenvector”.
2. Exercise 4 on page 5 of “The \$25,000,000 eigenvector”. You may use a program such as the Online Matrix Calculator to read off the eigenvalues of your matrix. Note that if you do this, the Online Matrix Calculator can also give you eigenvectors. But the eigenvector that it gives you will probably not be scaled so that its entries sum to 1.
3. The proof of Proposition 1 on page 3 was too fast for us. So this exercise and the next ones will fill in details that are missing.

The first missing point is: why is it true that a square matrix and its transpose have the same eigenvalues? Here is a sketch of one possible reason, with some details left to you.

Let A be any square matrix, and suppose someone hands you a diagonal matrix Σ and an orthogonal matrix U , and it just so happens that $A = U\Sigma U^{tr}$.

Let v be one of the columns of U , say that i th column. What is $U\Sigma U^{tr}v$?

Conclude from this that the eigenvectors of A are exactly the diagonal entries on Σ .

(You may use all of the calculations from class, and also all the facts there as well. You also can use the fact that an $n \times n$ matrix has exactly n eigenvalues (counting repetitions for the different eigenvectors). You should use the fact that the transpose of a product of two or more matrices is the same as the product of the transposes in backwards order.)

4. We now return to Proposition 1. Let’s take a square matrix A . By our calculations in class, we know that A can be factored as $A = U\Sigma U^{tr}$, where Σ is the diagonal matrix of eigenvalues, and U is an orthogonal matrix whose columns are corresponding eigenvectors.

Why is it true that $A^{tr} = U^{tr}\Sigma^{tr}U$?

And putting this together with the previous problem, show how to conclude that A and A^{tr} have the same eigenvectors.

5. Returning once again to Proposition 1, note that the *rows* of A^{tr} add to 1. Let e be the matrix of all 1’s. The text says that “it is easy to see that” e is an eigenvector of A^{tr} with eigenvalue 1. Why is this true?