

Q520: Third Homework: Probability

Due: Thursday, January 24

1. There are many ways to vary the Monty Hall problem. Here is one, based on adopting a specific policy for Monty's action after the contestant has selected a door. As you recall, our discussion in class assumed that Monty would always open a door and show that nothing was inside. We change this. We allow Monty to do nothing at that point. We also assume that Monty works in the following way:
  - a. If the contestant selected the door where the car actually is, then Monty opens one of the other two, each with a 50/50 chance.
  - b. If the contestant selected a door with nothing behind it, then 2/3 of the time Monty will do nothing, and 1/3 of the time Monty will open the empty door which the contestant did not pick.

Analyze this variation. The main thing is to decide whether it is a better policy to switch when given the opportunity.

2. Let  $A$  and  $B$  be events on some probability space. Assume that  $0 < \Pr(B) < 1$ . Prove that  $A$  and  $B$  are independent if and only if  $\Pr(A|B) = \Pr(A|\bar{B})$ .
3. Assume none of the probabilities is zero, prove that the following are equivalent:
  - a.  $\Pr(A \cap B|C) = \Pr(A|C) \cdot \Pr(B|C)$ .
  - b.  $\Pr(A|B \cap C) = \Pr(A|C)$ .

[To do this means doing two things: First, assume the equation of (a) and prove the equation in (b) using whatever algebra you need. Second, assume the equation of (b) and prove the equation in (a), again using whatever algebra you need. The kinds of algebra facts you need are things like  $x/y = x/z \times z/y$  provided  $z \neq 0$ . You will also need the overall assumptions of this problem that none of the probabilities involved are zero.]

4. Assume that  $A$  and  $B$  are independent given  $C$ .
  - a Prove that  $\bar{A}$  and  $B$  are again independent given  $C$ .
  - b It now *follows* from (a) that  $\bar{A}$  and  $\bar{B}$  are again independent given  $C$ . Explain this.
5. Consider what happens when we flip a fair coin 100 times. The sample space will be the set of 100-tuples of  $h$  and  $t$ 's. We'll write such a sequence as  $(s_1, \dots, s_{100})$ . Consider the following random variables:

$$X(s) = \begin{cases} 1 & \text{if } s_1, \dots, s_{99} \text{ are all } h \\ 0 & \text{otherwise} \end{cases}$$

$$Y(s) = \begin{cases} 1 & \text{if } s_{100} \text{ is } h \\ 0 & \text{otherwise} \end{cases}$$

Are  $X$  and  $Y$  independent random variables are not?

Most people's intuitions are that  $X$  and  $Y$  should not be independent. Why do you think this is?

6. (derived from Kathryn Blackmond Laskey's course) It is estimated that on a typical night, 1 in 1000 drivers approaching a certain roadblock has too much alcohol in his or her blood. If a driver has consumed too much alcohol, the test will give a positive result with probability 0.99. If the driver has not consumed too much alcohol, the test will give a negative result with probability 0.999. If a driver's test comes out positive, what is the probability that the driver has indeed consumed too much alcohol?

Let us suppose that on New Years Eve, the '1 in 1000' figure goes up to 20%. Calculate the probability that a driver who has tested positive at a roadblock is over the legal limit.

Comment on these results. Can you explain the difference between the answers in the different cases?