Logic and the Sizes of Sets

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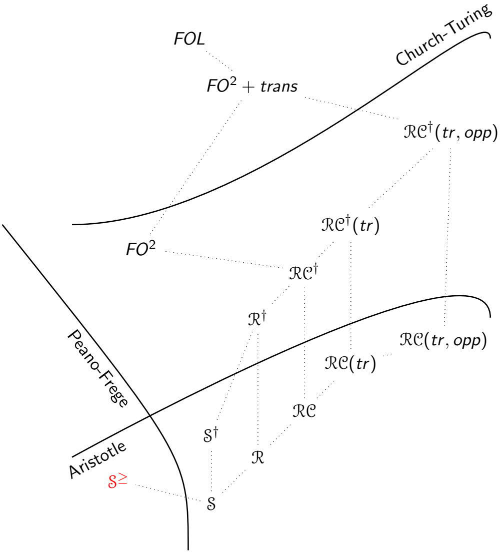
EASLLI 2014

Map of Some Natural Logics



first-order logic

FO^2 + "R is trans"

2 variable FO logic

† adds full N-negation

$\mathcal{RC}(tr)$ + opposites

$\mathcal{RC}$ + (transitive) comparative adjs

$\mathcal{R}$ + relative clauses

$\mathcal{S}$ + full N-negation

$\mathcal{R}$ = relational syllogistic

$\mathcal{S}^{\geq}$ adds $|p| \geq |q|$

$\mathcal{S}$: all/some/no p are q

Beyond first-order logic: cardinality

Read $\exists^{\geq}(X, Y)$ as "there are at least as many Xs as Ys".

$$\frac{\textit{All } Y \textit{ are } X}{\exists^{\geq}(X, Y)} \qquad \frac{\exists^{\geq}(X, Y) \quad \exists^{\geq}(Y, Z)}{\exists^{\geq}(X, Z)}$$

$$\frac{\textit{All } Y \textit{ are } X \quad \exists^{\geq}(Y, X)}{\textit{All } X \textit{ are } Y} \ \text{finiteness}$$

$$\frac{\textit{Some } Y \textit{ are } Y \quad \exists^{\geq}(X, Y)}{\textit{Some } X \textit{ are } X} \qquad \frac{\textit{No } Y \textit{ are } Y}{\exists^{\geq}(X, Y)}$$

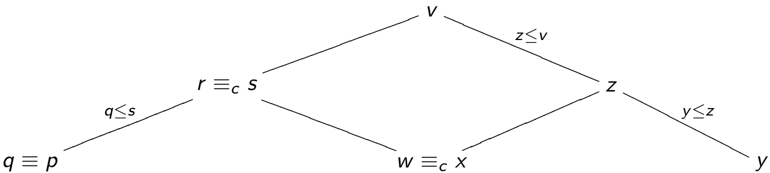
The point here is that by working with a weak basic system, we can say things which cannot be said in first-order logic.

An example of the main part of the proof

Suppose that Γ is the following set of sentences:

All p are q
There are at least as many q as p
All q are s
There are at least as many r as s
There are at least as many s as r
There are at least as many w as x
There are at least as many x as w
There are at least as many x as r
All y are z
There are at least as many w as z
All z are v
There are at least as many s as v

We define relations $\leq$ and $\leq_c$ in the obvious way, and draw a picture.
The lines are the $\leq_c$ relation, reading upward, with the stronger $\leq$ relations shown.



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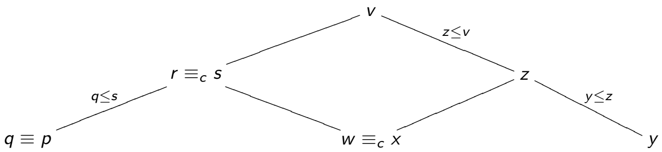

We start with distinct elements $*_p = *_q$, $*_r$, $*_s$, $*_w$, $*_x$, $*_y$, $*_z$.
We construct sets to interpret these variables going "bottom-up" using the listing of $\mathcal{V}/\equiv_c$:

$$[p],\ [w],\ [r],\ [y],\ [z],\ [v].$$

Each time we need fresh elements, we shall use numbers.

$$\begin{array}{lll} [\![p]\!] & = & \{*_p\} \\ [\![q]\!] & = & \{*_p\} \\ [\![w]\!] & = & \{*_w, 1\} \\ [\![x]\!] & = & \{*_x, 2\} \end{array} \qquad \begin{array}{lll} [\![r]\!] & = & \{*_r, 3, 4, 5, 6, 7\} \\ [\![s]\!] & = & \{*_p, *_s, 8, 9, 10, 11\} \\ [\![y]\!] & = & \{*_y, 12, \ldots, 22, 23\} \\ [\![z]\!] & = & \{*_y, *_z, 12, \ldots, 22, 23\} \\ [\![v]\!] & = & \{*_v, *_y, *_z, 12, \ldots, 23, 24, 25, \ldots, 39\} \end{array}$$

Every sentence φ follows from Γ iff it $\mathcal{M} \models \varphi$.

We just saw $\mathsf{S}^{\leq}$.
We now come to a logic which I'll call $\mathcal{S}^{\dagger}_{\leq}$.
Perhaps the largest known complete logic
about the sizes of subsets of a finite universe.

- *All*, *Some*,
  There are at least as many x as y, written $\exists^{\geq}(x, y)$
  There are more x than y, written $\exists^{>}(x, y)$
- Complemented variables x'

A lot of the action in the axiomatization has to do with assertions

$\exists^{\geq}(x, x')$	at least half of all objects are x's
$\exists^{\geq}(x', x)$	at least half of all objects are non-x's
	$\equiv$ at most half of all objects are x's
$\exists^{>}(x, x')$	more than half of all objects are x's
$\exists^{>}(x', x)$	more than half of all objects are non-x's
	$\equiv$ less than half of all objects are x's

$$\frac{}{\forall(p,p)}\ \text{(axiom)} \qquad \frac{\forall(n,p)\quad \forall(p,q)}{\forall(n,q)}\ \text{(Barbara)} \qquad \frac{\exists(p,q)}{\exists(p,p)}\ \text{(some)}$$

$$\frac{\exists(q,p)}{\exists(p,q)}\ \text{(conversion)} \qquad \frac{\forall(p,q)}{\forall(q',p')}\ \text{(anti)} \qquad \frac{\forall(p,p')}{\forall(p,q)}\ \text{(zero)}$$

$$\frac{\exists(p,n)\quad \forall(n,q)}{\exists(p,q)}\ \text{(Darii)} \qquad \frac{\forall(p',p)}{\forall(q,p)}\ \text{(one)} \qquad \frac{\forall(p,q)}{\exists^{\geq}(q,p)}\ \text{(subset-size)}$$

$$\frac{\exists^{\geq}(p,q)}{\exists^{\geq}(q',p')}\ \text{(card-mon)} \qquad \frac{\exists^{\geq}(p,q)}{\exists^{\geq}(q',p')}\ \text{(card-anti)} \qquad \frac{\forall(p,q)\quad \exists^{\geq}(p,q)}{\forall(q,p)}\ \text{(card-mix)}$$

$$\frac{\exists(p,p)\quad \exists^{\geq}(p,q)}{\exists(q,q)}\ \text{(card-}\exists\text{)} \qquad \frac{\forall(q,p)\quad \exists(p,q')}{\exists^{>}(p,q)}\ \text{(more)} \qquad \frac{\exists^{>}(p,q)}{\exists(p,q')}\ \text{(more-some)}$$

$$\frac{\exists^{>}(p,q)}{\exists^{\geq}(p,q)}\ \text{(more-at least)} \qquad \frac{\exists^{>}(n,p)\quad \exists^{\geq}(p,q)}{\exists^{>}(n,q)}\ \text{(more-left)} \qquad \frac{\exists^{>}(q,p)}{\exists^{>}(p',q')}\ \text{(more-anti)}$$

$$\frac{\exists(p,p)\quad \exists^{\geq}(q,q')}{\exists(q,q)}\ \text{(int)} \qquad \frac{\exists^{\geq}(p,p')\quad \exists^{\geq}(q',q)}{\exists^{\geq}(p,q)}\ \text{(half)} \qquad \frac{\exists^{>}(p,p')\quad \exists^{\geq}(q',q)}{\exists^{>}(p,q)}\ \text{(strict half)}$$

$$\frac{\exists^{\geq}(p,p')\quad \exists^{\geq}(q,q')\quad \exists(p',q')}{\exists(p,q)}\ \text{(maj)}$$

$$\frac{\exists(p,q)\quad \forall(p,q')}{\varphi}\ \text{(X)} \qquad \frac{\exists^{>}(p,q)\quad \exists^{\geq}(q,p)}{\varphi}\ \text{(X)}$$

The logic of most X are Y and not most X are Y

Our next-to-last logic strikes off in a different direction.

We take sentences of the form $M(X, Y)$ and $\neg M(X, Y)$.

We call this logic $\mathcal{L}(most)$.

A model of this tiny language is a structure $\mathcal{M} = (M, [\![\ \ ]\!])$
consisting of a finite set M
together with interpretations $[\![X]\!] \subseteq M$ of each X.

We then interpret our sentences in a model as follows

$$\mathcal{M} \models M(X, Y) \quad \text{iff} \quad |[\![X]\!] \cap [\![Y]\!]| > \tfrac{1}{2}|[\![X]\!]|$$

$$\mathcal{M} \models \neg M(X, Y) \quad \text{iff} \quad |[\![X]\!] \cap [\![Y]\!]| \leq \tfrac{1}{2}|[\![X]\!]|$$

Are there any valid principles at all?

$$\frac{M(X,Y) \quad M(Y,Z)}{M(X,Z)} \text{ ???}$$

$$\frac{M(X,Y)}{M(X,Y)} \text{ ???}$$

$$\frac{M(Y,X)}{M(X,Y)} \text{ ???}$$

$$\frac{M(X,Y)}{M(X,X)} \text{ ???}$$

An example of the kind of question we are interested in

Let

$$\Gamma = \left\{ \begin{array}{rr} M(X,Y) & M(Z,Y) \\ M(Y,X) & M(Y,W) \\ M(X,Z) & \neg M(W,Y) \\ \neg M(Z,X) & M(Z,W) \\ M(Y,Z) & \end{array} \right\}$$

Is it true or not that

$$\Gamma \models M(W,Z) \text{ ?}$$

An example of the kind of question we are interested in

Let

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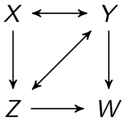
Is it true or not that

$$\Gamma \models M(W,Z)\ ?$$

I claim that the answer is no.
We shall take the graph below



and turn the nodes g into sets A_g so that $g \to h$ iff "most A_g are A_h."

The heart of the completeness argument

A majority graph is a finite simple graph $(G, \to)$ such that there exist finite sets A_g for $g \in G$ with the following property:

$g \to h$ if and only if "more than half of the A_g are A_h".

That is,

$$g \to h \quad \text{iff} \quad |A_g \cap A_h| > \frac{1}{2} \cdot |A_g|.$$

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That is,

$$g \to h \quad \text{iff} \quad |A_g \cap A_h| > \frac{1}{2} \cdot |A_g|.$$

A two-way edge in a graph is an edge $g \to h$ such that also $h \to g$.
A one-way edge in a graph is an edge $g \to h$ such that $h \not\to g$.

If G is a majority graph and there is a one-way edge from g to h, then $|A_h| > |A_g|$.

Observation by Chloe Urbanski

Thus G cannot have one-way cycles: there are no paths

$$g_1 \to g_2 \to \cdots \to g_n = g_1$$

such that $g_{i+1} \not\to g_i$. (There may be cycles with two-way edges.)

Theorem (Tri Lai 2013)

Every graph without one-way cycles is a majority graph.

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Every graph without one-way cycles is a majority graph.

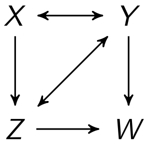
We can even get a stronger result.
For any $\alpha \in (0,1)$, we say that G is a
proportionality α-graph
if there are sets A_g for $g \in G$ such that

$$g \to h \quad \text{iff} \quad |A_g \cap A_h| > \alpha \cdot |A_g|.$$

Theorem (Tri Lai, Jörg Endrullis, and LM 2013)

For all $\alpha \in (0,1)$,
every graph without one-way cycles is a proportionality α-graph.

Our goal is to find sets for the graph below:



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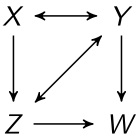


We begin with four subsets of $\{1, \ldots, 16\}$ each of size 8, with the property that distinct sets have intersections of size 4:

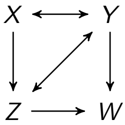
$$\begin{aligned} A_X &= \{1,2,3,4,5,6,7,8\} \\ A_Y &= \{1,2,3,4,9,10,11,12\} \\ A_Z &= \{1,2,5,6,9,10,13,14\} \\ A_W &= \{1,3,5,7,9,11,13,15\} \end{aligned}$$

For $i \neq j$, we write $A_i \sqcap A_j$ for the private intersection:

$$A_i \sqcap A_j = (A_i \cap A_j) \setminus \bigcup_{k \neq i,j} A_k$$

Illustration of how the proof goes

Our goal is to find sets for the graph below:



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$$A_i \sqcap A_j \quad = \quad (A_i \cap A_j) \setminus \bigcup_{k \neq i,j} A_k$$

For $i \neq j$, $A_i \sqcap A_j$ has size 1.
For example, $A_X \sqcap A_Z = \{6\}$.

So far, we have

$X \qquad Y$

$Z \qquad W$

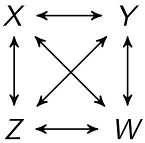
We replace each point x by three copies of itself, $3x-2$, $3x-1$, and $3x$.

$$\begin{aligned}
A_X &= \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24\} \\
A_Y &= \{1,2,3,4,5,6,7,8,9,10,11,12,25,26,27,28,29,30,31,32,33,34,35,36\} \\
A_Z &= \{1,2,3,4,5,6,13,14,15,16,17,18,25,26,27,28,29,30,37,38,39,40,41,42\} \\
A_W &= \{1,2,3,7,8,9,13,14,15,19,20,21,25,26,27,31,32,33,37,38,39,43,44,45\}
\end{aligned}$$

We then take three fresh points, 49, 50, and 51, add them to all sets A_i.

Then add one new point to A_Y, two new points to A_Z, and three to A_W.

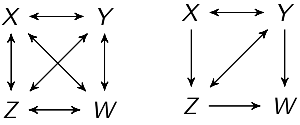
At this point, we have



$$
\begin{aligned}
A_X &= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 49, 50, 51\}\\
A_Y &= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 49, 50, 51, 52\}\\
A_Z &= \{1, 2, 3, 4, 5, 6, 13, 14, 15, 16, 17, 18, 25, 26, 27, 28, 29, 30, 37, 38, 39, 40, 41, 42, 49, 50, 51, 53, 54\}\\
A_W &= \{1, 2, 3, 7, 8, 9, 13, 14, 15, 19, 20, 21, 25, 26, 27, 31, 32, 33, 37, 38, 39, 43, 44, 45, 49, 50, 51, 55, 56, 57\}
\end{aligned}
$$

Now $|A_X| = 27$, $|A_Y| = 28$, $|A_Z| = 29$, and $|A_W| = 30$.

For $i \neq j$, $|A_i \cap A_j| = 15$, and $|A_i \sqcap A_j| = 3$.



current goal

$$
\begin{aligned}
A_X &= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 49, 50, 51\} \\
A_Y &= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 49, 50, 51, 52\} \\
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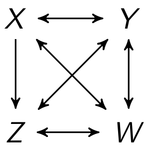
\end{aligned}
$$

We have already arranged that $A_X \to A_Y$ and $A_Y \to A_X$.

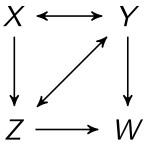
Here is how we arrange that $A_X \to A_Z$ and $A_Z \not\to A_X$.

Take the "private intersection" $A_X \sqcap A_Z = \{16, 17, 18\}$.
Remove 16 from A_X and A_Z, and return it as two fresh points $58 \in A_X$ and $59 \in A_Z$.

The point is that now $|A_X \cap A_Z| = 14$, and $\frac{14}{29} < \frac{1}{2} < \frac{14}{27}$.



current



goal

Similar tricks arrange all of our other requirements.

We get

$$
\begin{aligned}
A_X &= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 18, 22, 23, 24, 49, 50, 51, 58, 60, 61, 62\} \\
A_Y &= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 49, 50, 51, 52\} \\
A_Z &= \{1, 2, 3, 4, 5, 6, 13, 14, 15, 17, 18, 25, 26, 27, 28, 29, 30, 37, 38, 39, 40, 41, 42, 49, 50, 51, 53, 54, 59\} \\
A_W &= \{1, 2, 3, 7, 8, 9, 13, 14, 15, 25, 26, 27, 31, 32, 33, 37, 38, 39, 43, 44, 45, 49, 50, 51, 55, 56, 57, 63, 64, 65\}
\end{aligned}
$$

This exhibits our graph G as a majority graph.

$$
\begin{aligned}
A_X &= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 18, 22, 23, 24, 49, 50, 51, 58, 60, 61, 62\} \\
A_Y &= \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 49, 50, 51, 52\} \\
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\end{aligned}
$$

Recall our set

$$
\Gamma = \left\{ \begin{array}{rr} M(X,Y) & M(Z,Y) \\ M(Y,X) & M(Y,W) \\ M(X,Z) & \neg M(W,Y) \\ \neg M(Z,X) & M(Z,W) \\ M(Y,Z) & \end{array} \right\}
$$

We have built a model to see that

$$
\Gamma \not\models M(W,Z)
$$

Theorem (Tri Lai 2013)

Every graph without one-way cycles is a majority graph.

Theorem [LM]

Here is a complete logical system for this language.

$$\frac{M(X,Y)}{M(X,X)} \qquad \frac{M(X,Y)}{M(Y,Y)}$$

One of the infinitely many rules is

$$\frac{M(X,Y) \quad M(Y,Z) \quad M(Z,X) \quad \neg M(X,Z) \quad \neg M(Z,Y)}{M(Y,X)}$$

"There are no one-way cycles $X \to Y \to Z \to X$."

Variation: all, some, most

but no negation

$$\frac{}{\text{All } X \text{ are } X} \qquad \frac{\text{All } X \text{ are } Y \quad \text{All } Y \text{ are } Z}{\text{All } X \text{ are } Z}$$

$$\frac{\text{Some } X \text{ are } Y}{\text{Some } Y \text{ are } X} \qquad \frac{\text{Some } X \text{ are } Y}{\text{Some } X \text{ are } X} \qquad \frac{\text{Some } X \text{ are } Y \quad \text{All } Y \text{ are } Z}{\text{Some } X \text{ are } Z}$$

Can you think of any valid laws that add $M(X, Y)$ on top of All X are Y and Some X are Y?

Variation: all, some, most

but no negation

$$\frac{}{\text{All } X \text{ are } X} \qquad \frac{\text{All } X \text{ are } Y \quad \text{All } Y \text{ are } Z}{\text{All } X \text{ are } Z}$$

$$\frac{\text{Some } X \text{ are } Y}{\text{Some } Y \text{ are } X} \qquad \frac{\text{Some } X \text{ are } Y}{\text{Some } X \text{ are } X} \qquad \frac{\text{Some } X \text{ are } Y \quad \text{All } Y \text{ are } Z}{\text{Some } X \text{ are } Z}$$

$$\frac{\text{Most } X \text{ are } Y}{\text{Some } X \text{ are } Y}\, m_1 \qquad \frac{\text{Some } X \text{ are } X}{\text{Most } X \text{ are } X}\, m_2 \qquad \frac{\text{Most } X \text{ are } Y \quad \text{All } Y \text{ are } Z}{\text{Most } X \text{ are } Z}\, m_3$$

$$\frac{\text{Most } X \text{ are } Z \quad \text{All } X \text{ are } Y \quad \text{All } Y \text{ are } X}{\text{Most } Y \text{ are } Z}\, m_4$$

$$\frac{\text{All } Y \text{ are } X \quad \text{All } X \text{ are } Z \quad \text{Most } Z \text{ are } Y}{\text{Most } X \text{ are } Y}\, m_5$$

$$\frac{X_1 \triangleright_{A,B} Y_1 \quad Y_1 \triangleright_{B,A} X_2 \quad \cdots \quad X_n \triangleright_{A,B} Y_n \quad Y_n \triangleright_{B,A} X_1}{\text{Some } A \text{ are } B}\, \triangleright$$

$$\frac{X_1 \triangleright_{A,B} Y_1 \quad Y_1 \triangleright_{B,A} X_2 \quad \cdots \quad X_n \triangleright_{A,B} Y_n}{\text{Some } A \text{ are } B} \triangleright_{Y_n \triangleright_{B,A} X_1}$$

Examples:

$$\frac{\text{Most } Z \text{ are } X \quad \text{Most } Z \text{ are } Y}{\text{Some } X \text{ are } Y} \triangleright$$

Another example: From

$$\begin{array}{l} \text{Most } X \text{ are } B', \text{All } A' \text{ are } A, \text{Most } Y \text{ are } A', \text{All } B' \text{ are } B, \text{All } X \text{ are } Y \\ \text{Most } Y \text{ are } A'', \text{All } A'' \text{ are } A, \text{Most } X \text{ are } B'', \text{All } B'' \text{ are } B, \text{All } A'' \text{ are } X \end{array}$$

infer

$$\text{Some } A \text{ are } B.$$

As an example, Some X are X, All X are $Y \vdash$ Most X are Y via the tree below:

$$\dfrac{\dfrac{\text{Some } X \text{ are } X}{\text{Most } X \text{ are } X} \quad \text{All } X \text{ are } Y}{\text{Most } X \text{ are } Y}$$

Theorem (Jörg Endrullis & LM (2013))

The logical system for this language is complete.

Theorem

Infinitely many axioms are needed in the system.

Theorem

The decision problem for the consequence relation

$$\Gamma \vdash \varphi$$

is in polynomial time.

- Get a such complete logic for

All X are Y	Some X are Y	Most X are Y
No X are Y	$\exists^{\geq}(X, Y)$	

and sentential $\wedge$, $\vee$, and $\neg$.

- Alternatively, prove that there is no such logic.
- Investigate the algorithmic properties of the logic.