Uniform Functors on Sets

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Dedicated to Joseph Goguen on his 65th birthday

Abstract

This paper is a contribution to the study of uniformity conditions for endofunctors on sets initiated in Aczel [1] and pursued later in other works such as Turi [17]. The main results have been that the "usual" functors on sets are uniform in our sense, and that assuming the Anti-Foundation Axiom *AFA*, a uniform functor H has the property that its greatest fixed point H^* is a final coalgebra whose structure is the identity map. We propose a notion of uniformity whose definition involves notions from recent work in coalgebraic recursion theory such as completely iterative monads and completely iterative algebras (CIAs), see Adámek et al. [2, 3, 6] and Milius [11]. This simplifies many calculations and makes the definition of uniformity more natural than it had been. We also present several new results, including one which could be called a *Paranoia Theorem*: For a uniform H, the entire universe is a CIA: the structure is the inclusion of HV into the universe V itself.

1 Introduction

I have considered Joseph Goguen to be one of my main teachers for many years. My first encounter with him was in an undergraduate course in the theory of computation given at UCLA around 1979. What I remember most is that serious students had to both write research papers and take an oral final exam, and looking back I see it as both a serious didactic move and a way to take seriously the thoughts of students. After hearing of my interests in mathematics and linguistics, he suggested that I write on representing inexact concepts in Montague grammar, thereby mixing topics that he considered interesting: formal semantics and fuzzy logic. My short paper on this was one of the only ones I wrote during my undergraduate years. Later, I took a seminar that he and Charlotte Linde taught on natural language processing. Here I remember being surprised by their strong anti-Chomskyanism and advocacy of views that there was no "real world." I also remember his sense of humor as well as his more serious side.

A few years later, I found myself at Stanford's Center for the Study of Language and Information. Joseph had moved to SRI a few years earlier and was also at CSLI. I don't know how we started, but he and José Meseguer started meeting to decide on something to work on together. They pointed me to a conjecture of theirs on abstract data type computability which I settled and wrote up in a paper with the two of them. I remember this as "on-the-job training" in category theory. I also remember Joseph's delightful influence all over CSLI during that time. I mainly lots touch with him after that, though at some point after moving to UCSD we met again through a mutual friend, Martin Schapiro.

For me, one of the most long-lasting influences were various pointers to currents in the social sciences, along with indications that these deserved to be taken seriously in computer science and artificial intelligence. Another was the mingling and mixing of ideas from Western science and Eastern religions; despite being raised in Los Angeles and with Alan Watts' lectures regularly on my radio, I had met anyone who lived the ideas. I am still inspired by his wide-ranging concerns and penetrating insights into many subjects. In all of these, I am reminded of a character in the story of his namesake the Joseph of Genesis, the "man" in 37:15-17, someone who points people to important places and ideas. It is a pleasure to thank Joseph for his many years of direct inspiration to me and wish him many more years of work and fun.

1.1 What has happened to the study of recursive program schemes?

The title of our volume is "Semantics, Algebra, and Computation," and so I want to make the case that my contribution is related to all three points. 'Semantics' means meaning, and just as in the distantly-related areas of the semantics of natural language, the project in computer science is to give some sort of mathematical model of meaning. Today the semantic project attracts less attention and prestige than the study of algorithmic complexity. (This is especially true in the USA.) Still, the area is important because if one wants to be sure that computer programs 'do what they are supposed to', then one quickly needs formally specified and tractable notions of meaning. I think it is fair to say that the centerpiece of the semantic project concerning computation is the treatment of *recursion*, the main mechanism of 'looping' for computer programs and algorithms. The work reported here is an offshoot of *coalgebraic recursion theory*, an application of ideas from coalgebra and closely related fields to circular phenomena and more recently to recursive program schemes. Many of the mathematical tools that are now common in semantics were first introduced for the study of recursive program schemes. I would like to think that some of the notions in coalgebraic recursion theory also will enter the mainstream of semantic research. I also think that some of this work allows one to approach the semantics of computation from an even more algebraic perspective than previous studies. For example, one of Joseph Goguen's early papers mentions the use of initial continuous algebras in connection with recursion. As a result of very recent work, it turns out that one can dispense with the domain-theoretic underpinnings of continuous algebras (or more precisely, one has a clearer understanding of the principles that make continuous algebras work in the first place). But none of this is a main point in this paper, however, however, and so I will only touch on these matters in passing. For a longer discussion, one could see [12] and some papers referred to therein.

Given the importance of recursion and recursive program schemes, one has to ask why the subject is not pursued so intensively these days. Here are two possible answers: (a) the work that had been done was mathematically challenging, requiring expertise in both algebra and domain theory; and at the same time it was not clear that the results coming out of it justified one's mastery of the field. And (b) there were many easier things to do, and many of them had a closer connection to computer science practice.

This paper is actually about a different subject, but with a connection: our concern here is the notion of *uniformity* for functors on sets that goes back to Peter Aczel's book [1] on non-wellfounded sets. The book contains a short discussion of what it called the *Special Final Coalgebra Theorem*, a result that gives a sufficient condition for a functor on the category of sets (actually the category of classes) to have a final coalgebra whose carrier is the greatest fixed point of the functor and whose structure map is the identity. This is a natural matter to investigate from the point of view of the book. However, the particular condition given was difficult to understand and work with, and so it fell to other researchers to clarify the matter. This has been the subject of a number of other papers such as [15, 14, 18, 17]. This paper is another in the same line. It revisits the discussion in the light of concepts introduced by Adámek and his coworkers: mainly completely iterative monads and algebras. This paper formulates a new notion of uniformity for functors and studies it under AFA, and it also obtains some new results.

2 Background

In this section, we present the background that we need in two parts: background from coalgebra, and background from set theory. In both cases, the background will be unusual. From coalgebra, we need a set of definitions from a handful of recent papers. I doubt that most people who glance at this paper will have heard of any of these notions, and I also know that the spare presentation here will not really help one to get a feeling for the substantial work in the area. On the set theoretic side, most of the background concerns non-standard subjects such as non-wellfounded sets and functors on the category of classes.

The next two sections may be read in either order.

2.1 Background from coalgebra

Let $\mathcal{A}$ be category with a fixed finite coproduct operation $\oplus$.[1] An endofunctor $F : \mathcal{A} \to \mathcal{A}$ is *iteratable* [sic]: for each object X, the functor $H(\_) \oplus X$ has a final coalgebra. This condition of iteratabiltiy is satisfied by many functors of interest; it is perhaps most pertinent to this paper that results of Aczel and Mendler [4] later strengthened by Adámek et al [5] show that every endofunctor on the category of classes is iteratable. In the setting of this paper, it will be important to remember that the power set functor is iteratable on the category of classes but not on the category of sets. On the other hand the subfunctors $\mathcal{P}_\kappa$ are iteratable on the category of sets; here $\mathcal{P}_\kappa(X)$ is the set of subsets of X whose cardinality is at most κ.

If H is iteratable, then for each object X we have a final $H(\_) \oplus X$-coalgebra (TX, α_X). As the notation indicates, T extends to a functor and α to a natural transformation. T has many properties, but only a few are explicitly needed in this paper. For example, we need at one point that in the endofunctor category $[\mathcal{A}, \mathcal{A}]$, the functor $G \mapsto (H \cdot G) \oplus Id$ has (T, α) as a final coalgebra. Moreover, H brings not only T but also a free *completely iterative monad*. For our purposes, a completely iterative monad based on H is a monad (T, μ, η) together with natural transformations $\alpha : T \to HT \oplus Id$ and $\tau : HT \to T$ such that

1. For all objects X of $\mathcal{A}$, (TX, α_X) is a final coalgebra of $H(\_) \oplus X$.

2. $[\tau, \eta] : HT \oplus Id \to T$ is the pointwise inverse of α.

3. Every suitably guarded equation morphism has a unique solution.

[1]We are using the symbol $\oplus$ in this section rather than the more usual symbol + for coproducts. In this paper, + will denote the specific coproduct on sets or classes given by the Kuratowski pairing operation (see Section 2.2). We use the different notations to help the reader with this distinction.

Actually, the first point is the key and the other are consequences and/or strengthenings of it. We shall not need the precise formulation of the last point, so we omit it.

We shall always write κ for $\tau \cdot H\eta$. In general, an *ideal natural transformation into T* is one that factors through τ; so κ, for example, is ideal. We have the following facts:

Proposition 2.1 *The diagrams below commute:*

$$
\begin{array}{ccc}
HTT & \xrightarrow{\tau T} & TT \\
{\scriptstyle H\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\
HT & \xrightarrow[\tau]{} & T
\end{array}
\qquad\qquad
\begin{array}{ccc}
HT & \xrightarrow{\tau} & T \\
{\scriptstyle \kappa T}\downarrow & \nearrow{\scriptstyle \mu} & \\
TT & &
\end{array}
$$

Completely iterative algebras The following notion is studied in Milius [11] and other papers. Let $H : \mathcal{A} \to \mathcal{A}$ be an endofunctor. By a *flat equation morphism* in an object A (*of parameters*) we mean a morphism

$$e : X \to HX \oplus A\,.$$

Let $(A, a : HA \to A)$ be an H-algebra. We say that $s : X \to A$ is a *solution* of e in (A, a) if the square

$$
\begin{array}{ccc}
X & \xrightarrow{e} & HX \oplus A \\
{\scriptstyle s}\downarrow & & \downarrow{\scriptstyle Hs\oplus A} \\
A & \xleftarrow[{[a,A]}]{} & HA \oplus A
\end{array}
$$

commutes. (Note that we use often use the name of an object (such as A) as a name of the identity morphism on it.) And we call (A, a) a *completely iterative algebra* (or *CIA*) for H if every flat equation morphism in A has a unique solution in it.

Proposition 2.2 (AMV [6], Milius [11]) *Concerning CIAs and completely iterative monads:*

1. *If (A, a^{-1}) is a final coalgebra for H, then (A, a) is a CIA for H.*

2. *(TX, τ_X) is a CIA for H.*

3. *For every CIA (A, a) for H, the solution to the flat equation morphism α_A is an Eilenberg-Moore algebra of the monad T. We write this solution morphism as $\widetilde{a} : TA \to A$.*

4. *Moreover, for every CIA (A, a) for H, the triangle*

$$
\begin{array}{ccc}
HA & \xrightarrow{a} & A \\
{\scriptstyle \kappa_A}\downarrow & \nearrow{\scriptstyle \widetilde{a}} & \\
TA & &
\end{array}
$$

   commutes.

Definition Let H be iteratable, let T be the associated monad, and let (A, a^{-1}) be a final

H-coalgebra. Recall that $\widetilde{a} : TA \to A$ is the Eilenberg-Moore algebra structure associated to A; it is the solution to the flat equation morphism $\alpha_A : TA \to HTA + A$ with parameters in A. For any morphism of the form $f : B \to A$, we let $[\![f]\!] : TB \to A$ be given by

$$[\![f]\!] \quad = \quad \widetilde{a} \cdot Tf.$$

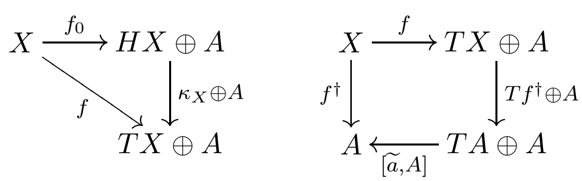
Lemma 2.3 *Once again, let (A, a^{-1}) be a final H-coalgebra. Here are some properties of the morphisms $[\![f]\!]$, where $f : B \to A$.*

1. $[\![f]\!] \cdot \eta_B = f$.
2. $[\![f]\!] \cdot \tau_B = a \cdot H[\![f]\!]$.
3. $[\![\mathsf{id}_A]\!] \cdot T[\![f]\!] = [\![[\![f]\!]]\!] = [\![f]\!] \cdot \mu_B$.
4. $[\![f]\!] = [\![\mathsf{id}_A]\!] \cdot Tf$.

The proofs are routine calculations using naturality and the definition of an Eilenberg-Moore algebra of a monad.

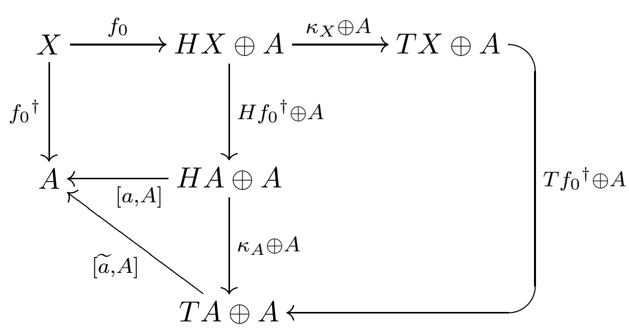
We also need what would be considered a folkloric result. In the statement below and in the sequel, recall that final coalgebra morphisms are always categorical isomorphisms.

Lemma 2.4 *Let (A, a^{-1}) be a final H-coalgebra, so that (A, a) is a CIA for H. Let $f : X \to TX \oplus A$ factor as on the left below.*



Then there is a unique $f^\dagger : X \to A$ such that $f^\dagger = [\widetilde{a}, A] \cdot (Tf^\dagger \oplus A) \cdot f$.

Proof We have already mentioned the result in Milius [11] to the effect that (A, a) is a CIA for H. Thus there is a unique morphism ${f_0}^\dagger$ making the square in the upper left below commute:



The triangle commutes using Proposition 2.1, and the square on the right by naturality of κ. So the outside of the figure commutes, showing that ${f_0}^\dagger$ is a morphism with the properties requested in our result. And if $g : X \to A$ is any morphism making the outside of the figure commute, then the square at the upper left commutes. Thus we have $g = {f_0}^\dagger$. This establishes the uniqueness of solutions. $\dashv$

We also need a version of this result formulated in terms of the earlier notation.

Lemma 2.5 *Let (A, a^{-1}) be a final H-coalgebra, and let $f : X \to TX$ factor through $\kappa_X : HX \to TX$. Then there is a unique $f^\dagger : X \to A$ such that*

$$f^\dagger \quad = \quad [\![f^\dagger]\!] \cdot f.$$

Proof Apply Lemma 2.4 to $\mathsf{inl} \cdot f : X \to TX \oplus A$. There is a unique $g : X \to A$ such that

$$g \quad = \quad [\widetilde{a}, A] \cdot (Tg \oplus A) \cdot \mathsf{inl} \cdot f \quad = \quad \widetilde{a} \cdot Tg \cdot f \quad = \quad [\![g]\!] \cdot f.$$

We take g for the needed morphism $f^\dagger$. For the uniqueness, if $g = [\![g]\!] \cdot f$, then $g = [\widetilde{a}, A] \cdot (Tg \oplus A) \cdot \mathsf{inl} \cdot f$; hence we are done by Lemma 2.4. ⊣

We emphasize that the background in this section only contains a hint of a more extensive subject that is currently an active area. Not only have I omitted many motivational points connected to recursive program schemes, first- and second-order substitution, the very interesting notion of an *Elgot algebra*, and the like. I also have not even mentioned all of the results that this paper will call upon. the results that we are going to use directly. One place to read about all of this an more is Stefan Milius' and my paper [12].

2.2 Background from set theory

We remind the reader of the basic facts of set theory which will be relevant in this paper.

The Kuratowski ordered pair (a, b) of two sets a and b is $\{\{a\}, \{a, b\}\}$. In terms of this one defines and studies relations, functions, and the like. One also defines versions of the natural numbers by: $0 = \emptyset$, $1 = \{\emptyset\}$, etc. Finally, we shall fix a coproduct operation $+$ on sets by

$$\begin{array}{rcl} a + b & = & (\{0\} \times a) \cup (\{1\} \times b) \\ & = & \{(0, x) : x \in a\} \cup \{(1, y) : y \in b\} \end{array}$$

For sets a and b, the coproduct injections $\mathsf{inl} : a \to a + b$ and $\mathsf{inr} : b \to a + b$ are then given by

$$\begin{array}{rcl} \mathsf{inl}(x) & = & (0, x) \\ \mathsf{inr}(y) & = & (1, y) \end{array}$$

Henceforth in this paper, the symbol $+$ is used for this operation on sets (extended in the natural way to classes).

For any set a, $\bigcup a$ is the set of elements of elements of a. The *transitive closure* of a is $a \cup \bigcup a \cup \bigcup\bigcup a \cup \cdots$. This is a set, and it is the $\subseteq$-least set which includes a.

If $a \subseteq b$, we write $i_{a,b}$ for the inclusion map of a into b. If $b = V$, then we generally drop it from the notation. So if $a \subseteq b$, we have $i_a = i_b \cdot i_{a,b}$.

The axioms of set theory are not about sets as much as they are about the *universe of sets*. One of the intuitive principles of the theory is that arbitrary collections of mathematical objects "should be" sets. Due to paradoxes, this intuitive principle is not directly formalized in standard set theories. In a sense, the axioms one does have are intended to give enough sets to constitute a mathematical universe while not having so many as to risk inconsistency. But it is natural in this connection to consider some collections of objects which are demonstrably not sets. These are called *proper classes*. The term *class* informally refers to a collection of mathematical objects. Classes are usually not first-class objects in set theory (certainly they are not in the most standard set theory, ZFC). Instead, a statement about classes is regarded as a paraphrase for some other (more complicated and usually less intuitive) statement about sets. This is probably not a good place to discuss the details of the formalization; one clear source is Chapter 1 of Azriel Levy's book [10] on set theory. For our purposes, classes may be taken as definable subcollections of sets. For example, if a is any set, then the class of all sets which do not contain a as an element is $\{x : a \notin x\}$. The class V of all sets is $\{x : x = x\}$. The definability here is in the first-order logic with just a symbol $\in$ for membership, and the quantifiers range over sets (not classes). If C is a class, the *power class of* C,

$$\mathcal{P}(C) \quad = \quad \{x : x \text{ is a set, and } (\forall y)(y \in x \to \varphi_C(y))\},$$

where φ_C is the formula that defines the class C.

We are interested in functors H on sets and classes which are *monotone* in the sense of preserving inclusions among objects: if $a \subseteq b$, then $Ha \subseteq Hb$.

Each set-based, monotone operation H on classes has a least fixed point H_* and a greatest fixed point H^*. For the least fixed point, we first define classes H_α by transfinite recursion: $H_0 = \emptyset$, $H_{\alpha+1} = H(H_\alpha)$ and for limit λ, $H_\lambda = \bigcup_{\beta<\lambda} H_\beta$. Then the class H_* is defined by $x \in H_*$ iff $(\exists\alpha)x \in H_\alpha$. The assumption that H be set-based, together with the Replacement Axiom, implies that H_* is a fixed point of H, and it is easy to see by induction that each H_α is a subset of it. So H_* is the least fixed point. In categorical terms, (H_*, id) is an initial H-algebra on the category of classes. We are especially concerned with the dual concept, greatest fixed points. As shown in Aczel [1],

$$H^* \quad = \quad \bigcup\{b : b \text{ is a set and } b \subseteq Hb\}.$$

H^* might well be a proper class.

For example, by Cantor's Theorem there are no sets which are fixed points of the power set functor, but on classes, the least fixed point exists and indeed is the class WF of wellfounded sets. Another fixed point is the class V of all sets. Saying that $\mathcal{P}V = V$ just means that every set of sets is a set, and that every set is a set of sets. (So this would contradict any axiom of *urelements*, and indeed usually set theories implicitly do not allow for urelements.) Note that i_V, $i_{\mathcal{P}V}$, $\mathcal{P}i_V$, and $\mathcal{P}i_{\mathcal{P}V}$ all denote the same operation, the identity on the universe.

Here are some further examples to orient the reader. The identity functor has the universe V as its greatest fixed point on the category of classes. The identity has not greatest fixed point on sets. But even on classes, the greatest fixed point is not the carrier of a final coalgebra structure, since that would be a mere singleton set. But consider the variant functor $H(a) = 1 \times a$. Here there are some differences, even though H is naturally isomorphic to the identity. Whether H has any fixed points at all is a question that is sensitive to the underlying set theory. Under the Foundation Axiom, the empty set $\emptyset$ is the only fixed point of H. Under the Anti-Foundation Axiom (formulated shortly), H has one additional fixed point (which therefore is the greatest fixed point): there is a unique set a such that $a = \{(0, a)\}$ (this uses AFA). And so $b = \{a\}$ satisfies $b = \{0\} \times \{a\} = 1 \times b$. Moreover, b is the only set with this property except for $\emptyset$.

In any case, the overall point is that properties of the greatest fixed points of various operations are sensitive to the underlying set theory. The topics of this paper are certain classes which form either final coalgebras or CIAs for various functors. Again, such classes do not exist in the usual set theory ZFC, due mainly to the Foundation Axiom. In this connection, and in connection with other coalgebraic notions, it is more natural to work in the set theory ZFA obtained from ZFC by replacing the Foundation Axiom with a 'dual' statement, the Anti-Foundation Axiom first formulated by Forti and Honsell and then popularized in Peter Aczel's book [1].

The Anti-Foundation Axiom The *Anti-Foundation Axiom (AFA)* is the assertion that for every set b and every $e : b \to \mathcal{P}b$, there exists a unique $s : b \to V$ such that $s = \mathcal{P}s \cdot e$:

$$\begin{array}{ccc} b & \xrightarrow{e} & \mathcal{P}b \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle \mathcal{P}s} \\ V & = & \mathcal{P}V \end{array} \tag{1}$$

The map s is called the *solution* to the *system* e.

To see how this is used, we mentioned above that under *AFA*, there is a unique set $a = \{(0, a)\}$. To see this, we let $b = \{v, w, x, y, z\}$ and consider $e : b \to \mathcal{P}b$ given by

$$\begin{array}{lcl} e(v) & = & \{w\} \\ e(w) & = & \{x, y\} \\ e(x) & = & \{z\} \end{array} \qquad \begin{array}{lcl} e(y) & = & \{v, z\} \\ e(z) & = & \emptyset \end{array}$$

Then if s is as in the statement of *AFA*, we have $s(v) = \{s(w)\}$, $s(w) = \{s(x), s(y)\}$, …, $s(z) = \emptyset$. So $s(x) = \{0\}$, $s(y) = \{s(v), 0\}$, and $s(w) = \{\{0\}, \{s(v), 0\}\} = (0, s(v))$. Finally, $s(v) = \{(0, s(v)\}$. Thus $s(v)$ is a set which solves $a = \{(0, a)\}$. It is not hard to check that it is the only solution, because any solution to this equation gives a solution to the "flat system" e by unraveling a bit.

Lemma 2.6 (Turi [17], see also [14]) *AFA is equivalent to the assertion that* $(V, i_V) = (V, \mathsf{id}_V)$ *is a final* $\mathcal{P}$*-coalgebra.*

Our overall setting in this paper is ZFA. (Actually, many of the results do not actually use *AFA*, especially those before Section 3.1. But the main results of the paper do use it.)

Remark The formulation of *AFA* in (1) above does not include any specific morphism between V and $\mathcal{P}V$. This is basically the way *AFA* is presented in Aczel's book [1], for example, and also my book with Jon Barwise [8]. The disadvantage of this kind of formalization is that it hides the fact that there are two different possible assertions:

$$\begin{array}{ccc} b & \xrightarrow{e} & \mathcal{P}b \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle \mathcal{P}s} \\ V & \xrightarrow[(i_{\mathcal{P}V})^{-1}]{} & \mathcal{P}V \end{array} \qquad \text{vs.} \qquad \begin{array}{ccc} b & \xrightarrow{e} & \mathcal{P}b \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle \mathcal{P}s} \\ V & \xleftarrow[i_{\mathcal{P}V}]{} & \mathcal{P}V \end{array}$$

When one reworks our statement of *AFA* using the first formulation, one can sense the connection to final coalgebras and Lemma 2.6. The second formulation would be closer to what we find in Lemma 3.3.

The main problem for this papers and all previous ones on "uniformity" for functors is to propose a condition guaranteeing that the greatest fixed point of a monotone H be a final coalgebra together with the identity. This paper proposes and studies one such condition.

3 Standard Functors and Monads

At this point, we have all of the background we need to begin our study The first concept we need is that of a standard functor on sets or classes. An endofunctor H is *standard* if H preserves inclusion maps in the sense that $Hi_{a,b} = i_{Ha,Hb}$. This notion was introduced in a slightly stronger form in Adámek and Trnková's book [7]; Theorem 3.4.5 of that book shows that every functor on sets is naturally isomorphic to a standard functor in their sense.

Proposition 3.1 *The coproduct + derived from the Kuratowski pair has the property that for all classes c, the endofunctor $\_ + c$ is standard.*

The proof is an easy calculation. Of course, the functors $c + \_$ are also standard. Here is a consequence of these: Let $x \subseteq x'$ and $y \subseteq y'$. Then the diagram

$$\begin{array}{ccc} x & \xrightarrow{\mathsf{inl}} & x + y \\ {\scriptstyle i_{x,x'}}\downarrow & & \downarrow{\scriptstyle i_{x,x'}+i_{y,y'}} \\ x' & \xrightarrow[\mathsf{inl}]{} & x' + y' \end{array} \tag{2}$$

commutes.

Definition Let T be the free completely iterative monad on H. T is *standard* if for each a, $Ta = HTa + a$, and moreover $\alpha_a = \mathsf{id}_{Ta}$.

Lemma 3.2 *Let T be the free completely iterative monad on a standard functor H. If T is a standard monad, then T is a standard functor.*

Proof Let $a \subseteq b$, and write i for $i_{a,b}$. We know that Ti is the unique map such that $Ti \cdot \tau_a = \tau_b \cdot HTi$. (This follows from the Substitution Theorem of [3, 2, 14] applied to $\eta_b \cdot i$.) But if we take Ti to be $i_{Ta,Tb}$, then the equation is satisfied because

$$i_{Ta,Tb} \cdot \tau_a \quad = \quad \tau_b \cdot i_{HTa,HTb} \quad = \quad \tau_b \cdot Hi_{Ta,Tb}.$$

In this we are using the fact that τ is inl for a standard functor, and also equation (2). So by uniqueness, $Ti = i_{Ta,Tb}$. ⊣

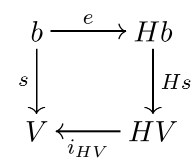
3.1 Establishing that $\mathcal{P}$ generates a standard iterative monad

We check here that under *AFA*, $\mathcal{P}$ generates a standard iterative monad. The general idea of our work is to use this fact to show that many other functors also generate standard iterative monads. In fact, our definition of uniformity effects such a reduction.

Lemma 3.3 (See [14]) *Let H be standard. The following are equivalent:*

1. (H^*, id) *is a final H-coalgebra.*

2. (V, i_{HV}) *is a coalgebra-final H-algebra: for every class b and every $e : b \to Hb$, there exists a unique $s : b \to V$ such that $s = i_{HV} \cdot Hs \cdot e$:*



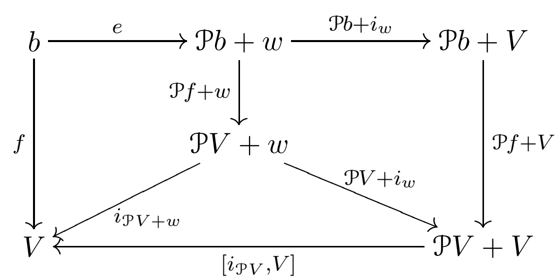
Proof We show first that (2) implies (1). Consider $e : b \to Hb$ and some associated s. Let $c = s[b]$ be the image of b under s. Then $Hs[Hb] \subseteq H(s[b]) = Hc$ (see, e.g., Proposition 5.1.2 of [13]). The condition in our lemma implies that $c \subseteq Hs[Hb]$, and so we see that $c \subseteq Hc$. By the monotonicity of H, we have $c \subseteq H^*$. Let $t : b \to c$ be such that $i_c \cdot t = s$. Then $i_{c,H^*} \cdot t$ is a coalgebra morphism from (b, e) to (H^*, id). The uniqueness part of finality follows from the observation that if f is a coalgebra morphism from (b, e) to (H^*, id), and if $c = f[b]$, then $c \subseteq H(c)$.

Now we prove that (1) implies (2). If (H^*, id) is final, then for every $e : b \to Hb$ we associate $i_{H^*} \cdot e^*$, where $e^* : b \to H^*$ is the final H-coalgebra morphism for e. The uniqueness is as above. ⊣

In the next proposition, and in the rest of this paper, we let G_w be the constant functor with value w.

Proposition 3.4 *For every set w, $((\mathcal{P} + G_w)^*, \mathsf{id})$ is a final coalgebra for $\mathcal{P} + G_w$.*

Proof We apply Lemma 3.3. Let $e : b \to \mathcal{P}b + w$. Consider the diagram below:



The map f comes from the fact that $(V, (i_{\mathcal{P}V})^{-1})$ is a CIA for $\mathcal{P}$. (So note that *AFA* is used here.) Thus the overall outside commutes. The right square easily commutes. For the triangle, we use the general fact that $i_{a+b} = [i_a, i_b]$. (In fact, for classes a, b, and c such that $a \subseteq c$, and $b \subseteq c$, and $a + b \subseteq c$, we have $i_{a+b,c} = [i_{a,c}, i_{b,c}]$.) We conclude that the left square above commutes. This is the existence of the needed f in Lemma 3.3, and the uniqueness comes from the CIA structure. ⊣

It follows from Proposition 3.4 that $\mathcal{P}$ generates a standard iterative monad on the category of classes. The same work, with obvious changes, shows that the subfunctors $\mathcal{P}_k$ determine standard iterative monads on the category of sets.

4 The class TV and the map χ

As we now know, the power set functor determines a free completely iterative monad

$$T \quad = \quad (\mathcal{P}, T^{\mathcal{P}}, \mu^{\mathcal{P}}, \eta^{\mathcal{P}}).$$

This monad is indeed standard. It also comes with additional natural transformations $\alpha^{\mathcal{P}}$ and $\tau^{\mathcal{P}}$. Because this is the most common monad in the rest of the paper, we drop the superscripts on all of this data related to it.

By AFA the inverse of inclusion gives a final coalgebra $(i_{\mathcal{P}V})^{-1} : V \to \mathcal{P}V$. Because so much of the rest of this paper uses the map $[\![i_{\mathcal{P}V}]\!]$, we shorten the notation to write

$$\chi \quad = \quad [\![i_{\mathcal{P}V}]\!] : TV \to V.$$

For a mnemonic on this, think of χ for χrunch. As we shall see, it takes elements of TV and collapses them back to sets. Those familiar with the *Mostowski collapse* in set theory might think of χ as a kind of non-wellfounded version of that map.

It is worthwhile to get a feeling for the class TV. To understand it better, we us Proposition 3.4, taking V for w. So TV is the greatest fixed point of the functor which takes a class X to

$$\mathcal{P}(X) + V \quad = \quad (\{0\} \times \mathcal{P}(X)) \cup (\{1\} \times V).$$

Hence TV is the largest collection C of sets with the property that each member of C is of one of the following forms:

1. $(0, x)$ for some subset $x \subseteq C$.

2. $(1, x)$ for some set x.

Remark Aczel's book [1] at some points mentions classes like $V[A]$. The book didn't say what this means in any detail, and I fear that some authors quoted these points without understanding matters. $V[A]$ corresponds to $T(A)$ (or better, to $\mathcal{P}TA$). Even without AFA, one should read $V[A]$ by taking the greatest fixed point of $X \mapsto \mathcal{P}(X) + A$ and then considering $\mathcal{P}X$.

Note as well that $\eta : Id \to T$ is defined by $\eta_X(a) = (1, a)$ for all classes X and all sets $a \in X$. As for τ, standardness implies that its components are all inclusions.

We now turn to χ. The elements of TV *code* sets as follows:

1. $(0, x)$ codes the set of sets coded by the elements of x.

2. $(1, x)$ codes x itself.

The map χ is the *decoding* map.

Example 4.1 Here are some examples of χ at work:

1. For all sets a, $\chi(1, a) = a$, and thus $\chi(0, \{(1, a)\}) = \{a\}$.

2. $\chi(0, \emptyset) = \emptyset$.

3. $\chi(0, \{(0, \emptyset)\}) = \{\chi(0, \emptyset)\} = \{\emptyset\}$.

4. $\chi(0, \{(0, \emptyset), (1, x)\}) = \{\chi(0, \emptyset), \chi(1, x)\} = \{\emptyset, x\}$.

5. For all sets a and b,
$$(0, \{(0, \{(1, a)\}), (0, \{(1, a), (1, b)\})\})$$
belongs to TV, and χ applied to it is the ordered pair (a, b).

In all of these, we omit mention of α since it is the identity.

We record the following application of Lemma 2.3:

Proposition 4.2 *Concerning $\chi : TV \to V$:*

1. $\chi \cdot \eta_V = \mathsf{id}_V$.
2. $\chi \cdot \tau_V = i_{\mathcal{P}V} \cdot \mathcal{P}\chi$.
3. $\chi \cdot T\chi = [\![\chi]\!] = \chi \cdot \mu_V$.

When we turn to the related functors $\mathcal{P}_\kappa$, we see that they also generate standard iterative monads on sets. In addition, each $\mathcal{P}_\kappa$ has the set H_κ as the carrier of a final coalgebra $(H_\kappa, i_{\mathcal{P}H_\kappa, H_\kappa})$[2]. All of the results above go through for them, mutatis mutandis.

5 Uniformity

As our title indicates, this paper is about notions of uniformity for functors on sets and classes. We propose a new definition in Section 5.1 below. Before that, we want to mention the previous notions of uniformity in the literature, and the motivation for them.

The first place where some notion of "uniform functor" may be found is Aczel's book [1] on non-wellfounded sets. His definition is in terms of the "expanded universe ... [which] has an atom x_i for each pure set i." In our terminology, this is exactly $\mathcal{P}T^{\mathcal{P}}$. Were his definition to be translated into our notation, it would look similar to ours. It would involve for each class A a map $\pi_A : HA \to T^{\mathcal{P}}A$ with some properties. However, the resulting π is not required to be a natural transformation (and indeed, it was not realized until several years later that T was a functor, etc.). As a compensation, the definition requires another property on π. Incidentally, I have not worked extensively with Aczel's definition, but it seems to be hard to check that the uniform functors in his sense are closed under composition.

We also emphasize that the first motivation for uniformity is to provide a sufficient condition on a monotone functor H that its greatest fixed point H^* be a final H-coalgebra along with the identity as a structure map.

The first work to formulate uniformity in terms of natural transformations is that of Turi [17] (also presented in Turi and Rutten [18]). Their definition is similar enough to ours to mention it in full. Their definition is in terms of a different monad on sets, the monad W given by WX

[2] We do apologize for any notational confusion that could result from our use of H for a functor and to designate an operation from cardinals to sets, and for that matter from our use of κ both for a cardinal and for a natural transformation.

is the *least fixed point* of $X \mapsto \mathcal{P}X + X$. They require of a functor H that there be a natural transformation $\rho : H \to \mathcal{P}W$ such that

$$\begin{array}{ccc} HV & \xrightarrow{\rho_V} & \mathcal{P}WV \\ {\scriptstyle i_{FV}}\downarrow & & \downarrow{\scriptstyle \mathcal{P}\epsilon_V} \\ V & = & \mathcal{P}V \end{array}$$

(The use of $\mathcal{P}WV$ corresponds to our requirement that that natural transformations involved in uniformity be *ideal*.) The main difference is that we use the monad T, a larger monad than W; hence more functors are uniform in our sense. (For example, the constant functors whose value are non-wellfounded sets are uniform in our sense but not in Turi and Rutten's sense.)

Once again, it is worthwhile mentioning that their motivation for uniformity again is the same as Aczel's. However, they recognize that there is an intuition:

> Intuitively, an endofunctor on SET is uniform on maps [their terminology] if it is completely determined by is action on objects (i.e., classes). Most endofunctors are thus uniform on maps. For instance, consider the endofunctor $X \mapsto A \times X$ mapping a class X to its product with a fixed class A. Given a function $f : X \to Y$, the value of $A \times f$ at an element (a, x) of $A \times X$ is the pair $(a, f(x)) \in A \times Y$ which is obtained by applying f to the $x \in X$ in $A \times X$. This suggests that the class X should be regarded as a class of variables and that, in general, the action of a functor F uniform on maps on a function f should simply be the substitution of the variables x occurring in FX by $f(x)$. (Turi [17] p. 211, and also Turi and Rutten [18] Section 5.5.)

For other approaches, see Devlin [9] and also Moss and Danner [15].

The upshot is that there are two intuitions at work in the definition of uniformity, or at least two different goals. One is to search for condition on functors F which guarantees that the greatest fixed point F^* of F be a final coalgebra with the identity as the structure map. I would like to emphasize, especially for readers with a background in category theory, that this kind of question is not "preserved under natural isomorphisms of functors". The identity functor will never be uniform under any reasonable definition, but functors like $1 \times x$ will turn out uniform under *AFA*.

A second intuition is mentioned in the quoted paragraph above. We could say that this has to do with the class TV and way that set theory is used to represent natural mathematical operations, and also with the matter of coding sets by elements of TV. The overall thrust of set theory as a foundational study is that natural mathematical operations are representable in a first-order way in the universe of sets. It is not always easy to spell out what this means, and most textbooks never get around to it. What we are doing in the definition of uniformity is to spell out the representability of natural mathematical operations, but not in terms of first-order logic but in terms of the iterative monad of the power set.

5.1 Our definition

We now come to the main definition in this paper. We continue to write T for the monad determined by the power set functor, omitting the superscript $\mathcal{P}$ in most places. We also remind the reader that an *ideal* natural transformation is one which factors through τ.

Definition A functor H is *uniform* if there is an ideal natural transformation $\pi : H \to T$

such that for all classes a,

$$[\![i_a]\!] \cdot \pi_a \quad = \quad i_{Ha}.$$

For a cardinal κ, H is *κ-uniform* if there is an ideal natural transformation $\pi : H \to T_\kappa$ such that for all classes a,

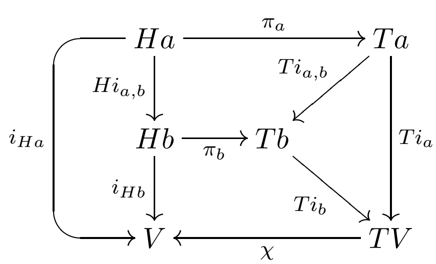
$$[\![i_a]\!]_\kappa \cdot \pi_a \quad = \quad i_{Ha,H_\kappa}.$$

Uniformity is equivalent to standardness plus the identity $\chi \cdot \pi_V = i_{HV}$. This says that if we encode HV as a subclass of TV and then collapse back to V via χ, we have an inclusion. The reason why we want to do any encoding has to do with co-recursion: given $e : a \to Ha$, we want to use get a solution satisfying an appropriate recursion principle. There is no evident way to do this without extra maps. We use π to get a related map $e' : a \to T(a)$. Having this, we can use Lemma 2.5 below to get a map $a \to V$.

Lemma 5.1 *Let $\pi : H \to T^{\mathcal{P}}$ be an ideal natural transformation. The following are equivalent:*

1. *H is uniform.*
2. *H is standard, and $\chi \cdot \pi_V = i_{HV}$.*

Proof First, assume that H is uniform via π. Then in particular, $\chi{\cdot}\pi_V = i_{HV}$. The interesting point is to check that H is standard. Let $a \subseteq b$. In the diagram below,



everything commutes except the region on the left: the top uses naturality of π; the triangle on the right is by applying T to the fact that $i_a = i_b \cdot i_{a,b}$; and uniformity is used in the overall outside and in the bottom square. So we see that $i_{Ha} = i_{Hb} {\cdot} Hi_{a,b}$. But now we notice a general fact: if x and y are any sets, and $f : x \to y$ is such that $i_x = i_y \cdot f$, then $x \subseteq y$ and $f = i_{x,y}$. It now follows that $Ha \subseteq Hb$ and that $Hi_{a,b} = i_{Ha,Hb}$, as desired.

Going the other way, suppose H is standard, and $\chi \cdot \pi_V = i_{HV}$. Let a be any class. Return to the diagram above, and replace b by V. Then our assumption that $\chi \cdot \pi_V = i_{HV}$ implies that the bottom square commutes, and the region on the left is by standardness. It follows that we have the desired uniformity equation $[\![i_a]\!] \cdot \pi_a = i_{Ha}$. ⊣

The second formulation is often easier to check, since standardness is usually immediate for functors. We use Lemma 5.1 without further mention.

Our main results The main results of this paper are as follows: the uniform functors contain the power set functor and the constants, and they are closed under a number of natural operations including composition and iteration. A uniform H has the property that H^* together with the identity is a final H-coalgebra, and V together with the inclusion of HV into it is a CIA for H. The same generally holds for κ-uniform functors, except that the only constant functors which are κ-uniform are those for sets in H_κ. If H is κ-uniform, then H^* is a subset of H_κ.

The rest of this paper is devoted to proofs of these assertions, and some additional discussion.

5.2 Examples and closure properties

Example 5.2 We establish the uniformity of the power set functor $\mathcal{P}$. This functor is easily standard. Let $\pi : \mathcal{P} \to T$ be $\tau \cdot \mathcal{P}\eta$ from the iterative monad determined by $\mathcal{P}$. Note that π is ideal. Furthermore,

$$\begin{array}{rcl} \chi \cdot \tau_V \cdot \mathcal{P}\eta_V & = & i_{\mathcal{P}V} \cdot \mathcal{P}\chi \cdot \mathcal{P}\eta_V \\ & = & i_{\mathcal{P}V} \cdot \mathcal{P}\mathsf{id}_V \\ & = & i_{\mathcal{P}V} \end{array}$$

We used Proposition 4.2.

Example 5.3 More generally, let H be a functor on $Class$ such that $H(a) \subseteq \mathcal{P}(a)$ and such that the inclusions give a natural transformation $\nu : H \to \mathcal{P}$. Then $\pi \cdot \nu$ shows H to be uniform, as $\chi \cdot \pi_V \cdot \nu_V = i_{\mathcal{P}V} \cdot \nu_V = i_{\mathcal{P}V} \cdot i_{HV,\mathcal{P}V} = i_{HV}$. This remark applies to functors of the form $\mathcal{P}_\kappa$, the set of subsets of a given set of cardinality at most κ.

We also have the easy fact that if H is κ-uniform, then H is uniform. And if $\kappa < \lambda$ and H is κ-uniform, then H is λ-uniform. This is because the obvious inclusions give morphisms of ideal monads.

Example 5.4 Let w be a set; we show that the constant functor G_w with value w is uniform. Let $\overline{w}$ be the transitive closure of w. Since $\overline{w} \subseteq \mathcal{P}(\overline{w})$, we have an inclusion $i_{\overline{w},\mathcal{P}(\overline{w})}$. To shorten our notation, we abbreviate this as i in this example. We regard i as a natural transformation between constant functors. We also have a natural transformation $G_{\overline{w}} \to \mathcal{P}G_{\overline{w}} \to \mathcal{P}G_{\overline{w}} + Id$. By a finality result concerning T in the functor category, we have a natural transformation $\pi_0 : G_{\overline{w}} \to T^{\mathcal{P}}$ such that $\pi_0 = \tau^{\mathcal{P}} \cdot \mathcal{P}\pi_0 \cdot i$. It follows easily from this that $\chi \cdot \pi_0(V)$ is the inclusion $i_{G_{\overline{w}}V} = i_{\overline{w}}$. And the desired ideal natural transformation is $\pi_0 \cdot j$, where j is the inclusion $i_{w,\overline{w}}$ considered as a natural transformation.

More generally, if $w \in H_\kappa$, then the constant functor with value w is κ-uniform.

Example 5.5 The identity functor I is *not* uniform. Here are two ways to see this. First, we argue directly, by contradiction. Suppose we had an ideal $\pi : I \to T$ such that $\chi \cdot \pi = \mathsf{id}_V$. Then for all sets a, $\mathsf{id}_a = \chi \cdot i_{T(a),T(V)} \cdot \pi_a$. In short, for all $x \in a$, $x = \chi(\pi_a x)$. Let $a = \{0, 1\}$. Then $\pi_a(0)$ must be $(0, \emptyset)$, as π is ideal, and $\chi^{-1}[\emptyset] = \{(0, \emptyset), (1, \emptyset)\}$. Let $f : a \to a$ be the transposition $f(0) = 1$ and $f(1) = 0$. By naturality, $\pi_a \cdot f = Tf \cdot \pi_a$. Applying this to 0, we see that $\pi_a(1) = Tf(0, \emptyset) = (0, \emptyset)$. But then we would have $1 = \chi \cdot \pi_a(1) = \chi(0, \emptyset) = \emptyset$; this is a contradiction.

A less elementary way to establish the non-uniformity is to use a result from later that for uniform H, the greatest fixed point H^* gives a final coalgebra with the identity map. For 1, we have $I^* = V$. But the final coalgebras of I are the singleton sets. So for this reason, I is not uniform.

Example 5.6 In contrast to this, the functor $H(a) = a + 0$ *is* uniform; this is the same as

$$H(a) \quad = \quad 1 \times a \quad = \quad \{(0, x) : x \in a\}.$$

The natural transformation $\pi : H \to T$ is given by

$$\pi_a(0, x) \quad = \quad (0, \{(0, \{(0, 0)\}), (0, \{(0, 0), (1, x)\})\}).$$

Similar to what we have seen in Example 4.1, part 5, for all sets x, $\chi(\pi_V(0,x)) = (0,x)$:

$$\begin{array}{ll} & \chi((0,\{(0,\{(0,0)\}),(0,\{(0,0),(1,x)\})\})) \\ = & \{\chi(0,\{(0,0)\}),\chi(0,\{(0,0),(1,x)\})\} \\ = & \{\{0\},\{0,x\}\} \qquad\qquad (*) \\ = & (0,x) \end{array}$$

The calculations in the line marked $(*)$ were performed in Example 4.1. Moreover, π is an ideal natural transformation because each $\pi_a(0,x)$ is an ordered pair beginning with 0; more formally, consider $\pi^* : H \to \mathcal{P}T$ given by

$$\pi^*_a(0,x) \quad = \quad \{(0,\{(0,0)\}),(0,\{(0,0),(1,x)\})\}.$$

Then $\pi = \tau \cdot \pi^*$. The most tedious part of the verification has to do with the naturality of π^*. Let $f : a \to b$. Note that $Tf(0,0) = (0,0)$, and for all $x \in a$, $Tf(1,x) = (1,fx)$. It follows that

$$Tf(0,\{(0,0),(1,x)\}) \quad = \quad \{(0,\mathcal{P}Tf\{(0,0),(1,x)\}\} \quad = \quad \{(0,\{(0,0),(1,fx)\}\}.$$

Therefore

$$\begin{array}{lll} \pi^*_b(1,fx) & = & \{(0,\{(0,0)\}),(0,\{(0,0),(1,fx)\})\} \\ & = & \{Tf(0,\{(0,0))\},Tf(0,\{(0,0),(1,x)\})\} \\ & = & \mathcal{P}Tf\{(0,\{(0,0)),(0,\{(0,0),(1,x)\}\}\} \\ & = & \mathcal{P}Tf(\pi^*_a(0,x)) \end{array}$$

The point is that we can "implement" the pairing machinery in a way which is recoverable by χ. In a similar fashion, we also have the following result:

Theorem 5.7 *If F and G are uniform, and a is any set, then the following functors are also uniform: $F+G$, $F \times G$, $1+F$, F^a.*

Proof See [14] for the many calculations involving the coding machinery. ⊣

There area also similar results for the κ-uniform functors.

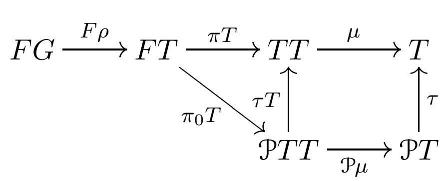
5.3 Closure under composition and iteration

Theorem 5.8 *If F and G are uniform, then $F \cdot G$ is also uniform.*

Proof Let F be uniform via π, and G uniform via ρ. To see that $F \cdot G$ is uniform, let $\pi * \rho = \pi T \cdot F\rho$, and consider the natural transformation

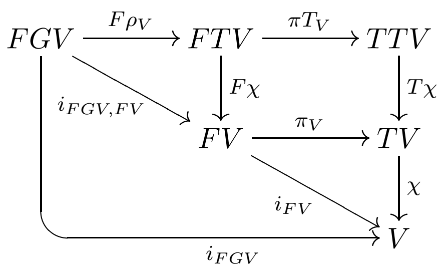
$$\mu \cdot (\pi * \rho).$$

The following diagram shows π to be an ideal natural transformation:



Here π_0 is a natural transformation with the property that $\pi = \tau \cdot \pi_0$; this gives the the triangle above. The commutativity of the square is a part of Proposition 2.1.

Consider next the following diagram:



The upper triangle is obtained by applying F to the uniformity equation for G and using the standardness of F as well. Everything else commutes easily. We now use Proposition 4.2 to calculate:

$$\chi \cdot \mu_V \cdot \pi T_V \cdot F\rho_V \quad = \quad \chi \cdot T\chi \cdot \pi T_V \cdot F\rho_V \quad = \quad i_{FGV}.$$

This completes the proof that FG is uniform. $\dashv$

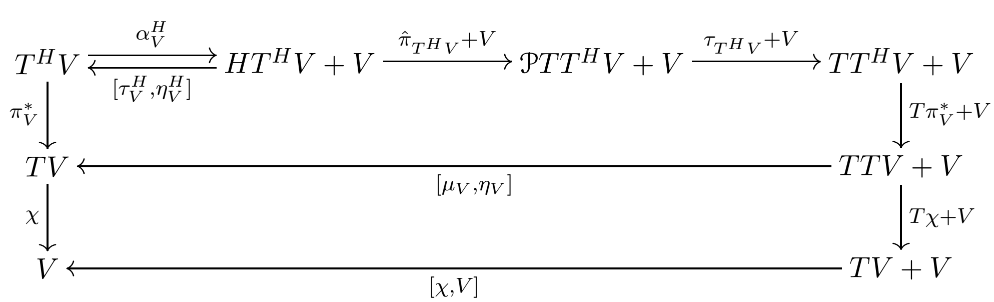
Theorem 5.9 *Let (T^H, μ^H, η^H) be a standard iterative monad which is free on a uniform functor H. Then T is also uniform.*

Proof In this proof we have the monad of H and also the monad of $\mathcal{P}$. As in our statement, we write the data coming from the first free completely iterative monad with the superscript H, and we continue our practice of dropping the superscripts on the free completely iterative monad of $\mathcal{P}$.

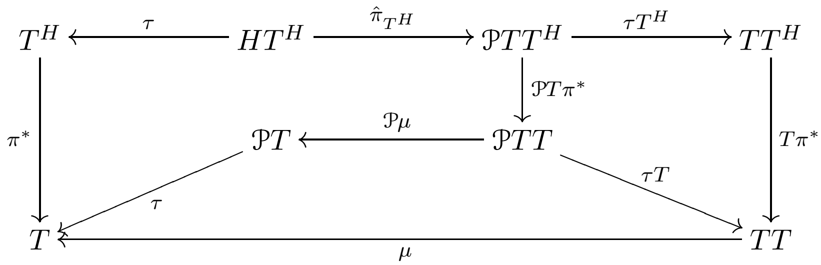
We know that T^H is standard by Lemma 3.2. Let H be uniform via $\pi : H \to T$. Let $\hat{\pi} : H \to \mathcal{P}T$ be such that $\pi = \tau \cdot \hat{\pi}$. By the fundamental freeness result of [2], there is a unique ideal monad morphism $\pi^* : T^H \to T$ such that $\pi = \pi^* \cdot \kappa$. We check that $\chi \cdot \pi^*_V = i_{T^H V}$.

Consider the following diagram:

(3)



We claim that both halves commute. The bottom uses Proposition 4.2. For the top, it is best to begin at $HT^H V + V$ and argue separately for the two components. The right component commutes due to the fact that π^* is a monad morphism; specifically, $\pi^* \cdot \eta^H = \eta$. The left component is more involved. We drop V and consider the following diagram in the endofunctor category:

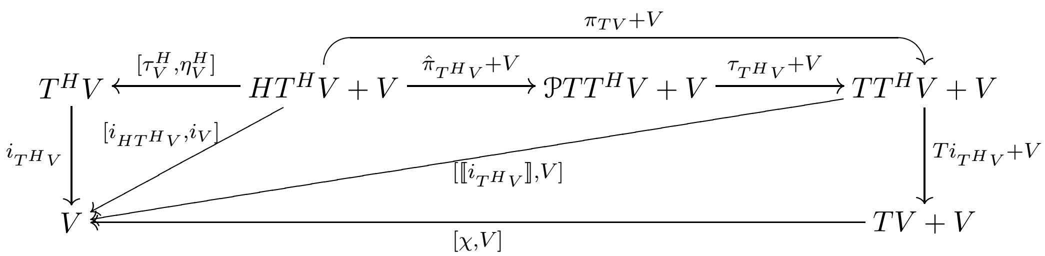


For the pentagonal region in the upper left, we appeal to Lemma 6.10 of [12]. The region on the right commutes by naturality of τ. The bottom square is an instance of Proposition 2.1.

At this point we know that (3) commutes. We conclude that $g = \chi \cdot \pi_V^*$ satisfies

$$[\chi, V] \cdot Tg \cdot (\pi_V + V) \cdot \alpha_V^H \quad = \quad g.$$

By our Solution Lemma 2.4, there is exactly one g which satisfies this equation. We check that $i_{T^H V}$ also satisfies it. We note that the diagram below commutes:



In the topmost region, we have used the fact that $\pi = \tau \cdot \hat{\pi}$. To see that the triangle on the left commutes, recall that $\alpha_V^H = [\tau_V^H, \eta_V^H]^{-1}$ is the identity and that $i_{HT^H V+V} = [i_{HT^H V}, i_V]$. We have used the fact that π is uniform in the middle region, and on the right we have the definition of $[\![i_{T^H V}]\!]$.

This concludes the proof that $\chi \cdot \pi_V^* = i_{T^H V}$. ⊣

This last result is new.

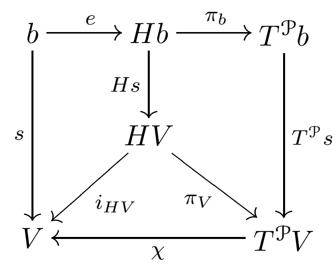
6 Consequences of Uniformity

Theorem 6.1 below is an adaptation of the analogous result from [14], and ultimately the ideas come from Turi [17], following Aczel [1]. We remind the reader that *AFA* is needed in the results of this section.

Theorem 6.1 *Let H be uniform. Then (H^*, id) is a final H-coalgebra, where H^* is the greatest fixed point of H.*

Proof H is standard, so we may use Lemma 3.3. Since we are assuming *AFA*, Lemma 2.5

applies. Let $e : b \to Hb$. There is a unique $s = f^\dagger : b \to V$ such that $s = [\![s]\!] \cdot \pi_b \cdot e$. Consider the following diagram.



All the parts clearly commute except the left, and thus this does commute. This part shows that $s = i_{HV} \cdot Hs \cdot e$. For the uniqueness, note that s with our desired property determines a solution to $\pi_b \cdot e$. ⊣

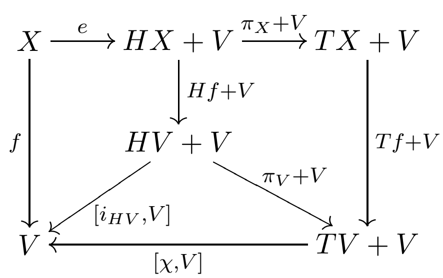
Corollary 6.2 *If H is uniform, then H generates a standard iterative monad by taking for each class a, $Ta = (H + G_a)^*$, the greatest fixed point of $H + G_a$.*

Proof We know that for all sets a, $H + G_a$ is uniform and standard. So the result follows from Theorem 6.1. ⊣

Corollary 6.3 *If H is κ-uniform, then $H^* \subseteq H_\kappa$. In particular, there is a final coalgebra for H which is a subset of the set of sets of hereditary cardinality at most κ.*

Theorem 6.4 *If H is uniform, then (V, i_{HV}) is a CIA for H.*

Proof The proof is virtually the same as that of Theorem 6.1, so we merely indicate the idea and exhibit the diagram. Let $e : X \to HX + V$. Consider the diagram below:



The map f comes from Lemma 2.4 applied to $(\pi_X + V) \cdot e$. The rest of the proof is the same. ⊣

7 Concluding Remarks

The main point of this paper has been to rework the theory of uniformity using some of the machinery introduced in coalgebraic recursion theory in past years, including the notions of a completely iterative monad and a completely iterative algebra. As we have seen, there are two different intuitions at work, two different goals for the study. In a sense, one wants to find functors with the nice property that their greatest fixed points are final coalgebras, and then the technical details lead one to propose definitions that are about functors working by a general form of substitution.

As it happens, the notions of uniformity that attempt to get at the intuition that a functor is determined "by substitution" in some sense single out a smaller class than those which give final coalgebras by considering greatest fixed points. To see this, consider the distribution functor $\mathcal{D}$ on sets given by $\mathcal{D}(X)$ is the set of all partial functions μ from X to $(0,1]$ such that $\Sigma_{x\in X}\ \mu(x) = 1$. On morphisms, $\mathcal{D}$ works by marginalization (summing). The details are technical but it seems intuitively clear that $\mathcal{D}$ is not uniform in our sense (or under any definition stated in terms of natural transformations and maps like χ). At the same time, it is the case that the greatest fixed point of $\mathcal{D}$ is a final coalgebra with the identity, and the universe is a CIA for it with the inclusion. For $\mathcal{D}$ itself, this is easy to see as $\mathcal{D}^*$ is a singleton $x = \{\{(x,1)\}\}$. Things are more interesting for variants such as $H(x) = \mathcal{D}(x) + A$ for a fixed set A. We show by example that (H^*, id) is a final coalgebra, invoking Lemma 3.3. Let $b = \{w, x, y, z\}$, let $a \in A$, and consider $e : b \to Hb$ given on the left below:

$$
\begin{array}{lcllcl}
e(w) & = & \mathsf{inl}\ \{(x,1/3),(y,1/3),(z,1/3)\} & f(w) & = & (0,\{(x,2/3),(z,1/3))\} \\
e(x) & = & \mathsf{inl}\ \{(x,1)\} & f(x) & = & (0,\{(x,1)\}) \\
e(y) & = & \mathsf{inl}\ \{(y,1)\} & & & \\
e(z) & = & \mathsf{inr}\ a & f(z) & = & (1,a)
\end{array}
$$

To get the desired $s : b \to V$, we must identify x and y (since they are bisimilar in e); this is the reason why uniformity in the sense of this paper fails). We do this in the system f. This system has a unique solution s^*, by standard techniques. We then extend s^* to the desired s by $s(y) = s^*(x)$.

The results here extend to show that every functor built from $\mathcal{D}$ and the polynomial-forming operations (except of course for the identity functor) has the properties of interest in this paper. One can even imagine re-working the definition of uniformity in this paper to allow $\mathcal{D}$ and related functors to be uniform. However, doing this in an ad hoc manner gives no insight to help with a search for the most general uniformity notion.

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