Basic Syllogistic Logics

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All men are mortal.
Socrates is a man.

Socrates is mortal.

Some men are mortal.
Socrates is a man.

Socrates is mortal.

All frogs are reptiles.
All reptiles are animals.

All frogs are animals.

All frogs are reptiles.
All frogs are animals.

All reptiles are animals.

All sagatricians are maltnomans.
All sagatricians are aikims.

All maltnomans are aikims.

The first point is that there is an exact definition of *validity* for arguments.
The second point here is that the *form* is as important, even more important, than the particular words.

$$\frac{\begin{array}{c}\textit{All } X \textit{ are } Y. \\ \textit{All } Y \textit{ are } Z. \\ \textit{All } W \textit{ are } X.\end{array}}{\textit{All } W \textit{ are } Z.}$$

So valid arguments can have more than two *premises*.
Our plan is to start with sentences containing all, some, and no.

Probably the key point of logic is that there is a distinction between

syntax and semantics.

The idea is that syntax is the raw symbols.

Syntax is usually painful: think of computer programming.
I'll try to avoid it as much as possible to concentrate on the ideas.

The semantics is where we get the meaning.
So in our examples, we need some context or model to give a meaning.
In our examples, the syntax will start with some variables p, q, n, n_1, ...
The our sentences are expressions of the form
All p are q, *Some p are q*, and *No p are q*

To say whether

All sagatricians (s) are maltnomans (m).

is true or not needs a model.
This is given by a few things:

First, a set M called the universe.
Second, for the words sagatrician and maltnoman,
we need sets $[\![sagatrician]\!] \subseteq M$ and $[\![maltnomanan]\!] \subseteq M$.

DEFINITION

For the language of All,
a model $\mathcal{M}$ is a set M together with sets $[\![p]\!] \subseteq M$
for all nouns p.

Syntax: We begin with atoms (nouns) p, q, X, Y, Z, etc.
The sentences in our first fragment are the expressions *All p are q*
We usually use letters like φ for sentences.

Semantics: A model $\mathcal{M}$ is a set M,
and for each variable p we have an interpretation $[\![p]\!] \subseteq M$.

$$\mathcal{M} \models \textit{All p are q} \qquad \text{iff} \qquad [\![p]\!] \subseteq [\![q]\!]$$

$\mathcal{M} \models \varphi$ is read as "$\mathcal{M}$ satisfies φ."

A statement like $\mathcal{M} \models$ *All p are q* could also be read as

All p are q is true in $\mathcal{M}$

Is All X are Y true or not?

Is All X are Y true or not?

Without a model (also called a context), the question makes no sense.

So let's take an example model, and ask whether our sentence is true in that model or not.

Let $M = \{1, 2, 3, 4, 5\}$
Let $[\![X]\!] = \{1, 2, 4\}$.
Let $[\![Y]\!] = \{3, 4\}$.

Is All X are Y true or not?

Without a model (also called a context), the question makes no sense.

So let's take an example model, and ask whether our sentence is true in that model or not.

Let $M = \{1, 2, 3, 4, 5\}$
Let $[\![X]\!] = \{1, 2, 4\}$.
Let $[\![Y]\!] = \{3, 4\}$.

In this model, All X are Y is false!

But if we change the model by re-setting $[\![Y]\!]$ to $\{1, 2, 3, 4\}$, then our sentence is true.

One fine point on the definition is that if $[\![X]\!]$ is the empty set $\emptyset$, then our sentence All X are Y is *true*!
So in this room now,

All people in the room over 7 feet tall are standing

is (on this definition) true.

This strange point will lead us to various issues over the next days.

The standard reply is to say that it's true because there are *no exceptions*.

But we again admit that the semantics of All that we are giving is not what most people would agree to in cases where $[\![X]\!] = \emptyset$.

At this point, we know how to give the semantics of single sentences.

φ follows from $\psi_1, \ldots, \psi_n$

This means that every model that makes all of the ψs true also makes φ true.
We write this as

$$\psi_1, \ldots, \psi_n \models \varphi$$

and we also say that the ψ's semantically imply φ.

To argue that $\psi_1, \ldots, \psi_n \models \varphi$ we need some reasoning.
Usually, we do this in English and in an informal way,
just as one would do ordinary mathematical reasoning.

But to argue that $\psi_1, \ldots, \psi_n \not\models \varphi$ we can produce a counterexample.

The main thing is that we have a rigorous definition, using a semantic notion (models).

We use letters like Γ (Greek letter Gamma) for sets of sentences.

$\Gamma \models \varphi$

This means that every model of all the sentences in Γ is also a model of φ.

Whenever Γ is a set that we have listed out, say
$\Gamma = \{\psi_1, \psi_2, \ldots, \psi_{104}\}$.
then usually we would write $\Gamma \models \varphi$ as

$$\psi_1, \psi_2, \ldots, \psi_{104} \models \varphi$$

rather than as

$$\{\psi_1, \psi_2, \ldots, \psi_{104}\} \models \varphi.$$

That is, we drop the set braces on the left of the $\models$ symbol.
We do this to make things a little more readable.

$$\overbrace{\psi_1, \psi_2, \ldots, \psi_n}^{\text{premises}} \quad \models \quad \overbrace{\varphi}^{\text{conclusion}}$$

The intuition is that

$$\psi_1, \psi_2, \ldots, \psi_n \models \varphi$$

means that

any circumstance in which the premises $\psi_1, \psi_2, \ldots, \psi_n$ are all true is also a circumstance in which the conclusion φ is true

$$\frac{\textit{All frogs are reptiles.} \quad \textit{All frogs are animals.}}{\textit{All reptiles are animals.}}$$

We can take $M = \{1, 2, 3, 4, 5, 6\}$.
$[\![F]\!] = \{1, 2\}$,
$[\![R]\!] = \{1, 2, 3, 4\}$.
$[\![A]\!] = \{1, 2, 4, 5, 6\}$.
In this context, the assumptions are true but the conculsion is false.
So the argument is invalid.

All frogs are reptiles, *All frogs are animals*, $\not\models$ *All reptiles are animals.*

Note the difference between syntax and semantics.
$\models$ is intended to mean follows by general-purpose reasoning.

We can check whether our definitions match with our intuitions.
In the case of our very simple fragment, this mostly is right.

The main exception is that people usually wouldn't say
All X are Y
in a context where they know that there are no X.

A secondary point is that a computer should be able to decide whether

$$\psi_1, \ldots, \psi_n \models \varphi$$

or not.

The entailment problem should be decidable.

Another way to make these points:
the definitions and theory should be "tight" enough so that the decision can be made *without semantics* (!),
by only looking at the *form* of the argument.

DEFINITION

Let Γ be a set of sentences $\{\psi_1, \ldots, \psi_n\}$.
A proof tree over Γ is a tree labeled with sentences,
and with the following property:
Every node is either labeled with a sentence from Γ,
or matches one of the rules of our system (see the next slide).

We draw proof trees with the root at the bottom and the leaves at the top.

$\Gamma \vdash \varphi$

This means that there is a proof tree over Γ whose root is labeled φ,
We say that φ is provable from Γ in our system.

$$\frac{}{\textit{All } p \textit{ are } p} \qquad\qquad \frac{\textit{All } p \textit{ are } n \quad \textit{All } n \textit{ are } q}{\textit{All } p \textit{ are } q}$$

Here is an example: Let Γ be the set

$$\{\textit{All A are B}, \textit{All Q are A}, \textit{All B are D}, \textit{All C are D}, \textit{All A are Q}\}$$

Let φ be *All Q are D*. Here is a proof tree showing that $\Gamma \vdash \varphi$:

$$\dfrac{\textit{All Q are A} \qquad \dfrac{\textit{All A are B} \quad \textit{All B are D}}{\textit{All A are D}}}{\textit{All Q are D}}$$

All of the leaves belong to Γ.
Note also that some elements of Γ are not used as leaves.
This is permitted according to our definition.

The proof tree above shows that $\Gamma \vdash \varphi$.

Another proof tree for the same assertion $\Gamma \vdash \varphi$

$$\dfrac{\textit{All Q are A} \qquad \dfrac{\dfrac{\textit{All A are B} \quad \textit{All B are B}}{\textit{All A are B}} \qquad \textit{All B are D}}{\textit{All A are D}}}{\textit{All Q are D}}$$

One of the leaves is justified not because it belongs to Γ,
but because it matches the reflexivity rule.

$$\dfrac{}{\textit{All p are p}} \qquad\qquad \dfrac{\textit{All p are n} \quad \textit{All n are q}}{\textit{All p are q}}$$

What are we doing here?

The idea is that proof trees are our model of basic reasoning using the words all, some, no.

A proof tree is like a caricature of a real proof.

It can be examined (and even constructed) by a person or computer
who has no understanding of anything but the rules!

There are several hopes about this work:
⋆ The whole thing will "scale up" to include many more words.
(This would call on linguistic semantics to provide the correct notion of context.)
⋆ The formal relation $\vdash$ should have something to do with $\models$ (logic)
⋆ The proof system $\vdash$ should have something to do with actual human reasoning (psychology)
⋆ A computer should be able to work with $\vdash$ without understanding anything.

A computer could check whether a purported tree actually satisfies our definition, even if it didn't "understand" *All*.
So one important question is: what is the relation between

$$\Gamma \vdash A \quad \text{and} \quad \Gamma \models A \quad ?$$

Soundness Lemma

If $\Gamma \vdash \varphi$, then $\Gamma \models \varphi$.

This means that proof trees do not lead us astray:
if $\Gamma \vdash \varphi$, then in any context where the sentences of Γ all hold, φ too must hold.

Our proof system will not lead us to believe that bogus syllogisms are in fact valid.

Here is the basic idea of why the Soundness Lemma holds.
The two most basic facts about $\subseteq$ are:

1. $X \subseteq X$ for all sets X.
2. For all sets X, Y, and Z: if $X \subseteq Y$ and $Y \subseteq Z$, then $X \subseteq Z$.

(Probably the third would be that $\emptyset \subseteq X$ for all X.)

Let's go back to our example proof tree.

$$\dfrac{\textit{All } Q \textit{ are } A \qquad \dfrac{\textit{All } A \textit{ are } B \qquad \textit{All } B \textit{ are } D}{\textit{All } A \textit{ are } D}}{\textit{All } Q \textit{ are } D}$$

Take any model, say $\mathcal{M}$.
Assume that in $\mathcal{M}$, $[\![A]\!] \subseteq [\![B]\!]$, etc.
We have to show that in this same model $\mathcal{M}$, $[\![Q]\!] \subseteq [\![D]\!]$.

The idea is to use our proof tree and specialize it to $\mathcal{M}$:

$$\dfrac{[\![Q]\!] \subseteq [\![A]\!] \qquad \dfrac{[\![A]\!] \subseteq [\![B]\!] \qquad [\![B]\!] \subseteq [\![D]\!]}{[\![A]\!] \subseteq [\![D]\!]}}{[\![Q]\!] \subseteq [\![D]\!]}$$

And then going downward mirrors *intuitively valid reasoning in the model*.

Since the model $\mathcal{M}$ was arbitrary (had no special features),
the conclusion $\Gamma \models \varphi$ holds.

$$\Gamma = \left\{ \begin{array}{l} \text{All A are B,} \\ \text{All A are C,} \\ \text{All B are C,} \\ \text{All C are B,} \\ \text{All C are D,} \\ \text{All B are E,} \\ \text{All D are G,} \\ \text{All F are G,} \\ \text{All G are F} \end{array} \right\}$$

We see that $\Gamma \vdash$ All B are G.

Do you think that $\Gamma \vdash$ *All D are E*?

Is there an algorithm to tell yes or no?

DEFINITION

A preorder is a pair $(P, \leq)$,
where P is a set
and $\leq$ is a relation on it with the following properties:

REFLEXIVE $p \leq p$

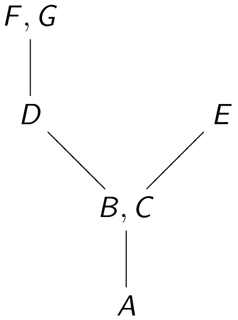
TRANSITIVE If $p \leq q$ and $q \leq r$, then $p \leq r$.

We need not have the following property:

ANTI-SYMMETRIC if $p \leq q$ and $q \leq p$, then $p = q$.

An anti-symmetric preorder is a partially ordered set (poset).

$$\Gamma = \left\{ \begin{array}{l} \text{All A are B}, \\ \text{All A are C}, \\ \text{All B are C}, \\ \text{All C are B}, \\ \text{All C are D}, \\ \text{All B are E}, \\ \text{All D are G}, \\ \text{All F are G}, \\ \text{All G are F} \end{array} \right\}$$

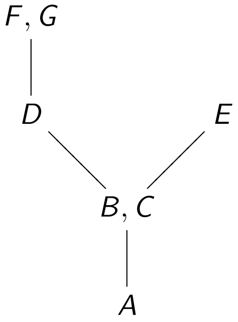


The set P here is $\{A, \ldots, G\}$.
The order $\leq$ is given by

$$X \leq Y \quad \text{iff} \quad \Gamma \vdash \text{All } X \text{ are } Y.$$

Downsets in preorders

In a preorder, $\downarrow p = \{x : x \leq p\}$.



$\downarrow F = \downarrow G = \{A, B, C, D, F, G\}$

$\downarrow D = \{A, B, C, D\}, \downarrow E = \{A, B, C, E\}$

$\downarrow B = \{A, B, C\} = \downarrow C$

$\downarrow A = \{A\}$

$\downarrow$ is monotone: if $p \leq q$, then $\downarrow p \subseteq \downarrow q$.

Suppose that $\Gamma \models$ *All X are Y*.
Let M be the set of variables.
(Yes, the model is built from the syntax!)

Define $A \leq B$ to mean that $\Gamma \vdash$ *All A are B*.
Check that this is reflexive and transitive, using the logic.
The semantics is via downsets:

$$[\![A]\!] \quad = \quad \downarrow A \quad = \quad \{B : B \leq A\}$$

By transitivity, $\mathcal{M} \models \Gamma$.
In more detail, suppose Γ contains All C are D.
Then if $W \leq C$, then also $W \leq D$.

Suppose that $\Gamma \models$ *All X are Y*.
Let M be the set of variables.
(Yes, the model is built from the syntax!)

Define $A \leq B$ to mean that $\Gamma \vdash$ *All A are B*.
Check that this is reflexive and transitive, using the logic.
The semantics is via downsets:

$$[\![A]\!] \quad = \quad \downarrow A \quad = \quad \{B : B \leq A\}$$

By transitivity, $\mathcal{M} \models \Gamma$.
In more detail, suppose Γ contains All C are D.
Then if $\mathrm{W} \leq C$, then also $\mathrm{W} \leq D$.

An example of how the proof works

All A are B
All A are C
All B are C
All C are B
All C are D
All B are E
All D are G
All F are G
All G are F

$$[\![A]\!] = \{A\}$$
$$[\![B]\!] = \{A, B, C\}$$
$$[\![C]\!] = \{A, B, C\}$$
$$[\![D]\!] = \{A, B, C, D\}$$
$$[\![E]\!] = \{A, B, C, E\}$$
$$[\![F]\!] = \{A, B, C, D, F, G\}$$
$$[\![G]\!] = \{A, B, C, D, F, G\}$$

Getting back to the question of whether or not $\Gamma \vdash \textit{All D are E}$.

Since $[\![D]\!]$ is not a subset of $[\![E]\!]$, $\Gamma \not\models \textit{All D are E}$.
By soundness, $\Gamma \nvdash \textit{All D are E}$.

Syntax: *All p are q*, *Some p are q*

Semantics: A model $\mathcal{M}$ is a set M,
and for each noun p we have an interpretation $[\![p]\!] \subseteq M$.

$$\begin{array}{lll} \mathcal{M} \models \textit{All p are q} & \text{iff} & [\![p]\!] \subseteq [\![q]\!] \\ \mathcal{M} \models \textit{Some p are q} & \text{iff} & [\![p]\!] \cap [\![q]\!] \neq \emptyset \end{array}$$

Proof system:

$$\frac{}{\textit{All p are p}} \qquad \frac{\textit{All p are n} \quad \textit{All n are q}}{\textit{All p are q}}$$

$$\frac{\textit{Some p are q}}{\textit{Some q are p}} \qquad \frac{\textit{Some p are q}}{\textit{Some p are p}} \qquad \frac{\textit{All q are n} \quad \textit{Some p are q}}{\textit{Some p are n}}$$

Some n are n, *All n are p*, *All n are q* $\vdash$ *Some p are q*.

The proof tree is

$$\dfrac{\textit{All n are q} \qquad \dfrac{\dfrac{\textit{All n are p} \quad \textit{Some n are n}}{\textit{Some n are p}}}{\textit{Some p are n}}}{\textit{Some p are q}}$$

Suppose that $\Gamma \models$ *Some p are q*.

We shall construct a particular model $\mathcal{M}$ of Γ
(hence of *Some p are q*),
and then read off a proof *Some p are q*.

We take for M the set of pairs

$$\{x, y\}$$

such that $\Gamma \vdash$ *Some x are y*.
Note that if $\{x, y\} \in M$, then also $\{x\} \in M$.

We declare

$$\{x, y\} \in [\![u]\!] \quad \text{iff} \quad x \leq u \text{ or } y \leq u.$$

This defines a model $\mathcal{M}$.

$\mathcal{M} \models \Gamma$

This is fairly routine.

Thus $\mathcal{M} \models$ *Some p are q*

Suppose that $\{x, y\} \in [\![p]\!] \cap [\![q]\!]$.

At this point, we can prove the completeness of the system.
Again, we assume that $\Gamma \models$ *Some p are q*.
and we show that $\Gamma \vdash$ *Some p are q*.

(What about $\Gamma \models$ *All p are q*?)

End of the proof of completeness for the language of All and Some

$\{x, y\} \in [\![u]\!]$ IFF $x \leq u$ OR $y \leq u$

We know that there is some $\{x, y\} \in [\![p]\!] \cap [\![q]\!]$.
There are four possibilities:

1. $x \leq p$ and $y \leq q$
2. $x \leq p$ and $x \leq q$
3. $y \leq p$ and $x \leq q$
4. $y \leq p$ and $y \leq q$

Let's just go through the first case: $x \leq p$ and $y \leq q$
The proof tree below shows that $\Gamma \vdash$ *Some p are q*:

$$\dfrac{\dfrac{\vdots}{\textit{All } x \textit{ are } p} \qquad \dfrac{\dfrac{\vdots}{\textit{All } y \textit{ are } q} \qquad \dfrac{\vdots}{\textit{Some } x \textit{ are } y}}{\text{Some } x \text{ are } q}}{\textit{Some } p \textit{ are } q}$$

$$\frac{}{\textit{All } p \textit{ are } p} \qquad \frac{\textit{All } p \textit{ are } n \quad \textit{All } n \textit{ are } q}{\textit{All } p \textit{ are } q}$$

$$\frac{\textit{Some } p \textit{ are } q}{\textit{Some } q \textit{ are } p} \qquad \frac{\textit{Some } p \textit{ are } q}{\textit{Some } p \textit{ are } p} \qquad \frac{\textit{All } q \textit{ are } n \quad \textit{Some } p \textit{ are } q}{\textit{Some } p \textit{ are } n}$$

We now know that this is complete for the language of All and Some.

$$\frac{}{\textit{All } p \textit{ are } p} \qquad \frac{\textit{All } p \textit{ are } n \quad \textit{All } n \textit{ are } q}{\textit{All } p \textit{ are } q}$$

$$\frac{\textit{Some } p \textit{ are } q}{\textit{Some } q \textit{ are } p} \qquad \frac{\textit{Some } p \textit{ are } q}{\textit{Some } p \textit{ are } p} \qquad \frac{\textit{All } q \textit{ are } n \quad \textit{Some } p \textit{ are } q}{\textit{Some } p \textit{ are } n}$$

We now know that this is complete for the language of All and Some.

We can also add names to the fragment, interpreted as points in M.

$$\frac{}{J \textit{ is } J} \qquad \frac{M \textit{ is } J}{J \textit{ is } M}$$

$$\frac{J \textit{ is } M \quad M \textit{ is } F}{J \textit{ is } F} \qquad \frac{J \textit{ is an } X \quad J \textit{ is a } Y}{\textit{Some } X \textit{ are } Y}$$

$$\frac{\textit{All } X \textit{ are } Y \quad J \textit{ is an } X}{J \textit{ is a } Y} \qquad \frac{M \textit{ is an } X \quad J \textit{ is } M}{J \textit{ is an } X}$$

The languages $\mathcal{S}$ and $\mathcal{S}^\dagger$ add noun-level negation

Let us add complemented atoms $\overline{p}$ on top of the language of All and Some, with interpretation via set complement: $[\![\overline{p}]\!] = M \setminus [\![p]\!]$.

We always have $\overline{\overline{p}} = p$.

So we have

$$\mathcal{S} \quad \left.\begin{array}{l} \left\{\begin{array}{l} \textit{All } p \textit{ are } q \\ \textit{Some } p \textit{ are } q \\ \textit{All } p \textit{ are } \overline{q} \equiv \textit{No } p \textit{ are } q \\ \textit{Some } p \textit{ are } \overline{q} \equiv \textit{Some } p \textit{ aren't } q \end{array}\right. \\ \\ \quad \textit{Some non-}p \textit{ are non-}q \end{array}\right\} \quad \mathcal{S}^\dagger$$

Let Γ be

$\{All\ B'\ are\ X, All\ X\ are\ Y, All\ Y\ are\ B, All\ B\ are\ X, All\ Y\ are\ C\}.$

We claim that $\Gamma \models All\ A\ are\ C.$

Let Γ be

$\{All\ B'\ are\ X, All\ X\ are\ Y, All\ Y\ are\ B, All\ B\ are\ X, All\ Y\ are\ C\}$.

We claim that $\Gamma \models All\ A\ are\ C$.

Here is the reasoning, done informally.
Since all B and all B' are X, everything whatsoever is an X.
And since all $X \leq Y \leq B$, we see that everything is a B.
But also $B \leq X \leq Y \leq C$.
In particular, all A are C.

But the last two premises and the fact that all X are Y
also imply that all B are C.
So all A are C.

$$\frac{}{\textit{All } p \textit{ are } p} \qquad \frac{\textit{Some } p \textit{ are } q}{\textit{Some } p \textit{ are } p} \qquad \frac{\textit{Some } p \textit{ are } q}{\textit{Some } q \textit{ are } p}$$

$$\frac{\textit{All } p \textit{ are } n \quad \textit{All } n \textit{ are } q}{\textit{All } p \textit{ are } q} \qquad \frac{\textit{All } n \textit{ are } p \quad \textit{Some } n \textit{ are } q}{\textit{Some } p \textit{ are } q}$$

$$\frac{\textit{All } q \textit{ are } \overline{q}}{\textit{All } q \textit{ are } p}\ \textit{Zero} \qquad \frac{\textit{All } \overline{q} \textit{ are } q}{\textit{All } p \textit{ are } q}\ \textit{One}$$

$$\frac{\textit{All } p \textit{ are } \overline{q}}{\textit{All } q \textit{ are } \overline{p}}\ \textit{Antitone} \qquad \frac{\textit{Some } p \textit{ are } \overline{p}}{\varphi}\ \textit{Ex falso quodlibet}$$

An example of what can be done in this logic

We saw this yesterday as one of our goal examples

All xenophobics are yodelers.
All zookeepers are non-yoders.

All zookeepers are non-xenophobics.

Show that

$$\{\textit{All } B \textit{ are } X, \textit{All } B' \textit{ are } X\} \vdash \textit{All } A \textit{ are } X.$$

Let Γ be

$$\{\textit{All } B' \textit{ are } X, \textit{All } X \textit{ are } Y, \textit{All } Y \textit{ are } B, \textit{All } B \textit{ are } X, \textit{All } Y \textit{ are } C\}.$$

Here is a derivation showing that $\Gamma \vdash$ *All A are C*.

$$\dfrac{\dfrac{\textit{All } B' \textit{ are } X \quad \dfrac{\textit{All } X \textit{ are } Y \quad \textit{All } Y \textit{ are } B}{\textit{All } X \textit{ are } B}}{\dfrac{\textit{All } B' \textit{ are } B}{\textit{All } A \textit{ are } B}} \quad \dfrac{\textit{All } B \textit{ are } X \quad \dfrac{\textit{All } X \textit{ are } Y \quad \textit{All } Y \textit{ are } C}{\textit{All } X \textit{ are } C}}{\textit{All } B \textit{ are } C}}{\textit{All } A \textit{ are } C}$$

1. $\textit{Some } X \textit{ are } X' \vdash S$ (a contradiction fact)
2. $\textit{All } X \textit{ are } Z, \textit{No } Z \textit{ are } Y \vdash \textit{No } Y \textit{ are } X$ (Celarent)
3. $\textit{No } X \textit{ are } Y \vdash \textit{No } Y \textit{ are } X$ (E-conversion)
4. $\textit{Some } X \textit{ are } Y, \text{No } Y \text{ are } Z \vdash \textit{Some } X \textit{ are } Z'$ (Ferio)
5. $\textit{All } Y \textit{ are } Z, \textit{All } Y \textit{ are } Z' \vdash \textit{No } Y \textit{ are } Y$ (complement inconsistency)

The system uses

$$\frac{\textit{Some } p \textit{ are } \overline{p}}{\varphi} \textit{ Ex falso quodlibet}$$

and this is prima facie weaker than reductio ad absurdum.

(*Reductio* will be discussed next time in detail.
It is closer to what Aristotle used, and for more expressive fragments, it is needed.)

I won't go through the completeness in this lecture

$$\frac{}{\textit{All } p \textit{ are } p} \qquad \frac{\textit{Some } p \textit{ are } q}{\textit{Some } p \textit{ are } p} \qquad \frac{\textit{Some } p \textit{ are } q}{\textit{Some } q \textit{ are } p}$$

$$\frac{\textit{All } p \textit{ are } n \quad \textit{All } n \textit{ are } q}{\textit{All } p \textit{ are } q} \qquad \frac{\textit{All } n \textit{ are } p \quad \textit{Some } n \textit{ are } q}{\textit{Some } p \textit{ are } q}$$

$$\frac{\textit{All } q \textit{ are } \overline{q}}{\textit{All } q \textit{ are } p}\ \textit{Zero} \qquad \frac{\textit{All } \overline{q} \textit{ are } q}{\textit{All } p \textit{ are } q}\ \textit{One}$$

$$\frac{\textit{All } p \textit{ are } \overline{q}}{\textit{All } q \textit{ are } \overline{p}}\ \textit{Antitone} \qquad \frac{\textit{Some } p \textit{ are } \overline{p}}{\varphi}\ \textit{Ex falso quodlibet}$$

You can read it in the rest of this slide set, and in the course notes.

Definition

An orthoposet is a tuple $(P, \leq, 0, ')$ such that

POSET	$\leq$ is a reflexive, transitive, and antisymmetric relation on the set P.
ZERO	$0 \leq p$ for all $p \in P$.
ANTITONE	If $x \leq y$, then $y' \leq x'$.
INVOLUTIVE	$x'' = x$.
INCONSISTENCY	If $x \leq y$ and $x \leq y'$, then $x = 0$.

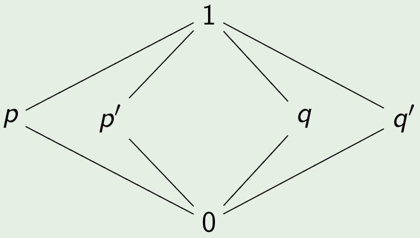
A Key Point

Orthoposets need not have a meet or join operation.

Example

For all sets X we have an orthoposet $(\mathcal{P}(X), \subseteq, \emptyset, ')$, where $a' = X \setminus a$ for all subsets a of X.

Example



$(x')' = x$, $0' = 1$, $1' = 0$.

Orthoposets: two examples

Example

For all sets X we have an orthoposet $(\mathcal{P}(X), \subseteq, \emptyset, ')$, where $a' = X \setminus a$ for all subsets a of X.

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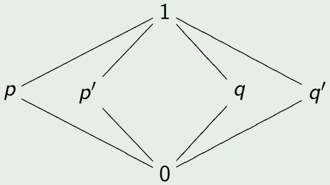


$(x')' = x$, $0' = 1$, $1' = 0$.

The idea

$$\frac{\text{boolean algebra}}{\text{propositional logic}} = \frac{\text{orthoposet}}{\text{logic of All, Some and }'}$$

The details concerning completeness are somewhat different, and the whole thing would take about 10 minutes.

Let Γ be any set of sentences in the fragment.
Let $\mathcal{V}$ be the set of variables.
We already know the preorder $\leq$:

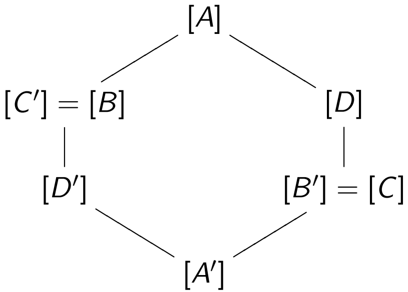
$$X \leq \mathrm{Y} \quad \text{iff} \quad \Gamma \vdash \textit{All } X \textit{ are } Y.$$

(so Some plays no role)
We have an induced equivalence relation $\equiv$,
and we take $\mathcal{V}_\Gamma$ to be the quotient $\mathcal{V}/\!\equiv$.
If there is some X such that $X \leq X'$, then set 0 to be $[X]$.
We finally define $[X]' = [X']$.
If there is no X such that $X \leq X'$, we add fresh elements 0 and 1 to $\mathcal{V}/\!\equiv$.
It is not hard to check that we have an orthoposet $\mathcal{V}_\Gamma$.

Let $\Gamma =$
$\{$All B are A, All B' are A, All C' are B, All C are B', All C are $D\}$.
Then

$$\begin{array}{ll} [A] = \{A\} & [A'] = \{A'\} \\ [B] = \{B, C'\} & [B'] = \{B', C\} \\ [C] = \{B', C\} & [C'] = \{B, C'\} \\ [D] = \{D\} & [D'] = \{D'\} \end{array}$$

Here is a picture of the orthoposet $\mathcal{V}_\Gamma$:



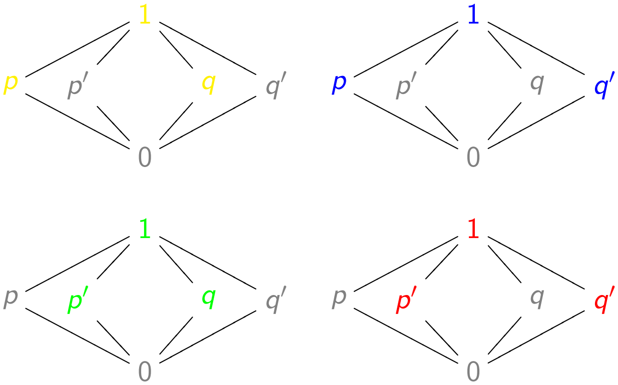
A point of a orthoposet $P = (P, \leq, 0, ')$ is a subset $\mathcal{S} \subseteq P$ with the following properties:

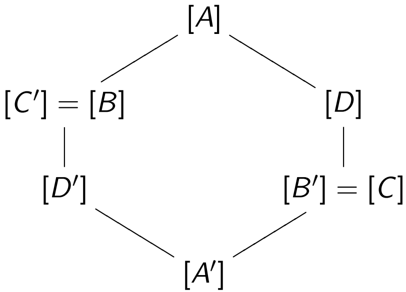
up-closed If $p \in S$ and $p \leq q$, then $q \in \mathcal{S}$.

complete For all p, either $p \in S$ or $p' \in \mathcal{S}$.

pairwise compatible For all $p, q \in \mathcal{S}$, $p \not\leq q'$.

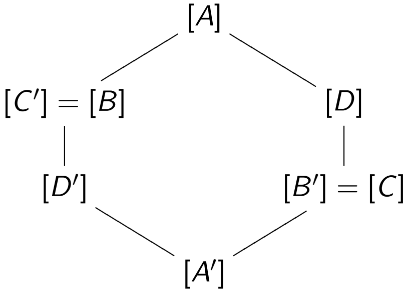
Look back at the Chinese lantern.
There are four points here: the sets marked •, •, •, and •:




[A]
[C′] = [B]
[D]
[D′]
[B′] = [C]
[A′]

What are the points?

There are three points.



$\mathcal{S} = \{[D'], [B], [A]\}$, $\mathcal{T} = \{[B'], [D], [A]\}$, $\mathcal{U} = \{[B], [D], [A]\}$.

Let $X = \{1, 2, 3\}$, and let $\mathcal{P}(X)$ be the power set orthoposet. Then $\mathcal{S}$ is a point, where

$$\mathcal{S} \quad = \quad \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

It is easy to check that the points on this $\mathcal{P}(X)$ are exactly $\mathcal{S}$ as above and the three principal ultrafilters.
$\mathcal{S}$ shows that a point of a boolean algebra need not be a filter.

The Extension Lemma for pairwise consistent sets

Lemma

Let $\mathcal{S} \subseteq P$ be pairwise consistent: $(\forall p, q \in \mathcal{S}) p \not\leq q'$.
Then for all $x \in P$, either $\mathcal{S} \cup \{x\}$ or $\mathcal{S} \cup \{x'\}$ is again pairwise consistent.

Proof.

Suppose not. Then x and x' figure in to problems both times.
There is some $p \in \mathcal{S}$ such that $p \leq x'$.
There is some $q \in \mathcal{S}$ such that $q \leq x'' = x$.
And now: $q \leq x \leq p'$. Ooops! □

The Extension Lemma for pairwise consistent sets

Lemma

Let $\mathcal{S} \subseteq P$ be pairwise consistent*: $(\forall p, q \in \mathcal{S}) p \not\leq q'$.*
Then for all $x \in P$, either $\mathcal{S} \cup \{x\}$ or $\mathcal{S} \cup \{x'\}$ is again pairwise consistent.

Lemma

If $p \not\leq q$, then $\{p, q'\}$ is pairwise consistent.
Thus there is a point $\mathcal{S}$ containing p but not q.

Pairwise consistent sets extend to points

Lemma

For a subset S_0 of an orthoposet $P = (P, \leq, 0, ')$, the following are equivalent:

1. *S_0 is a subset of a point $\mathcal{S}$ in P.*
2. *S_0 is pairwise compatible.*

Proof.

Clearly (1) $\Longrightarrow$ (2).
For the more important direction, use Zorn's Lemma to get $\mathcal{S} \supseteq S_0$ which is pairwise compatible, and maximal with this property.
For all p, either p or p' belongs to $\mathcal{S}$. [By maximality.]
Check easily that $\mathcal{S}$ is up-closed:
If $p \in \mathcal{S}$, $p \leq q$, but $q \notin S$, then $q' \in \mathcal{S}$.
And now $p \leq q = (q')'$, so $\mathcal{S}$ is not pairwise compatible. □

Let $P = (P, \leq, ')$ be an orthoposet.
Let $points(P)$ be the set of points of P.
We have an orthoposet

$$(\mathcal{P}(points(P)), \subseteq, \emptyset, ')$$

Let $m : P \to \mathcal{P}(points(P))$ be given by

$$m(p) \quad = \quad \{\mathcal{S} : p \in \mathcal{S}\}.$$

Theorem

m is a strict morphism of orthoposets:
$m(0) = \emptyset$,
$m(p') = (m(p))'$,
and $p \leq q$ *iff* $m(p) \subseteq m(q)$.

Let $P = (P, \leq, ')$ be an orthoposet.
Let $points(P)$ be the set of points of P.
We have an orthoposet

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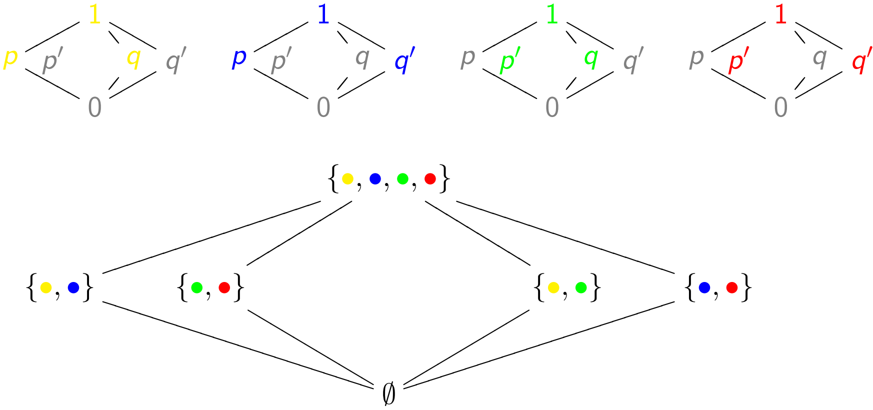
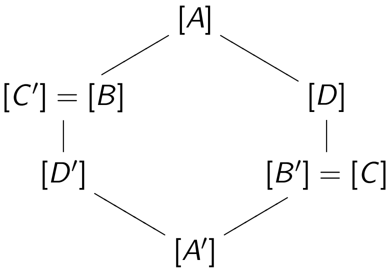
Corollary

Every orthoposet is isomorphic to a sub-orthoposet of a power set orthoposet.


1
p
p′
q
q′
0
1
p
p′
q
q′
0
1
p
p′
q
q′
0
1
p
p′
q
q′
0
{•, •, •, •}
{•, •}
{•, •}
{•, •}
{•, •}
∅

$\mathcal{S} = \{[D'], [B], [A]\}$, $\mathcal{T} = \{[B'], [D], [A]\}$, $\mathcal{U} = \{[B], [D], [A]\}$.



$$\begin{array}{ll} m([A]) = \{\mathcal{S}, \mathcal{T}, \mathcal{U}\} & m([A']) = \emptyset \\ m([B]) = \{\mathcal{S}, \mathcal{U}\} & m([B']) = \{\mathcal{T}\} \\ m([D]) = \{\mathcal{T}, \mathcal{U}\} & m([D']) = \{\mathcal{S}\} \end{array}$$

This pretty much solves our earlier problem of getting a model of Γ where $[\![B]\!] \not\subseteq [\![D]\!]$.

But how?

Sources the Representation Theorem

N. Zierler and M. Schlessinger

Boolean embeddings of orthomodular sets and quantum logic.
Duke Mathematical Journal 32 (1965), 251–262.

F. Katrnoška

On the representation of orthocomplemented posets.
Comment. Math. Univ. Carolinae 23 (1982), 489–498.

C. S. Calude, P. H. Hertling, K. Svozil

Embedding quantum universes into classical ones.
Foundations of Physics, 29, 3 (1999), 349-379.

Lemma

Let Γ be consistent in $\mathcal{L}(\text{all}, \text{some}, ')$.
There is a canonical model $\mathcal{M} = (M, [\![\ ]\!])$ such that

1. $\mathcal{M} \models \Gamma$.
2. *If $\mathcal{M} \models$ All X are Y, then $\Gamma \vdash$ All X are Y.*

Proof.

Let $\mathcal{V}_\Gamma$ be the syntactic orthoposet for Γ. Let $M = \mathit{points}(\mathcal{V}_\Gamma)$.
The interpretation $[\![\ ]\!] : \mathcal{V} \to \mathcal{P}(M)$ is given by

$$\mathcal{V} \xrightarrow{n} \mathcal{V}_\Gamma \xrightarrow{m} \mathcal{P}(\mathit{points}(\mathcal{V}_\Gamma)) = \mathcal{P}(M)$$

Key point If Γ contains Some U are V, need a point including $\{[\text{U}], [\text{V}]\}$.
If none exists, then wlog $\text{U} \leq \text{V}'$. But then Γ is inconsistent. □

LEMMA

Let Γ be consistent in $\vdash$.
There is a canonical model $\mathcal{M} = (M, [\![\;]\!])$ such that

1. $\mathcal{M} \models \Gamma$.
2. *If $\mathcal{M} \models$ All X are Y, then $\Gamma \vdash$ All X are Y.*

THIS GIVES HALF OF COMPLETENESS:

If $\Gamma \models$ All X are Y,
then $\mathcal{M} \models$ All X are Y,
and so $\Gamma \vdash$ All X are Y.

For Some sentences, we need a little more.

If Γ is consistent and $\Gamma \models$ Some X are Y, then $\Gamma \vdash$ Some X are Y

Lemma (Ian Pratt-Hartmann 2007)

There is some existential sentence in Γ, say Some A are B, such that

$$\Gamma_{all} \cup \{\textit{Some A are B}\} \models \textit{Some X are Y}.$$

If Γ is consistent and $\Gamma \models$ Some X are Y, then $\Gamma \vdash$ Some X are Y

Fix A and B as in the lemma.
Consider the model $\mathcal{M} = \mathcal{M}(\mathcal{V}_{\Gamma_{all}})$ of points on $\mathcal{V}_{\Gamma_{all}}$. $\mathcal{M} \models \Gamma_{all}$.
Consider $\{[A], [B], [X']\}$.
If this set were a subset of a point $\mathcal{S}$, then consider $\{\mathcal{S}\}$ as a one-point submodel of $\mathcal{M}$.
In the submodel, $\Gamma_{all} \cup \{$Some A are $B\}$ would hold,
and yet Some X are Y would fail, since $[\![X]\!] = \emptyset$.
Therefore $\{[A], [B], [X']\}$ is not pairwise compatible.
There are six cases:

$$\begin{array}{ll} A \leq A' & A \leq B' \\ A \leq X & B \leq B' \\ B \leq X & X' \leq X \end{array}$$

Only two are significant.

If Γ is consistent and $\Gamma \models$ Some X are Y, then $\Gamma \vdash$ Some X are Y

Next, consider $\{A, B, Y'\}$.
The same analysis gives two other cases: $A \leq Y$ and $B \leq Y$.
Putting these together with the other two gives four pairs.
The case when $A \leq X$ and $B \leq Y$ is representative:

$$\dfrac{\dfrac{\vdots}{\textit{All } B \textit{ are } Y} \qquad \dfrac{\dfrac{\dfrac{\vdots}{\textit{All } A \textit{ are } X} \quad \textit{Some } B \textit{ are } A}{\textit{Some } B \textit{ are } X}}{\textit{Some } X \textit{ are } B}}{\textit{Some } X \textit{ are } Y}$$

The other cases are similar. This completes the proof.

Read $\exists^{\geq}(X, Y)$ as "there are at least as many Xs as Ys".

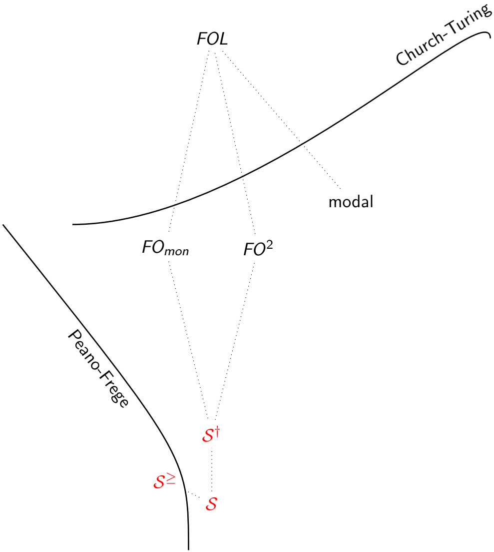
$$\frac{\textit{All } Y \textit{ are } X}{\exists^{\geq}(X, Y)} \qquad \frac{\exists^{\geq}(X, Y) \quad \exists^{\geq}(Y, Z)}{\exists^{\geq}(X, Z)}$$

$$\frac{\textit{All } Y \textit{ are } X \quad \exists^{\geq}(Y, X)}{\textit{All } X \textit{ are } Y}$$

$$\frac{\textit{Some } Y \textit{ are } Y \quad \exists^{\geq}(X, Y)}{\textit{Some } X \textit{ are } X} \qquad \frac{\textit{No } Y \textit{ are } Y}{\exists^{\geq}(X, Y)}$$

The point here is that by working with a weak basic system, we can go beyond the expressive power of first-order logic.

Picture



monadic FOL
2 variable fragment

† adds full N-negation
We have discussed these

$$\frac{}{\forall(n, n)}\ (\mathrm{T}) \qquad \frac{\forall(n, p) \quad \forall(p, q)}{\forall(n, q)}\ (\mathrm{B})$$

$$\frac{}{\forall(\text{red } x, x)}\ (\mathrm{Adj}_1) \qquad \frac{\forall(n, \text{red } x) \quad \forall(n, y)}{\forall(n, \text{red } y)}\ (\mathrm{Adj}_2)$$

$$\frac{\exists(n, p)}{\exists(n, n)}\ (\mathrm{I}) \qquad \frac{\exists(n, q) \quad \forall(q, p)}{\exists(p, n)}\ (\mathrm{D}) \qquad \frac{\exists(x, \text{red } y)}{\exists(\text{red } x, \text{red } y)}\ (\mathrm{Adj}_3)$$

$$\frac{\exists(\text{red } x, \text{blue } y) \quad \forall(\text{red } x, \text{green } z)}{\exists(\text{red } x, \text{blue } z)}\ (\mathrm{Adj}_4)$$

$$\frac{\exists(\text{red } x, \text{blue } y) \quad \forall(\text{red } x, \text{green } z)}{\exists(\text{blue } x, \text{green } y)}\ (\mathrm{Adj}_5)$$