Relational Reasoning in Natural Language

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Adding transitive verbs

the work on $\mathcal{R}$, $\mathcal{R}^{\dagger}$, and other systems is joint with Ian Pratt-Hartmann

The main language in this lecture uses "see" or r as variables for transitive verbs.

All p are q	*All p aren't q* $\equiv$ *No p are q*
Some p are q	*Some p aren't q*
All p see all q	*All p don't see all q* $\equiv$ *No p sees any q*
All p see some q	*All p don't see some q* $\equiv$ *No p sees all q*
Some p see all q	*Some p don't see any q*
Some p see some q	*Some p don't see some q*

The interpretation is the natural one, using the subject wide scope readings in the ambiguous cases.

This is $\mathcal{R}$.
(The first system of its kind was Nishihara, Morita, Iwata 1990.)

Another language called $\mathcal{R}^{\dagger}$ has complemented atoms $\overline{p}$ on top of $\mathcal{R}$.

Semantics

$$[\![r]\!] \subseteq M \times M$$

Issue: is there one verb, or many?

For this fragment, we might as well restrict attention to just one verb.
But when we move to the fragment with relative clauses, this will not do.

Are there any interesting inferences?

$$\left\{ \begin{array}{l} \text{No } x \text{ see any } y \\ \text{All } z \text{ see some } y \end{array} \right\} \models \text{No } x \text{ are } z$$

$$\left\{ \begin{array}{l} \text{All } x \text{ see all } y \\ \text{All } p \text{ are } y \\ \text{Some } p \text{ are } q \end{array} \right\} \models \text{All } x \text{ see some } q$$

Towards the syntax for $\mathcal{R}$

All p are q	$\forall(p, q)$
Some p are q	$\exists(p, q)$
All p r all q	$\forall(p, \forall(q, r))$
All p r some q	$\forall(p, \exists(q, r))$
Some p r all q	$\exists(p, \forall(q, r))$
Some p r some q	$\exists(p, \exists(q, r))$
No p are q	$\forall(p, \overline{q})$
Some p aren't q	$\exists(p, \overline{q})$
All p don't r all q $\equiv$	
No p r any q	$\forall(p, \forall(q, \overline{r}))$
All p don't r some q $\equiv$	
No p r all q	$\forall(p, \exists(q, \overline{r}))$
Some p don't r any q	$\exists(p, \forall(q, \overline{r}))$
Some p don't r some q	$\exists(p, \exists(q, \overline{r}))$

Towards the syntax for $\mathcal{R}$

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All p r some q	$\forall(p, \exists(q, r))$
Some p r all q	$\exists(p, \forall(q, r))$
Some p r some q	$\exists(p, \exists(q, r))$
No p are q	$\forall(p, \overline{q})$
Some p aren't q	$\exists(p, \overline{q})$
No p r any q	$\forall(p, \forall(q, \overline{r}))$
No p r all q	$\forall(p, \exists(q, \overline{r}))$
Some p don't r any q	$\exists(p, \forall(q, \overline{r}))$
Some p don't r some q	$\exists(p, \exists(q, \overline{r}))$

set terms c

positive	p	$\forall(p, r)$	$\exists(p, r)$
negative	$\overline{p}$	$\exists(p, \overline{r})$	$\forall(p, \overline{r})$

Reading the set terms

$\forall(p, r)$	those who r all p
$\exists(p, r)$	those who r some p
$\forall(p, \overline{r})$	those who fail-to-r all $p \approx$ those who r no p
$\exists(p, \overline{r})$	those who fail-to-r some $p \approx$ those who don't r some p

TOWARDS THE SYNTAX FOR $\mathcal{R}$

All p are q	$\forall(p, q)$	
Some p are q	$\exists(p, q)$	
All p r all q	$\forall(p, \forall(q, r))$	
All p r some q	$\forall(p, \exists(q, r))$	
Some p r all q	$\exists(p, \forall(q, r))$	
Some p r some q	$\exists(p, \exists(q, r))$	simplifies to
No p are q	$\forall(p, \overline{q})$	
Some p aren't q	$\exists(p, \overline{q})$	$\forall(p, c)$ $\exists(p, c)$
No p sees any q	$\forall(p, \forall(q, \overline{r}))$	
No p sees all q	$\forall(p, \exists(q, \overline{r}))$	
Some p don't r any q	$\exists(p, \forall(q, \overline{r}))$	
Some p don't r some q	$\exists(p, \exists(q, \overline{r}))$	

set terms c

positive	p	$\forall(p, r)$	$\exists(p, r)$
negative	$\overline{p}$	$\exists(p, \overline{r})$	$\forall(p, \overline{r})$

Syntax of $\mathcal{R}$ and $\mathcal{R}^\dagger$

We start with one collection of unary atoms (for nouns) and another of binary atoms (for transitive verbs).

expression	variables	syntax
unary atom	p, q	
binary atom	r	
set term	c, d	$p \mid \exists(p,r) \mid \forall(p,r) \mid$ $\overline{p} \mid \exists(p,\overline{r}) \mid \forall(p,\overline{r})$
$\mathcal{R}$ sentence	φ	$\forall(p,c) \mid \exists(p,c)$
$\mathcal{R}^\dagger$ sentence	φ	$\forall(p,c) \mid \exists(p,c) \mid \forall(\overline{p},c) \mid \exists(\overline{p},c)$

Negations

We need one last concept, syntactic negation:

expression	syntax	negation
set term c	p	$\overline{p}$
	$\overline{p}$	p
	$\exists(p, r)$	$\forall(p, \overline{r})$
	$\forall(p, r)$	$\exists(p, \overline{r})$
	$\exists(p, \overline{r})$	$\forall(p, r)$
	$\forall(p, \overline{r})$	$\exists(p, r)$
$\mathcal{R}$ sentence φ	$\forall(p, c)$	$\exists(p, \overline{c})$
	$\exists(p, c)$	$\forall(p, \overline{c})$

Note that $\overline{\overline{p}} = p$, $\overline{\overline{c}} = c$ and $\overline{\overline{\varphi}} = \varphi$.

Semantics, again

We said before that the semantics was based on

$$[\![r]\!] \subseteq M \times M$$

This doesn't give the full semantics of the language, since our syntax now has set terms.

We define

$$\begin{array}{lcl} [\![\text{r all } p]\!] & = & \{x \in M : \text{for all } y \in [\![p]\!], x\, [\![r]\!]\, y\} \\ [\![\text{r some } p]\!] & = & \{x \in M : \text{for some } y \in [\![p]\!], x\, [\![r]\!]\, y\} \end{array}$$

For set terms involving $\bar{r}$, we take $[\![\bar{r}]\!] = (M \times M) \setminus [\![r]\!]$, and then

$$\begin{array}{lcl} [\![\bar{r} \text{ all } p]\!] & = & \{x \in M : \text{for all } y \in \ \mathit{semanticsp}, x\, [\![\bar{r}]\!]\, y\} \\ [\![\bar{r} \text{ some } p]\!] & = & \{x \in M : \text{for some } y \in [\![p]\!], x\, [\![\bar{r}]\!]\, y\} \end{array}$$

By the way

Set terms like $\forall(p, r)$ use the relation in a way that is different from the Kripke semantics for modal logic.

$$\begin{array}{lcl} [\![\text{r all } p]\!] & = & \{x \in M : \text{for all } y \in [\![p]\!],\ x\,[\![r]\!]\,y\} \\ [\![\Box_r p]\!] & = & \{x \in M : \text{ for all } y \text{ such that } x[\![r]\!]y,\ y \in [\![p]\!]\} \end{array}$$

Review of the notation

$$\left\{ \begin{array}{l} \text{No } x \text{ see any } y, \\ \text{All } z \text{ see some } y \end{array} \right\} \models \text{No } x \text{ are } z$$

$$\forall(x, \forall(y, \overline{r})), \forall(z, \exists(y, r)) \models \forall(x, \overline{z})$$

$$\left\{ \begin{array}{l} \text{All } x \text{ see all } y, \\ \text{All } p \text{ are } y, \\ \text{Some } p \text{ are } q \end{array} \right\} \models \text{All } x \text{ see some } y$$

$$\forall(x, \forall(y, r)), \forall(p, y), \exists(p, q) \models \forall(x, \exists(y, r))$$

The hardest two families of inferences

$$\frac{\forall(x_1, \exists(x_2, r)) \quad \cdots \quad \forall(x_n, \exists(y, r)) \quad \forall(x_1, \forall(y, r))}{\forall(x_1, \exists(y, r))} \text{ (H)}$$

$$\frac{\forall(y_1, \exists(y_2, r)) \ \cdots \ \forall(y_n, \exists(z, r)) \quad \forall(z, \forall(y_1, r)) \quad \forall(z, x) \quad \exists(x, x)}{\exists(x, \forall(y_1, r))} \text{ (HH)}$$

To see that (H) is semantically valid, argue by cases as to whether $\exists x_1$ or not.

For (HH), argue by cases as to whether $\exists y_1$ or not.

Relational syllogistic logic

p and q range over unary atoms,
c over set terms, and t over binary atoms or their negations.

$$\frac{\exists(p,q) \qquad \forall(q,c)}{\exists(p,c)} \qquad \frac{\forall(p,q) \qquad \forall(q,c)}{\forall(p,c)}$$

$$\frac{\forall(p,q) \qquad \exists(p,c)}{\exists(q,c)} \qquad \frac{}{\forall(p,p)} \qquad \frac{\exists(p,c)}{\exists(p,p)}$$

$$\frac{\forall(q,\bar{c}) \qquad \exists(p,c)}{\exists(p,\bar{q})} \qquad \frac{\forall(p,\bar{p})}{\forall(p,c)} \qquad \frac{\exists(p,\exists(q,t))}{\exists(q,q)}$$

$$\frac{\forall(p,\forall(n,t)) \qquad \exists(q,n)}{\forall(p,\exists(q,t))} \qquad \frac{\exists(p,\exists(q,t)) \qquad \forall(q,n)}{\exists(p,\exists(n,t))}$$

$$\frac{\forall(p,\exists(q,t)) \qquad \forall(q,n)}{\forall(p,\exists(n,t))} \qquad \begin{array}{c}[\varphi] \\ \vdots \\ \dfrac{\exists(p,\overline{p})}{\overline{\varphi}}\ \text{RAA}\end{array}$$

Relational syllogistic logic

Most are monotonicty principles

$$\exists(p^{\uparrow}, q^{\uparrow}) \qquad \forall(p^{\downarrow}, q^{\uparrow})$$
$$\exists(p^{\uparrow}, \forall(q^{\downarrow}, t)) \qquad \exists(p^{\uparrow}, \exists(q^{\uparrow}, t))$$
$$\forall(p^{\downarrow}, \forall(q^{\downarrow}, t)) \qquad \forall(p^{\downarrow}, \exists(q^{\uparrow}, t))$$

Plus also RAA and

$$\frac{}{\forall(p, p)} \qquad \frac{\exists(p, c)}{\exists(p, p)} \qquad \frac{\forall(p, \bar{p})}{\forall(p, c)} \qquad \frac{\exists(p, \exists(q, t))}{\exists(q, q)}$$

$$\frac{\forall(q, \bar{c}) \qquad \exists(p, c)}{\exists(p, \bar{q})} \qquad (\star) \; \frac{\forall(p, \forall(n, t)) \qquad \exists(q, n)}{\forall(p, \exists(q, t))}$$

Of these, $(\star)$ is the most interesting.

Let's see some examples

Here is a derivation showing that

$$\forall(x, \forall(y, r)), \forall(p, y), \exists(p, q) \vdash \forall(x, \exists(y, r))$$

$$\dfrac{\dfrac{\forall(x, \forall(y, r)) \qquad \dfrac{\exists(p, q) \quad \forall(p, y)}{\exists(p, y)}}{\forall(x, \exists(p, r))} \qquad \forall(p, y)}{\forall(x, \exists(y, r))}$$

Let's see some examples

Here is a formal proof showing that

$$\forall(x, \overline{x}) \vdash \forall(y, \forall(x, r))$$

In words, if there are no xs, then all y's have any relation whatsoever to all of them.
Note that this does not follow from the rule

$$\frac{\forall(p, \bar{p})}{\forall(p, c)}$$

Let's see some examples

Here is a formal proof showing that

$$\forall(x, \overline{x}) \vdash \forall(y, \forall(x, r))$$

In words, if there are no xs, then all y's have any relation whatsoever to all of them.
Here is a derivation:

$$\dfrac{\dfrac{\dfrac{[\exists(y, \exists(x, \overline{r}))]^1}{\exists(x, x)} \qquad \forall(x, \overline{x})}{\exists(x, \overline{x})}}{\forall(y, \forall(x, r))} \text{ (RAA)}^1$$

Note: the general definition of $\Gamma \vdash \varphi$ now must talk about uncancelled leaves in the derivation.

Example of a proof in this system

What do you think?

$$\textit{All } X \textit{ see all } Y, \textit{All } X \textit{ see some } Z, \textit{All } Z \textit{ see some } Y \models \textit{All } X \textit{ see some } Y$$

Example of a proof in this system

What do you think?

$$\textit{All } X \textit{ see all } Y, \textit{All } X \textit{ see some } Z, \textit{All } Z \textit{ see some } Y \models \textit{All } X \textit{ see some } Y$$

The conclusion does indeed follow.
We should have a formal proof.

Example of a proof in this system

What do you think?

$$\textit{All } X \textit{ see all } Y, \textit{All } X \textit{ see some } Z, \textit{All } Z \textit{ see some } Y \models \textit{All } X \textit{ see some } Y$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\color{red}{\textit{Some } X \textit{ see no } Y}}{\exists X} \quad \textit{All } X \textit{ see some } Z}{\textit{Some } X \textit{ see some } Z}}{\exists Z} \quad \textit{All } Z \textit{ see some } Y}{\textit{Some } Z \textit{ see some } Y}}{\exists Y} \quad \textit{All } X \textit{ see all } Y}{\textit{All } X \textit{ see some } Y} \quad \color{red}{\textit{Some } X \textit{ see no } Y}}{\textit{Some } X \textit{ aren't } X}$$

$\exists X$ abbreviates *Some* X *are* X

But now

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\textit{Some X see no Y}]}{\exists X} \quad \textit{All X see some Z}}{\textit{Some X see some Z}}}{\exists Z} \quad \textit{All Z see some Y}}{\textit{Some Z see some Y}}}{\exists Y} \quad \textit{All X see all Y}}{\textit{All X see some Y}} \quad [\textit{Some X see no Y}]}{\dfrac{\textit{Some X aren't X}}{\textit{All X see some Y}}\ RAA}
$$

This shows that

$$\textit{All X see all Y}, \textit{All X see some Z}, \textit{All Z see some Y} \vdash \textit{All X see some Y}$$

An important consequence of RAA: Proof by Cases

If $\Gamma, \varphi \vdash \psi$, and $\Gamma, \overline{\varphi} \vdash \psi$, then $\Gamma \vdash \psi$

For the proof, note first that $\Gamma, \varphi, \overline{\psi} \vdash \bot$.

So by RAA, $\Gamma, \overline{\psi} \vdash \overline{\varphi}$.

Take a derivation showing that $\Gamma, \overline{\varphi} \vdash \psi$,
replace all leaves $\overline{\varphi}$ with derivations showing $\Gamma, \overline{\psi} \vdash \overline{\varphi}$.
In this way, we see that $\Gamma, \overline{\psi} \vdash \psi$.

Thus $\Gamma, \overline{\psi} \vdash \bot$.

So as desired, $\Gamma \vdash \psi$.

Another important consequence of RAA

A set Γ is consistent if $\Gamma \nvdash \perp$.
Γ is complete if for all φ, either $\varphi \in \Gamma$ or $\overline{\varphi} \in \Gamma$.

In any logical system with RAA, every consistent set Γ has a consistent and complete extension Δ.

(For the proof, one can either use Zorn's Lemma,
or else successively to Γ add each sentence or its negation.)

A negative result on this language $\mathcal{R}$

We are going to prove the completeness of the logic for $\mathcal{R}$, but before that, we argue that RAA is needed.

Theorem

There are no finite syllogistic logical systems which are sound and complete for $\mathcal{R}$.

However, there is a logical system (presented above) which uses reductio ad absurdum

$$\frac{\begin{array}{c}[\varphi] \\ \vdots \\ \exists(p, \overline{p})\end{array}}{\overline{\varphi}}\ \text{RAA}$$

and which is complete.

There are no finite, sound, and complete purely syllogistic logics $\vdash_*$ for $\mathcal{R}$

Suppose towards a contradiction that $\vdash_*$ did it.
We allow rules with arbitrarily many premises.

Fix $n \in \mathbb{N}$ greater than the number of premises in any rule in the system $\vdash_*$.
Let $Y_1, \ldots, Y_n$ be distinct variables.
Let Γ be the following set of $\mathcal{R}$-formulas:

All Y_i see some Y_{i+1}	$(1 \leq i < n)$
All Y_1 see all Y_n	
All Y_i are Y_i	$(1 \leq i < n)$
All Y_i aren't Y_j	$(1 \leq i \neq j \leq n)$

There are no finite, sound, and complete purely syllogistic logics $\vdash_*$ for $\mathcal{R}$

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All Y_i *see some* Y_{i+1}	$(1 \leq i < n)$
All Y_1 *see all* Y_n	
All Y_i *are* Y_i	$(1 \leq i < n)$
All Y_i *aren't* Y_j	$(1 \leq i \neq j \leq n)$

Observe that $\Gamma \models$ *All* Y_1 *see some* Y_n,
but this sentence is not in Γ.

Proof, continued

$$\Gamma = \begin{cases} \textit{All } Y_i \textit{ see some } Y_{i+1} & (1 \leq i < n) \\ \textit{All } Y_1 \textit{ see all } Y_n & \\ \textit{All } Y_i \textit{ are } Y_i & (1 \leq i < n) \\ \textit{All } Y_i \textit{ aren't } Y_j & (1 \leq i \neq j \leq n) \end{cases}$$

For $1 \leq i < n$, let $\Delta_i = \Gamma \setminus \{\textit{All } Y_i \textit{ see some } Y_{i+1}\}$.

Claim If $\varphi \in \mathcal{R}$ and $\Delta_i \models \varphi$, then $\varphi \in \Gamma$.

Proof, continued

$$\Gamma = \begin{cases} \textit{All } Y_i \textit{ see some } Y_{i+1} & (1 \leq i < n) \\ \textit{All } Y_1 \textit{ see all } Y_n & \\ \textit{All } Y_i \textit{ are } Y_i & (1 \leq i < n) \\ \textit{All } Y_i \textit{ aren't } Y_j & (1 \leq i \neq j \leq n) \end{cases}$$

For $1 \leq i < n$, let $\Delta_i = \Gamma \setminus \{\textit{All } Y_i \textit{ see some } Y_{i+1}\}$.

Claim If $\varphi \in \mathcal{R}$ and $\Delta_i \models \varphi$, then $\varphi \in \Gamma$.
It follows this claim that $\Gamma \not\vdash_* \textit{All } Y_1 \textit{ see some } Y_n$.

Why the claim establishes the result

Claim If $\varphi \in \mathcal{R}$ and $\Delta_i \models \varphi$, then $\varphi \in \Gamma$.

Here is why $\Gamma \not\vdash_*$ *All Y_1 see some Y_n*.
We show by induction on proof trees using $\vdash_*$ that all deductions from Γ must have an element of Γ on the root.

No rule of $\vdash_*$ has more than $n-1$ premises.
By induction hypothesis, the sentences just above the root are contained in Γ.
So by the claim, the root is in Γ.

Therefore the logic is not complete.

Claim: if $\varphi \in \mathcal{R}$ and $\Delta_i \models \varphi$, then $\varphi \in \Gamma$.

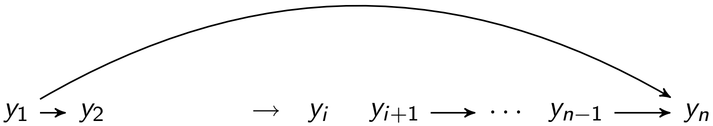
$$\Gamma = \begin{cases} \textit{All } Y_i \textit{ see some } Y_{i+1} & (1 \leq i < n) \\ \textit{All } Y_1 \textit{ see all } Y_n & \\ \textit{All } Y_i \textit{ are } Y_i & (1 \leq i < n) \\ \textit{All } Y_i \textit{ aren't } Y_j & (1 \leq i \neq j \leq n) \end{cases}$$

For $1 \leq i < n$, let $\Delta_i = \Gamma \setminus \{\textit{All } Y_i \textit{ see some } Y_{i+1}\}$.

Proof sketch We consider every sentence in the language $\mathcal{R}$.
We check that all are either in Γ or are falsified in some model of Δ_i.

All Y_i *are* Y_j	*All* Y_i *aren't* Y_j $\equiv$ *No* X *are* Y_j
Some Y_i *are* Y_j	*Some* Y_i *aren't* Y_j
All Y_i *see all* Y_j	*All* Y_i *don't see all* Y_j $\equiv$ *No* X *sees any* Y_j
All Y_i *see some* Y_j	*All* Y_i *don't see some* Y_j $\equiv$ *No* X *sees all* Y_j
Some Y_i *see all* Y_j	*Some* Y_i *don't see any* Y_j
Some Y_i *see some* Y_j	*Some* Y_i *don't see some* Y_j

The proof goes on



This structure satisfies Δ_i and makes a few false:

All Y_i are Y_j $\surd$	*No Y_j are Y_k* $\surd$ for $j = k$
Some Y_i are Y_j $\surd$	*Some Y_i aren't Y_j*
All Y_i r all Y_j	*All Y_i don't r all Y_j* $\equiv$ *No Y_i sees any Y_j*
All Y_i r some Y_j	*All Y_i don't r some Y_j* $\equiv$ *No Y_i sees all Y_j*
Some Y_i r all Y_j	*Some Y_i don't r any Y_j*
Some Y_i r some Y_j	*Some Y_i don't r some Y_j*

The proof goes on

The empty structure satisfies Δ_i and makes a few more false:

All Y_j *are* $Y_k \surd$	*No* Y_j *are* $Y_k \surd$ for $j = k$
Some Y_j *are* $Y_k \surd$	*Some* Y_j *aren't* $Y_k \surd$ for $j \neq k$
All Y_j *r all* Y_k	*All* Y_j *don't r all* $Y_k \equiv$ *No* Y_i *sees any* Y_j
All Y_j *r some* Y_k	*All* Y_j *don't r some* $Y_k \equiv$ *No* Y_i *sees all* Y_j
Some Y_j *r all* $Y_k \surd$	*Some* Y_j *don't r any* $Y_k \surd$
Some Y_j *r some* $Y_k \surd$	*Some* Y_j *don't r some* $Y_k \surd$

What remains

We list the remaining sentences.

All Y_j *r all* Y_{j+1}	*All* Y_j *don't r all* $Y_k \equiv$ *No* Y_i *sees any* Y_j
All Y_1 *r some* Y_n	*All* Y_j *don't r some* $Y_k \equiv$ *No* Y_i *sees all* Y_j

We only list the sentences not in Δ_i which are true in both of the two models we have seen so far.

The ones on the right are easy to falsify in a model of Δ_i.

What remains

We list the remaining sentences.

$$\textit{All } Y_j \textit{ r all } Y_{j+1}$$
$$\textit{All } Y_1 \textit{ r some } Y_n$$

We can also falsify each of these in a model of Δ_i.

What to do?

We should not be deterred by the negative result:
there are two ways to go:

1. Move from a pure syllogistic system to a more liberal type of logic. (We have already done this by adding RAA.)
2. Use infinitely many rules.
(I believe this is possible, but it is tedious.)

Relational syllogistic logic

p and q range over unary atoms,
c over set terms, and t over binary atoms or their negations.

$$\frac{\exists(p,q) \qquad \forall(q,c)}{\exists(p,c)} \qquad \frac{\forall(p,q) \qquad \forall(q,c)}{\forall(p,c)}$$

$$\frac{\forall(p,q) \qquad \exists(p,c)}{\exists(q,c)} \qquad \frac{}{\forall(p,p)} \qquad \frac{\exists(p,c)}{\exists(p,p)}$$

$$\frac{\forall(q,\bar{c}) \qquad \exists(p,c)}{\exists(p,\bar{q})} \qquad \frac{\forall(p,\bar{p})}{\forall(p,c)} \qquad \frac{\exists(p,\exists(q,t))}{\exists(q,q)}$$

$$\frac{\forall(p,\forall(n,t)) \qquad \exists(q,n)}{\forall(p,\exists(q,t))} \qquad \frac{\exists(p,\exists(q,t)) \qquad \forall(q,n)}{\exists(p,\exists(n,t))}$$

$$\frac{\forall(p,\exists(q,t)) \qquad \forall(q,n)}{\forall(p,\exists(n,t))} \qquad \begin{array}{c}[\varphi] \\ \vdots \\ \dfrac{\exists(p,\overline{p})}{\overline{\varphi}}\ \text{RAA}\end{array}$$

Completeness

We shall show that every set Γ of sentences which is consistent in R is satisfiable.

By what we saw earlier,
we may assume that Γ is $\mathcal{R}$-complete:
for every sentence θ of $\mathcal{R}$, either θ or $\overline{\theta}$ belongs to Γ.

The architecture of the completeness proof

Start with a consistent and complete Γ, and build a model $\mathcal{M}$ such that for positive sentences φ,

$$\mathcal{M} \models \varphi \quad \text{iff} \quad \Gamma \vdash \varphi$$

positive:

All p are q	$\forall(p, q)$
Some p are q	$\exists(p, q)$
All p r all q	$\forall(p, \forall(q, r))$
All p r some q	$\forall(p, \exists(q, r))$
Some p r all q	$\exists(p, \forall(q, r))$
Some p r some q	$\exists(p, \exists(q, r))$

negative:

No p are q	$\forall(p, \overline{q})$
Some p aren't q	$\exists(p, \overline{q})$
All p don't r all q $\equiv$	
No p r any q	$\forall(p, \forall(q, \overline{r}))$
All p don't r some q $\equiv$	
No p r all q	$\forall(p, \exists(q, \overline{r}))$
Some p don't r any q	$\exists(p, \forall(q, \overline{r}))$
Some p don't r some q	$\exists(p, \exists(q, \overline{r}))$

The architecture of the completeness proof

Start with a consistent and complete Γ, and build a model $\mathcal{M}$ such that for positive sentences φ,

$$\mathcal{M} \models \varphi \quad \text{iff} \quad \Gamma \vdash \varphi$$

From this, we show that if φ is negative and $\varphi \in \Gamma$, we again have $\mathcal{M} \models \varphi$.

For if not, then $\overline{\varphi}$ is positive and $\mathcal{M} \models \overline{\varphi}$.
So $\Gamma \vdash \overline{\varphi}$.
And now Γ is inconsistent.

Model construction

For such a consistent and R-complete set Γ, we shall define a model $\mathcal{M} = \mathcal{M}(\Gamma)$ as follows: we let

$$M = \{x_1, x_2 : \Gamma \vdash \exists(x, x)\} \cup \{\{p, q\} : p \neq q \text{ and } \Gamma \vdash \exists(p, q)\}.$$

We assume this union is disjoint, and we call the elements $\{p, q\}$ pair-elements.

So we have two copies of every variable x such that Γ entails the existence of x,

and also pair elements $\{p, q\}$ corresponding to sentences of the form $\exists(p, q)$ which are provable from Γ and such that $p \neq q$.

Our semantics will insure that the pair-element $\{p, q\}$ belongs to $[\![p]\!] \cap [\![q]\!]$,

and so this element will witness the truth of $\exists(p, q)$ in the model which we are constructing.

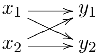
Model construction

The unary atoms x are interpreted in our models as follows:

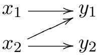
$$\begin{array}{ll} w_i \in [\![x]\!] & \text{iff } \Gamma \vdash \forall(w, x) \\ \{p, q\} \in [\![x]\!] & \text{iff } \Gamma \vdash \forall(p, x), \text{ or } \Gamma \vdash \forall(q, x) \end{array}$$

For the binary atom r, we need a lot more work.
First, suppose M contains x_1, x_2, y_1, and y_2.

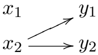
The picture of $[\![r]\!] \cap (\{x_1, x_2\} \times \{y_1, y_2\})$



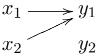
$\forall(x, \forall(y, r))$



$\times\forall(x, \forall(y, r))$
$\forall(x, \exists(y, r))$
$\exists(x, \forall(y, r))$

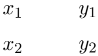


$\times\forall(x, \forall(y, r))$
$\times\forall(x, \exists(y, r))$
$\exists(x, \forall(y, r))$



$\times\forall(x, \forall(y, r))$
$\forall(x, \exists(y, r))$
$\times\exists(x, \forall(y, r))$

$\times\forall(x, \exists(y, r))$
$\times\exists(x, \forall(y, r))$
$\exists(x, \exists(y, r))$



$\times\exists(x, \exists(y, r))$

It depends on which sentences are provable from Γ.

The model in full

$x_i[\![r]\!]y_j$	iff	$x_i \rightarrow y_j$ according to the pictures from before
$\{x, y\}[\![r]\!]w_2$	iff	$\Gamma \vdash \forall(x, \forall(w, r))$ or $\Gamma \vdash \forall(x, \forall(y, r))$
$\{x, y\}[\![r]\!]w_1$	iff	$\{x, y\}[\![r]\!]w_2$, or $\Gamma \vdash \forall(x, \exists(w, r))$, or $\Gamma \vdash \forall(y, \exists(w, r))$
$u_1[\![r]\!]\{x, y\}$	iff	$\Gamma \vdash \forall(u, \forall(x, r))$ or $\Gamma \vdash \forall(u, \forall(y, r))$
$u_2[\![r]\!]\{x, y\}$	iff	$u_1[\![r]\!]\{x, y\}$, or $\Gamma \vdash \exists(u, \forall(x, r))$, or $\Gamma \vdash \exists(u, \forall(y, r))$
$\{x, y\}[\![r]\!]\{p, q\}$	iff	$\Gamma \vdash \forall(x, \forall(p, r))$, or $\Gamma \vdash \forall(x, \forall(q, r))$, or $\Gamma \vdash \forall(y, \forall(p, r))$, or $\Gamma \vdash \forall(y, \forall(q, r))$

Facts

Concerning the relation $[\![r]\!]$:

1. $x_1[\![r]\!]y_2$ iff $\Gamma \vdash \forall(x, \forall(y, r))$.
2. $x_1[\![r]\!]y_1$ iff $\Gamma \vdash \forall(x, \forall(y, r))$ or $\Gamma \vdash \forall(x, \exists(y, r))$.
3. $x_2[\![r]\!]y_2$ iff $\Gamma \vdash \forall(x, \forall(y, r))$ or $\Gamma \vdash \exists(x, \forall(y, r))$.
4. $x_2[\![r]\!]y_1$ iff for some i and j, $x_i[\![r]\!]y_j$.

The easy part

Let Γ be an arbitrary set of $\mathcal{R}$-sentences.
Let φ be a positive sentence. If $\Gamma \vdash \varphi$, then $\mathcal{M}(\Gamma) \models \varphi$.
We argue by cases on φ.

Case 1: φ is $\forall(x, y)$. If $z_i \in [\![x]\!]$, then $z \leq x$. By monotonicity, $z \leq y$. So $z_i \in [\![y]\!]$. If $\{p, q\} \in [\![x]\!]$, then without loss of generality $\forall(p, x)$. Again, we see that $\{p, q\} \in [\![y]\!]$.

Case 2: φ is $\exists(x, y)$. This time $\{x, y\}$ is an element of our model. Our logic contains the identity axioms All x are x. By our semantics, $\{x, y\} \in [\![x]\!] \cap [\![y]\!]$. Thus the model overall satisfies Some x are y.

Case 3: φ is $\forall(x, \forall(y, r))$. Let $z_i \in [\![x]\!]$ and $w_j \in [\![y]\!]$; so we have $z \leq x$ and $w \leq y$.
By monotonicity, $\Gamma \vdash \forall(z, \forall(w, r))$.
So $z_i [\![r]\!] w_j$ for $1 \leq i, j \leq 2$.
We also must consider pair-elements $\{p, q\} \in [\![x]\!]$. Without loss of

MORE

Case 4: φ is $\forall(x, \exists(y, r))$. In this case, we can assume that $[\![x]\!] \neq \emptyset$. That is, $\Gamma \vdash \exists(x, x)$. Then $\Gamma \vdash \exists(y, y)$ as well. We shall show that every element of $[\![x]\!]$ is related to y_1. Let $z_i \in [\![x]\!]$, so that $z \leq x$. By monotonicity, $\Gamma \vdash \forall(z, \exists(y, r))$.
Then by our Facts, parts (2) and (4), we indeed have $z_i[\![r]\!]y_1$ and $z_i[\![r]\!]y_2$. Further, let $\{p, q\} \in [\![x]\!]$. Without loss of generality, $p \leq x$. By monotonicity, $\Gamma \vdash \forall(p, \exists(y, r))$. Then by the definition of $[\![r]\!]$, $\{p, q\}[\![r]\!]y_1$.

Case 5: φ is $\exists(x, \forall(y, r))$. By rule (I) of our logic, $\Gamma \vdash \exists(x, x)$. Let $w_j \in [\![y]\!]$. Then $\Gamma \vdash \forall(w, y)$. Hence $\Gamma \vdash \exists(x, \forall(w, r))$.
By our Facts, parts (3) and (4), $x_2[\![r]\!]w_1$, and also $x_2[\![r]\!]w_2$. We must also consider pair-elements of $[\![y]\!]$. Let $\{p, q\} \in [\![y]\!]$ so that $\Gamma \vdash \exists(p, q)$; and assume $\Gamma \vdash \forall(p, y)$. By monotonicity, $\Gamma \vdash \exists(x, \forall(p, r))$. By construction, $x_2[\![r]\!]\{p, q\}$. We conclude that x_2 is the required witness to $\exists(x, \forall(y, r))$.

Case 6: φ is $\exists(x, \exists(y, r))$. Here both $\Gamma \vdash \exists(x, x)$ and also $\exists(y, y)$. By Proposition 39, part (4), $x_2[\![r]\!]y_1$. So $\mathcal{M} \models \exists(x, \exists(y, r))$.

The hard part

Let Γ be complete and consistent. Let φ be a positive sentence. If $\mathcal{M}(\Gamma) \models \varphi$, then $\varphi \in \Gamma$.

We argue by cases on φ. In each case, we assume that $\mathcal{M}(\Gamma) \models \varphi$, and we then show $\Gamma \vdash \varphi$. Since Γ is complete, we indeed have $\varphi \in \Gamma$.

One fact which we shall use frequently is that if $[\![x]\!] \neq \emptyset$ in $\mathcal{M}(\Gamma)$, then $\Gamma \vdash \exists(x, x)$. For if $y_j \in [\![x]\!]$, they by the structure of the model, $\Gamma \vdash \exists(y, y)$ and also $\forall(y, x)$. Similar considerations apply to a pair-element $\{u, w\} \in [\![x]\!]$.

Case 1: φ is $\forall(x, y)$. We may assume that $\Gamma \vdash \exists(x, x)$; if not, then $\Gamma \vdash \varphi$ using (A). And then the structure of the model easily tells us that $\Gamma \vdash \varphi$.

Case 2: φ is $\exists(x, y)$. The argument is very close to what we do concerning $\exists(x, \exists(y, r))$ in Case 6 below.

The hard part, continued

Case 3: φ is $\forall(x, \forall(y, r))$.
By completeness, either $\Gamma \vdash \forall(x, \overline{x})$; or $\Gamma \vdash \forall(y, \overline{y})$; or else both $\Gamma \vdash \exists(x, x)$ and $\Gamma \vdash \exists(y, y)$.
In the first case, $\Gamma \vdash \varphi$ using the rule (A).

In the second case, we show easily that $\Gamma \vdash \varphi$.

In the last case, consider $\mathcal{M} = \mathcal{M}(\Gamma)$.
By the lemma which we have already seen, $\mathcal{M}(\Gamma) \models \varphi$.
In $\mathcal{M}$, $x_1 \in [\![x]\!]$ and $y_2 \in [\![y]\!]$.
Since $\mathcal{M} \models \varphi$, $x_1 [\![r]\!] y_2$.
By inspection of the model, $\Gamma \vdash \forall(x, \forall(y, r))$.

More

Case 4: φ is $\forall(x, \exists(y, r))$. As in the previous case, we may assume that $\Gamma \vdash \exists(x, x)$. Consider $\mathcal{M} = \mathcal{M}(\Gamma)$. In the model, $[\![x]\!] \neq \emptyset$ by definition of the model, and because $\mathcal{M} \models \varphi$, the same is true of y. In particular, x_1 is related to some element of $[\![y]\!]$. Say $x_1[\![r]\!]z_j$ where $z \leq y$. If $j = 1$, then by Proposition 39, part (2), we $\Gamma \vdash \forall(x, \exists(z, r))$ or $\Gamma \vdash \forall(x, \forall(z, r))$. In the first case, we are done by monotonicity. So we shall assume that $\Gamma \vdash \forall(x, \forall(z, r))$. Since z_j belongs to the model $\Gamma \vdash \exists(z, z)$. Therefore $\Gamma \vdash \forall(x, \exists(z, r))$, and as above we are done.

Now if $j = 2$, then by Proposition 39, part (1), we have $\Gamma \vdash \forall(x, \forall(z, r))$. Exactly as above, we reason that $\Gamma \vdash \forall(x, \exists(y, r))$.

The last possibility is that $x_1[\![r]\!]\{p, q\}$, where $\Gamma \vdash \exists(p, q)$. Without loss of generality, suppose that $\Gamma \vdash \forall(x, \forall(q, r))$ and also that $\Gamma \vdash \forall(p, y)$. The derivation in Example **??** shows that $\Gamma \vdash \forall(x, \exists(y, r))$.

Yet more

Suppose next that it is $x_2 \in [\![x]\!]$ which is related by $[\![r]\!]$ to y_2.
We have two alternatives: $\Gamma \vdash \exists(x, \forall(y, r))$ (and we are done);
or else $\Gamma \vdash \forall(x, \forall(y, r))$.
In this last case, we have already seen $\Gamma \vdash \exists(x, x)$, and we now have $\Gamma \vdash \exists(x, \forall(y, r))$.
Finally, suppose that $\{p, q\} \in [\![x]\!]$ and also that $\{p, q\}[\![r]\!]y_2$.

Either $\Gamma \vdash \forall(p, \forall(y, r))$ or $\Gamma \vdash \forall(q, \forall(y, r))$.
Since this pair-element $\{p, q\}$ belongs to our model, $\Gamma \vdash \exists(p, q)$.
So either $\Gamma \vdash \exists(p, \forall(y, r))$ or $\Gamma \vdash \exists(q, \forall(y, r))$.
But also, either $p \leq x$ or $q \leq x$.
Without loss of generality, $p \leq x$. By monotonicity,
$\Gamma \vdash \exists(x, \forall(y, r))$.

CONCLUDING THE PROOF

Case 6: φ is $\exists(x, \exists(y, r))$. In our final case, we must have $\Gamma \vdash \exists(x, x)$; also $\Gamma \vdash \exists(y, y)$.

Suppose that $z_1 \in [\![x]\!]$ and $w_1 \in [\![y]\!]$ are related by $[\![r]\!]$.
Thus $z \leq x$ and $w \leq y$. By examining the model,
$\Gamma \vdash \forall(z, \exists(w, r))$.
Since z_1 belongs to our model, $\Gamma \vdash \exists(z, z)$.
Thus $\Gamma \vdash \exists(z, \exists(w, r))$.
And by monotonicity again, $\Gamma \vdash \exists(x, \exists(y, r))$.

Next, suppose that $z_1 \in [\![x]\!]$ and $w_2 \in [\![y]\!]$ are related by $[\![r]\!]$.
The work here is quite similar, and we omit all the details.
The same goes for the case of $z_2 \in [\![x]\!]$ and $w_1 \in [\![y]\!]$, and also for the case of $z_2 \in [\![x]\!]$ and $w_2 \in [\![y]\!]$.
There are several more cases, owing to the possibility that the witnesses to $\exists(x, \exists(y, r))$ might include pair-elements.
These are all routine, and we omit these details.

This concludes the proof.

Completeness of the logical system
For $\Gamma \cup \{\varphi\} \subseteq \mathcal{R}$, $\Gamma \models \varphi$ iff $\Gamma \vdash \varphi$ in R.

The soundness is an easy induction on derivations.
For the completeness, we need only show that a consistent set Γ is satisfiable.
We may assume that Γ is $\mathcal{R}$-complete.
Consider $\mathcal{M} = \mathcal{M}(\Gamma)$.
By what we know $\mathcal{M}$ satisfies the positive sentences in Γ.

We claim that $\mathcal{M}$ satisfies the negative sentences in Γ as well.
For suppose that ψ is positive and $\overline{\psi}$ belongs to Γ.
If $\mathcal{M} \not\models \overline{\psi}$, we would have $\mathcal{M} \models \psi$.
Also by what we know, $\Gamma \vdash \psi$. But then Γ is inconsistent, a contradiction.

The claim shown, we now see that $\mathcal{M} \models \Gamma$.

Incorporating background information

Suppose we have a stock of background facts about verbs, such as

kissing entails touching

This background fact cannot be stated in any of the languages which we have so far studied.
Nevertheless, it can be made into a semantic requirement:
we would require of a model that

$$[\![\text{kiss}]\!] \subseteq [\![\text{touch}]\!].$$

Even though we cannot state our background fact as an axiom, it does yield rules of inference.

INCORPORATING BACKGROUND INFORMATION

More abstractly, suppose we have a rule like

$$r \Rightarrow s \tag{1}$$

and we restrict attention to the models where $[\![r]\!] \subseteq [\![s]\!]$.
We get rules like

$$\frac{\forall(d,\forall(c,r))}{\forall(d,\forall(c,s))} \qquad \frac{\forall(d,\exists(c,r))}{\forall(d,\exists(c,s))} \qquad \frac{\exists(d,\forall(c,r))}{\exists(d,\forall(c,s))} \qquad \frac{\exists(d,\exists(c,r))}{\exists(d,\exists(c,s))}$$

We add these to the system $\mathcal{R}$.

PROPOSITION

The system R together with the rules of $\Rightarrow$ a sound and complete logic: $\Gamma \vdash \varphi$ iff $\Gamma \models \varphi$.

$\mathcal{R}^\dagger$

expression	variables	syntax
unary atom	p, q	
binary atom	r	
set term	c, d	$p \mid \exists(p,r) \mid \forall(p,r) \mid$ $\overline{p} \mid \exists(p,\overline{r}) \mid \forall(p,\overline{r})$
$\mathcal{R}$ sentence	φ	$\forall(p,c) \mid \exists(p,c)$
$\mathcal{R}^\dagger$ sentence	φ	$\forall(p,c) \mid \exists(p,c) \mid \forall(\overline{p},c) \mid \exists(\overline{p},c)$

A stronger negative result on the larger language $\mathcal{R}^\dagger$

Theorem

There are no finite, purely syllogistic logical systems which are sound and complete for $\mathcal{R}$.

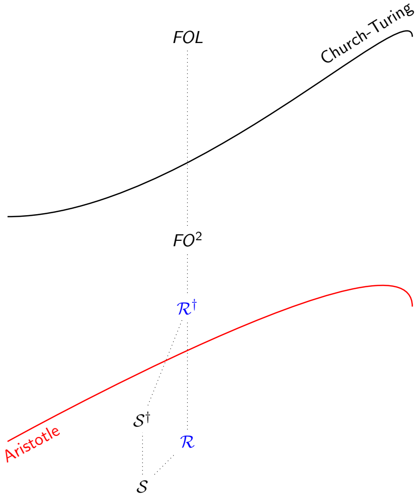
As we now know, there is such a system using RAA.

Theorem

There are no finite, sound and complete syllogistic logical systems for $\mathcal{R}^\dagger$, even allowing *RAA*.

(But see our next lecture set for a rejoinder.)

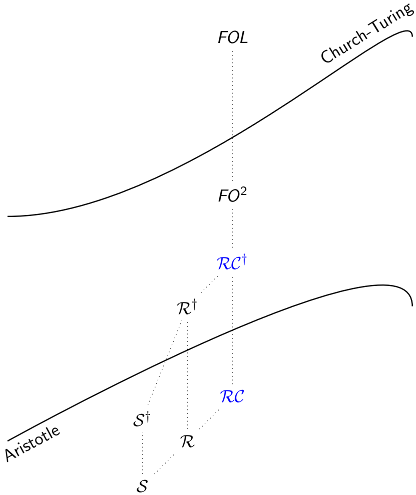
The Aristotle Boundary



† adds full *N*-negation

relational syllogistic

Next: relative clauses



† adds full *N*-negation

add relative clauses
= relativized quantifiers

Inference with relative clauses

What do you think about this one?

$$\frac{\textit{All skunks are mammals}}{\textit{All who fear all who respect all skunks fear all who respect all mammals}}$$

Inference with relative clauses

It follows, using an interesting antitonicity principle:

$$\frac{\textit{All skunks are mammals}}{\textit{All who respect all mammals respect all skunks}}$$

INFERENCE WITH RELATIVE CLAUSES

It follows, using an interesting antitonicity principle:

$$\dfrac{\dfrac{\textit{All skunks are mammals}}{\textit{All who respect all mammals respect all skunks}}}{\textit{All who fear all who respect all skunks fear all who respect all mammals}}$$

$\mathcal{RC}$ and $\mathcal{RC}^\dagger$

$\mathcal{RC}$ allows sentential subjects to be noun phrases containing subject relative clauses.

who r all p	*who r some p*
who don't r all p	*who don't r any p*

---

expression	syntax	
$\mathcal{RC}$ sentence	$\forall(d^+, c)$	$\exists(d^+, c)$
$\mathcal{RC}^\dagger$ sentence	$\forall(d, c)$	$\exists(d, c)$

d^+ is a positive set term, and c is an arbitrary set term.

The logical system RC

$$\frac{}{\forall(c^+, c^+)}\ (\mathrm{T}) \qquad \frac{\exists(c^+, d)}{\exists(c^+, c^+)}\ (\mathrm{I}) \qquad \frac{\forall(b^+, c^+) \qquad \forall(c^+, d)}{\forall(b^+, d)}\ (\mathrm{B})$$

$$\frac{\exists(b^+, c^+) \qquad \forall(c^+, d)}{\exists(b^+, d)}\ (\mathrm{D1}) \qquad \frac{\forall(b^+, c^+) \qquad \exists(b^+, d)}{\exists(c^+, d)}\ (\mathrm{D2})$$

$$\frac{\forall(p, q)}{\forall(\forall(q, r), \forall(p, r))}\ (\mathrm{J}) \quad \frac{\forall(p, q)}{\forall(\exists(p, r), \exists(q, r))}\ (\mathrm{K}) \quad \frac{\exists(p, q)}{\forall(\forall(p, r), \exists(q, r))}\ (\mathrm{L})$$

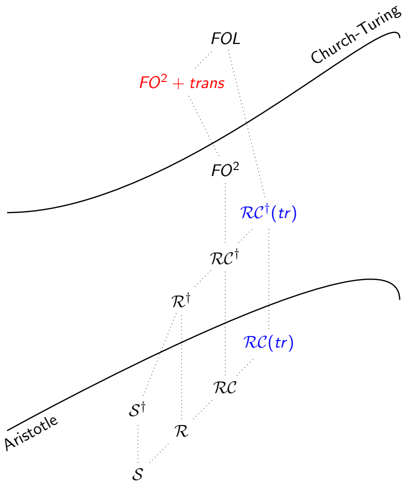
$$\frac{\exists(q, \exists(p, r))}{\exists(p, p)}\ (\mathrm{II}) \qquad \frac{\forall(p, \bar{p})}{\forall(c^+, \forall(p, r))}\ (\mathrm{Z})$$

Return of the skunks
Iterated relative clauses

$$\dfrac{\dfrac{\forall(s, m)}{\forall(\forall(m, r), \forall(s, r))}}{\forall(\forall(\forall(s, r), f), \forall(\forall(m, r), f))}$$

Next: comparative adjectives

used for inferences involving phrases like bigger than some kitten



!!

† adds full *N*-negation

tr adds comparatives, requiring transitivity

Comparative adjectives: $\mathcal{R}(tr)$

Every giraffe is taller than every gnu
Some gnu is taller than every lion
Some lion is taller than some zebra
Every giraffe is taller than some zebra

We extend $\mathcal{R}$ to a language $\mathcal{R}(tr)$ by taking a set **A** of comparative adjective phrases in the base.

In the semantics, we would require that for $a \in \mathbf{A}$, $[\![a]\!]$ must be a transitive relation (in every model $\mathcal{M}$):

$$\text{if } [\![a]\!](x, y) \text{ and } [\![a]\!](y, z), \text{ then } [\![a]\!](x, z).$$

The logical system R(tr) for $\mathcal{R}(tr)$

$$\frac{\forall(x,\forall(y,r)) \quad \exists(y,\forall(z,r))}{\forall(x,\forall(z,r))} \qquad \frac{\forall(x,\forall(y,r)) \quad \exists(y,\exists(z,r))}{\forall(x,\exists(z,r))}$$

$$\frac{\forall(x,\exists(y,r)) \quad \forall(y,\exists(z,r))}{\forall(x,\exists(z,r))} \qquad \frac{\forall(x,\exists(y,r)) \quad \forall(y,\forall(z,r))}{\forall(x,\forall(z,r))}$$

$$\frac{\exists(x,\forall(y,r)) \quad \exists(y,\forall(z,r))}{\exists(x,\forall(z,r))} \qquad \frac{\exists(x,\forall(y,r)) \quad \exists(y,\exists(z,r))}{\exists(x,\exists(z,r))}$$

$$\frac{\exists(x,\exists(y,r)) \quad \forall(y,\forall(z,r))}{\exists(x,\forall(z,r))} \qquad \frac{\exists(x,\exists(y,r)) \quad \forall(y,\exists(z,r))}{\exists(x,\exists(z,r))}$$

R(*tr*): an example

Every giraffe is taller than every gnu
Some gnu is taller than every lion
Some lion is taller than some zebra
Every giraffe is taller than some zebra

$$\dfrac{\dfrac{\forall(\text{giraffe}, \forall(\text{gnu}, \text{taller})) \quad \exists(\text{gnu}, \forall(\text{lion}, \text{taller}))}{\forall(\text{giraffe}, \forall(\text{lion}, \text{taller}))} \quad \exists(\text{lion}, \exists(\text{zebra}, \text{taller}))}{\forall(\text{giraffe}, \exists(\text{zebra}, \text{taller}))}$$

A point on the completeness

The proof turns out to be very similar to the completeness proof for the logical system for R.

Indeed, we only have to check that when we work in the bigger system R(tr),
the model that we built earlier is transitive.

And luckily, this is true!

Irreflexivity and finiteness

The additional requirement also results in the soundness of the following irreflexivity rule

$$\frac{}{\forall(p, \exists(p, \overline{r}))}\ \text{(Irr)}$$

As weak as this looks, adding it gives a complete system.

We also have a finiteness rule:

$$\frac{\exists(p, p)}{\exists(p, \forall(p, \overline{r}))}\ \text{(Fin)}$$

On top of irreflexivity and transitivity, this gives a complete system.

Comparative adjectives: $\mathcal{RC}(tr)$

Recall that $\mathcal{RC}$ has subject relative clauses.

We again wish to interpret this on structures with transitive relations interpreting adjectives.

In the logic, we add a few rules to the system RC for $\mathcal{RC}$:

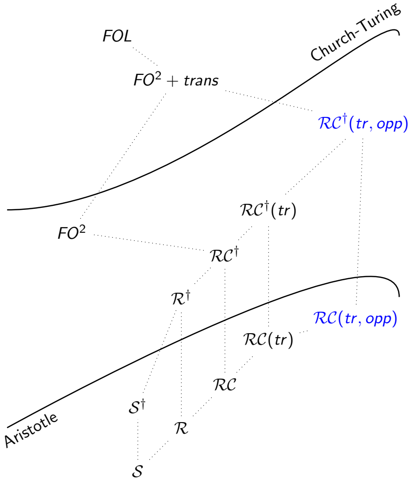
$$\frac{\forall(p, \exists(q, r))}{\forall(\exists(p, r), \exists(q, r))} \qquad \frac{\forall(p, \forall(q, r))}{\forall(\exists(p, r), \forall(q, r))}$$

$$\frac{\exists(p, \forall(q, r))}{\forall(\forall(p, r), \forall(q, r))} \qquad \frac{\exists(p, \exists(q, r))}{\forall(\forall(p, r), \exists(q, r))}$$

This system derives R(tr).

Next: relational converses

used for inferences relating bigger and smaller



† adds full *N*-negation
* adds relative clauses

opp adds opposites
of comparative adjectives

Converses of transitive relations

On top of all the other syllogistic systems we have seen

$$\frac{\forall(p, \forall(q, t))}{\forall(q, \forall(p, t^{-1}))} \qquad \frac{\exists(p, \forall(q, t))}{\forall(q, \exists(p, t^{-1}))}\ (\text{scope}) \qquad \frac{\forall(p, \exists(q, r^{-1}))}{\forall(\forall(q, r), \forall(p, r))}$$

$$\frac{\exists(\exists(p, r^{-1}), \exists(q, r))}{\exists(p, \exists(q, r))} \qquad \frac{\exists(\forall(p, r), \forall(q, r^{-1}))}{\forall(p, \forall(q, r^{-1}))} \qquad \frac{\exists(\forall(p, r), \exists(q, r^{-1}))}{\exists(q, \forall(p, r^{-1}))}$$

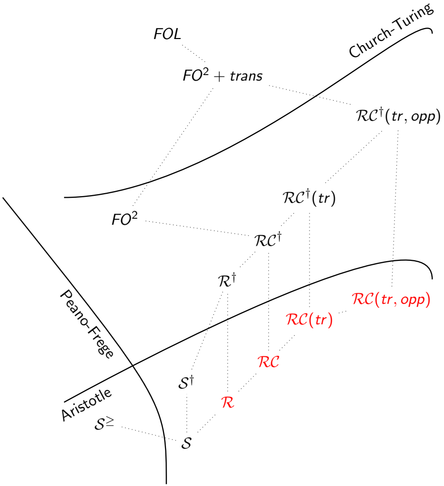
$$\frac{\forall(p, \exists(q, r)) \quad \forall(\exists(p, r^{-1}), \exists(n, r))}{\forall(p, \exists(n, r))}\ (\star) \qquad \frac{\forall(p, \exists(q, r)) \quad \forall(\exists(p, r^{-1}), \forall(n, r))}{\forall(p, \forall(n, r))}$$

(scope): if some p is bigger than all q,
then all q are smaller than some p or other.

($\star$): if every dog is bigger than some hedgehog,
and everything smaller than some dog is bigger than some cat,
then every dog is bigger than some cat.

Where we are

We covered these in this lecture.



first-order logic

FO^2 + "R is trans"

2 variable FO logic

† adds full N-negation

$\mathcal{RC}(tr)$ + opposites

$\mathcal{RC}$ + (transitive) comparative adjs

$\mathcal{R}$ + relative clauses

$\mathcal{S}$ + full N-negation

$\mathcal{R}$ = relational syllogistic

$\mathcal{S}^\geq$ adds $|p| \geq |q|$

$\mathcal{S}$: all/some/no p are q