Logics with Verbs, Relative Clauses, and Comparative Adjectives

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Where we are

Aristotle

Turing

\( FOL \)

\( FO^2 + \text{trans} \)

\( FO^2 \)

\( FOL \)

\( \mathcal{R} \)

\( \mathcal{S} \)

\( \mathcal{C} \)

\( \mathcal{C}^{\dagger}(tr) \)

\( \mathcal{C}^{\dagger}(tr, opp) \)

\( \mathcal{R}^{\dagger} \)

\( \mathcal{R}^{\dagger}\text{trans}, \text{opp} \)

\( \mathcal{R}^{\dagger}\text{trans} \)

\( \mathcal{R}^{\dagger}\text{opp} \)

\[ \mathcal{R}^{\ast} + \text{(transitive)} \]

comparative adjs

done

\[ \mathcal{R} + \text{relative clauses} \]

relational syllogistic

done

all these are today

done
I am going to introduce this topic in a slow way by cutting things down to a very small language and working up.

(This is my favorite thing to do for this subject!)

Working with the small language enables us to do a lot of the proofs, and to “get our hands dirty”.

After we do this, I will go more quickly and omit a huge number of details.
Does the conclusion follow?

All boys are males
All women see all males
All who see all boys see all males

The definition of “follow” here is in terms of models.

For this fragment, a model $\mathcal{M}$ should be

- a set $M$, the “universe”
- subsets of $M$ for the nouns: $\{\text{boys}\}$, $\{\text{males}\}$, and $\{\text{females}\}$
- a subset of $M \times M$ for the verb: $\{\text{see}\}$

In other words, it is a model like we had before, but with a set of pairs of individuals to interpret the verb.
We make set terms in the following way:

- \textit{see} is a set term.
- If \( x \) is a set term, so is \((\textit{see all } x)\).

We can read this in various ways to make it sound more like English.
Sometimes it would be as “see all who are \( x \)”.

So we get set terms like

- see all boys
- see all (see all women)
- see all (see all (see all women))
We make **set terms** in the following way:

- see is a set term.
- If $x$ is a set term, so is (see all $x$).

We can read this in various ways to make it sound more like English.
Sometimes it would be as “see all who are $x$.”

So we get set terms like

- see all boys
- see all (see all women)
- see all (see all (see all women))

Later on, we’ll call set terms **set terms**.
For our sentences, we take all expressions

$$\text{All } x \ y$$

where $x$ and $y$ are set terms.

In the semantics, we need an \textit{inductive definition} to interpret the set terms.

$$\left[ \text{see all } x \right] = \{ m \in M : \_ \_ \_ \_ \_ \_ \_ \}$$
For our sentences, we take all expressions

\[ \text{All } x \ y \]

where \( x \) and \( y \) are set terms.

In the semantics, we need an inductive definition to interpret the set terms.

\[
\begin{align*}
\left[ \text{see all } x \right] & = \{ m \in M : \text{________} \} \\
\left[ \text{see all } x \right] & = \{ m \in M : \text{for all } n \in \left[ x \right], (m, n) \in \left[ \text{see} \right] \}
\end{align*}
\]
Example of a model

\[
[males] = \{a, d, f\}
\]
\[
[females] = \{b, c, e\}
\]
\[
[boys] = \{a, d\}
\]

Technically, \( M = \{a, \ldots, f\}, \)
\[
[see] = \{(a, b), (a, c), (a, e), (c, d), (b, a), (b, b), (b, f), (c, f), (d, b), (d, d), (d, e), (d, f)\}\]
\[ \text{[males]} = \{a, d, f\} \]
\[ \text{[females]} = \{b, c, e\} \]
\[ \text{[boys]} = \{a, d\} \]

What is \text{[see all males]}?  
What is \text{[see all females]}?  
What is \text{[see all (see all females)]}?  
What is \text{[see all (see all (see all females))]}?
For sure we need the logic of *All* from earlier:

\[
\begin{align*}
\text{All } x & \text{ are } x \\
\text{All } x & \text{ are } y \quad \text{All } y & \text{ are } z \\
\hline
\text{All } x & \text{ are } z
\end{align*}
\]

*Barbara*

But note that we are using this with $x$, $y$, and $z$ as *set terms*.

But in fact we are missing at least one rule!
All x are x  \textit{axiom}

All y x

All (see all x)(see all y)  \textit{skunk}

All x are y  All y are z  \textit{Barbara}
**Logic**

\[ \text{All } x \text{ are } x \text{ axiom} \]

\[ \text{All } y \text{ x} \]

\[ \text{All } (\text{see all } x)(\text{see all } y) \text{ skunk} \]

**Example:** \( \text{All } x \text{ (see all } y), \text{ All } z \text{ y} \vdash \text{All } x \text{ (see all } z) \)

\[ \text{All } z \text{ y} \]

\[ \text{All } x \text{ (see all } y) \]

\[ \text{All } (\text{see all } y)(\text{see all } z) \]

\[ \text{All } x \text{ (see all } z) \]
Does the conclusion follow?

All boys are males
All who see all women also see all who see all boys
All who see all see all boys – they are all women
All women see all males
All who see all boys see all males

In our language, this is

All boys males
All (see all women) (see all (see all boys))
All (see all (see all boys)) women
All women (see all males)
All (see all boys) (see all males)
The canonical model \( \Gamma \) of a set of assertions in this logic

We are given a set \( \Gamma \) in this language. We aim to build a model \( M \) of \( \Gamma \) with the property that

\[ \text{if } M \models \varphi, \text{ then } \Gamma \vdash \varphi. \]

This would prove the completeness theorem for our logic, because if \( \Gamma \not\vdash \varphi \), then \( M \not\models \varphi \).

(In other words, every sentence which is true in all models of \( \Gamma \) (hence in \( M \)) would be provable from \( \Gamma \).)
The canonical model $\Gamma$ of a set of assertions in this logic

Let $\mathcal{M}$ be the set of all set terms.

Let

$$
\begin{align*}
\llbracket \text{females} \rrbracket &= \{ x : \Gamma \vdash \text{All} x \times \text{females} \} \\
\llbracket \text{males} \rrbracket &= \{ x : \Gamma \vdash \text{All} x \times \text{males} \} \\
\llbracket \text{boys} \rrbracket &= \{ x : \Gamma \vdash \text{All} x \times \text{boys} \}
\end{align*}
$$

The hardest point is to decide on the semantics of see
Let $\mathcal{M}$ be the set of all set terms.

Let

\[
\begin{align*}
\llbracket \text{females} \rrbracket & = \{ x : \Gamma \vdash \text{All } x \text{ females} \} \\
\llbracket \text{males} \rrbracket & = \{ x : \Gamma \vdash \text{All } x \text{ males} \} \\
\llbracket \text{boys} \rrbracket & = \{ x : \Gamma \vdash \text{All } x \text{ boys} \}
\end{align*}
\]

The hardest point is to decide on the semantics of see

\[
\llbracket \text{see} \rrbracket = \{ (x, y) : \Gamma \vdash \text{All } x \text{ see all } y \}\]
Does the conclusion follow?

All $p$ are $q$

\[ \text{All who see all who see all } p \text{ (see all } q) \]

The way we'll do this is to draw the canonical model of $\Gamma$, where

\[ \Gamma = \{ \text{All } p \text{ are } q \} . \]

To make the notation simpler,
let us write $p_0$ for $p$, and $p_{n+1}$ for see all $p_n$;
we adopt similar notation for $q$.
Then the canonical model of $\Gamma$ is shown below:

\[
\begin{array}{cccccc}
  p_0 & \leftarrow & p_1 & \leftarrow & p_2 & \leftarrow & p_3 & \leftarrow & p_4 & \cdots \\
  \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
  q_0 & \leftarrow & q_1 & \leftarrow & q_2 & \leftarrow & q_3 & \leftarrow & q_4 & \cdots \\
\end{array}
\]

Notice that the notation for subset arrows is different than the arrows for see.
Notice also that we have omitted the reflexive subset arrows on all of the nodes,
and also that the arrows which are shown alternate directions.
**Lemma:** \([x] = \{y : \Gamma \vdash \text{All } y \text{ are } x\}\)

The proof is by induction on \(x\).

**Lemma:** \(\mathcal{M} \models \Gamma\)

This follows from the last lemma.

**Lemma:** If \(\mathcal{M} \models \varphi\), then \(\Gamma \vdash \varphi\).

This is basically what we saw yesterday for the language of \(\text{All}\).
Solving the problem

Does this follow?

All boys males
All (see all women) (see all (see all boys))
All (see all (see all boys)) women
All women (see all males)
All (see all boys) (see all males)

We make the part of the canonical model just using the set terms above, and all the subterms:

boys  see all boys
men   see all men
women see all women
       see all see all women
see all boys

subset

see all see all boys

subset

see all males

subset

see all females

subset

males

subset

females

More

16/48
The logic gives us part of the “see” relation in a trivial way.

```
boys

see all boys

see all see all boys

see all males

see all females

males

females
```
What would we do if we had more than one verb?

- All women are football players
- All football fans love all football players
- All football fans like all women

Does this work?
In addition to our assumptions $\Gamma$, we can adopt a set $\Delta$ of **background assumptions** which need not be expressible in our language.

In this case, we might say

$$\Delta = \{ \text{loves} \sqsubseteq \text{likes} \}$$

Then we require that our models respect the background assumptions:

$[\text{loves}] \subseteq [\text{likes}]$. 

**Background knowledge**
BACKGROUND KNOWLEDGE: A NEW LOGICAL LAW

\[
\text{All } x \text{ love all } y \\
\text{All } x \text{ like all } y
\]

What does completeness say?

And why does it hold?
All x love all y
All x like all y

What does completeness say?

**Theorem: The following are equivalent**

- Every model of $\Gamma$ which respects the assumptions in $\Delta$ satisfies $\varphi$.
- There is a proof of $\varphi$ using the assumptions in $\Gamma$ and the logic previously described, augmented by the inference rules corresponding to the sentences in $\Delta$.

And why does it hold?
Adding transitive verbs

All work on $\mathcal{R}$ and related systems is joint with Ian Pratt-Hartmann

The next language uses “see” or $r$ as variables for transitive verbs.

\[
\begin{align*}
\text{All } p & \text{ are } q \\
\text{Some } p & \text{ are } q \\
\hline
\text{All } p & \text{ see all } q \\
\text{All } p & \text{ see some } q \\
\text{Some } p & \text{ see all } q \\
\text{Some } p & \text{ see some } q
\end{align*}
\]

\[
\begin{align*}
\text{All } p & \text{ aren’t } q \equiv \text{No } p & \text{ are } q \\
\text{Some } p & \text{ aren’t } q \\
\hline
\text{All } p & \text{ don’t see all } q \equiv \text{No } p & \text{ sees any } q \\
\text{All } p & \text{ don’t see some } q \equiv \text{No } p & \text{ sees all } q \\
\text{Some } p & \text{ don’t see any } q \\
\text{Some } p & \text{ don’t see some } q
\end{align*}
\]

The interpretation is the natural one, using the subject wide scope readings in the ambiguous cases.

This is $\mathcal{R}$.
(The first system of its kind was Nishihara, Morita, Iwata 1990.)

The language $\mathcal{R}^\dagger$ has complemented atoms $\overline{p}$ on top of $\mathcal{R}$. 
Example of a proof in the system for $R^+$

What do you think? Sound or unsound?

\[
\begin{align*}
\text{All } X \text{ see all } Y, \text{ All } X \text{ see some } Z, \text{ All } Z \text{ see some } Y \\
\models \quad \text{All } X \text{ see some } Y
\end{align*}
\]
What do you think? Sound or unsound?

\[ \text{All } X \text{ see all } Y, \text{ All } X \text{ see some } Z, \text{ All } Z \text{ see some } Y \]

\[ \models \text{ All } X \text{ see some } Y \]

The conclusion does indeed follow:
take cases as to whether or not there are \( Z \).

We should have a formal proof.
Example of a proof in this system

All $X$ see all $Y$, All $X$ see some $Z$, All $Z$ see some $Y$

$\vdash$ All $X$ see some $Y$

Some $X$ see no $Y$

Some $X$ are $X$  All $X$ see some $Z$

Some $X$ see some $Z$

Some $Z$ are $Z$  All $Z$ see some $Y$

Some $Z$ see some $Y$

Some $Y$ are $Y$  All $X$ see all $Y$

All $X$ see some $Y$  Some $X$ see no $Y$

Some $X$ aren’t $X$
But now

\[ \text{Some } X \text{ see no } Y \]

\[
\begin{align*}
\text{Some } X \text{ are } X & \quad \text{All } X \text{ see some } Z \\
\text{Some } X \text{ see some } Z & \\
\text{Some } Z \text{ are } Z & \quad \text{All } Z \text{ see some } Y \\
\text{Some } Z \text{ see some } Y & \\
\text{Some } Y \text{ are } Y & \quad \text{All } X \text{ see all } Y \\
\text{All } X \text{ see some } Y & \\
\text{All } X \text{ see some } Y & \quad \text{[Some } X \text{ see no } Y \]
\end{align*}
\]

\[ \text{Some } X \text{ aren’t } X \quad \text{RAA} \]

This shows that

\[ \text{All } X \text{ see all } Y, \text{ All } X \text{ see some } Z, \text{ All } Z \text{ see some } Y \vdash \text{ All } X \text{ see some } Y \]
Towards the syntax for $\mathcal{R}$

- All $p$ are $q$  \[ \forall(p, q) \]
- Some $p$ are $q$  \[ \exists(p, q) \]
- All $p$ r all $q$  \[ \forall(p, \forall(q, r)) \]
- All $p$ r some $q$  \[ \forall(p, \exists(q, r)) \]
- Some $p$ r all $q$  \[ \exists(p, \forall(q, r)) \]
- Some $p$ r some $q$  \[ \exists(p, \exists(q, r)) \]
- No $p$ are $q$  \[ \forall(p, \overline{q}) \]
- Some $p$ aren’t $q$  \[ \exists(p, \overline{q}) \]
- All $p$ don’t r all $q$  \[ \equiv \forall(p, \forall(q, \overline{r})) \]
- No $p$ r any $q$  \[ \forall(p, \forall(q, \overline{r})) \]
- All $p$ don’t r some $q$  \[ \equiv \forall(p, \exists(q, \overline{r})) \]
- No $p$ r all $q$  \[ \forall(p, \exists(q, \overline{r})) \]
- Some $p$ don’t r any $q$  \[ \exists(p, \forall(q, \overline{r})) \]
- Some $p$ don’t r some $q$  \[ \exists(p, \exists(q, \overline{r})) \]
Towards the syntax for \( \mathcal{R} \)

| All p are q | \( \forall(p, q) \) |
| Some p are q | \( \exists(p, q) \) |
| All p r all q | \( \forall(p, \forall(q, r)) \) |
| All p r some q | \( \forall(p, \exists(q, r)) \) |
| Some p r all q | \( \exists(p, \forall(q, r)) \) |
| Some p r some q | \( \exists(p, \exists(q, r)) \) |
| No p are q | \( \forall(p, \overline{q}) \) |
| Some p aren’t q | \( \exists(p, \overline{q}) \) |
| No p r any q | \( \forall(p, \forall(q, \overline{r})) \) |
| No p r all q | \( \forall(p, \exists(q, \overline{r})) \) |
| Some p don’t r any q | \( \exists(p, \forall(q, \overline{r})) \) |
| Some p don’t r some q | \( \exists(p, \exists(q, \overline{r})) \) |

Set terms c

| \( p \) | \( \forall(p, r) \) | \( \exists(p, r) \) |
| \( \overline{p} \) | \( \exists(p, \overline{r}) \) | \( \forall(p, \overline{r}) \) |
\(\forall (p, r)\) those who \(r\) all \(p\)

\(\exists (p, r)\) those who \(r\) some \(p\)

\(\forall (p, \bar{r})\) those who fail-to-\(r\) all \(p = \) those who \(r\) no \(p\)

\(\exists (p, \bar{r})\) those who fail-to-\(r\) some \(p = \) those who don’t \(r\) some \(p\)
To understand how they work, let us exhibit a rendering of the simplest set terms into more idiomatic English:

- $\forall(\text{boy, see})$ those who see all boys
- $\exists(\text{girl, admire})$ those who admire some girl(s)
- $\forall(\text{boy, see})$ those who fail-to-see all boys
  
  = those who see no boys
  
  = those who don’t see any boys
- $\exists(\text{girl, recognize})$ those who fail-to-recognize some girl
  
  = those who don’t recognize some girl or other
Towards the syntax for $\mathcal{R}$

<table>
<thead>
<tr>
<th>All $p$ are $q$</th>
<th>$\forall(p, q)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Some $p$ are $q$</td>
<td>$\exists(p, q)$</td>
</tr>
<tr>
<td>All $p$ $r$ all $q$</td>
<td>$\forall(p, \forall(q, r))$</td>
</tr>
<tr>
<td>All $p$ $r$ some $q$</td>
<td>$\forall(p, \exists(q, r))$</td>
</tr>
<tr>
<td>Some $p$ $r$ all $q$</td>
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<tr>
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<td>$\exists(p, \exists(q, r))$</td>
</tr>
<tr>
<td>No $p$ are $q$</td>
<td>$\forall(p, \overline{q})$</td>
</tr>
<tr>
<td>Some $p$ aren’t $q$</td>
<td>$\exists(p, \overline{q})$</td>
</tr>
<tr>
<td>No $p$ sees any $q$</td>
<td>$\forall(p, \forall(q, \overline{r}))$</td>
</tr>
<tr>
<td>No $p$ sees all $q$</td>
<td>$\forall(p, \exists(q, \overline{r}))$</td>
</tr>
<tr>
<td>Some $p$ don’t $r$ any $q$</td>
<td>$\exists(p, \forall(q, \overline{r}))$</td>
</tr>
<tr>
<td>Some $p$ don’t $r$ some $q$</td>
<td>$\exists(p, \exists(q, \overline{r}))$</td>
</tr>
</tbody>
</table>

set terms $c$ [positive $p$ $\forall(p, r)$ $\exists(p, r)$ negative $\overline{p}$ $\exists(p, \overline{r})$ $\forall(p, \overline{r})$]

simplifies to $\forall(p, c) \; \exists(p, c)$
We start with one collection of unary atoms (for nouns) and another of binary atoms (for transitive verbs).

<table>
<thead>
<tr>
<th>expression</th>
<th>variables</th>
<th>syntax</th>
</tr>
</thead>
<tbody>
<tr>
<td>unary atom</td>
<td>$p, q$</td>
<td></td>
</tr>
<tr>
<td>binary atom</td>
<td>$r$</td>
<td></td>
</tr>
<tr>
<td>positive set term</td>
<td>$c^+$</td>
<td>$p \mid \exists(p, r) \mid \forall(p, r)$</td>
</tr>
<tr>
<td>set term</td>
<td>$c, d$</td>
<td>$p \mid \exists(p, r) \mid \forall(p, r) \mid \overline{p} \mid \exists(p, \overline{r}) \mid \forall(p, \overline{r})$</td>
</tr>
<tr>
<td>$\mathcal{R}$ sentence</td>
<td>$\varphi$</td>
<td>$\forall(p, c) \mid \exists(p, c)$</td>
</tr>
<tr>
<td>$\mathcal{R}^\dagger$ sentence</td>
<td>$\varphi$</td>
<td>$\forall(p, c) \mid \exists(p, c) \mid \forall(\overline{p}, c) \mid \exists(\overline{p}, c)$</td>
</tr>
<tr>
<td>$\mathcal{R}C$ sentence</td>
<td>$\varphi$</td>
<td>$\forall(d^+, c) \mid \exists(d^+, c)$</td>
</tr>
<tr>
<td>$\mathcal{R}C^\dagger$ sentence</td>
<td>$\varphi$</td>
<td>$\forall(d, c) \mid \exists(d, c)$</td>
</tr>
</tbody>
</table>
We need one last concept, syntactic negation:

<table>
<thead>
<tr>
<th>expression</th>
<th>syntax</th>
<th>negation</th>
</tr>
</thead>
<tbody>
<tr>
<td>positive set term $c$</td>
<td>$p$</td>
<td>$\overline{p}$</td>
</tr>
<tr>
<td></td>
<td>$\overline{p}$</td>
<td>$p$</td>
</tr>
<tr>
<td></td>
<td>$\exists(p, r)$</td>
<td>$\forall(p, \overline{r})$</td>
</tr>
<tr>
<td></td>
<td>$\forall(p, r)$</td>
<td>$\exists(p, \overline{r})$</td>
</tr>
<tr>
<td></td>
<td>$\exists(p, \overline{r})$</td>
<td>$\forall(p, r)$</td>
</tr>
<tr>
<td></td>
<td>$\forall(p, \overline{r})$</td>
<td>$\exists(p, r)$</td>
</tr>
<tr>
<td>$\mathcal{R}$ sentence $\varphi$</td>
<td>$\forall(p, c)$</td>
<td>$\exists(p, \overline{c})$</td>
</tr>
<tr>
<td></td>
<td>$\exists(p, c)$</td>
<td>$\forall(p, \overline{c})$</td>
</tr>
</tbody>
</table>

Note that $\overline{\overline{p}} = p$, $\overline{\overline{c}} = c$ and $\overline{\varphi} = \varphi$. 
Consider the model $\mathcal{M}$ with $M = \{w, x, y, z\}$, $\llbracket \text{cat} \rrbracket = \{w, x, y\}$, $\llbracket \text{dog} \rrbracket = \{z\}$, with $\llbracket \text{see} \rrbracket$ shown below on the left, and $\llbracket \text{likes} \rrbracket$ on the right:

Then $\llbracket \exists (\text{dog, see}) \rrbracket$ is the set of entities that see some dog, namely $\{x, y, z, w\}$.
Similarly, $\llbracket \exists (\text{dog, likes}) \rrbracket = \{w, y\}$.

It follows that

$$\llbracket \forall (\exists (\text{dog, likes}), \text{see}) \rrbracket = \{x\}.$$ 

Since $\llbracket \text{cat} \rrbracket$ contains $x$, we have

$$\mathcal{M} \models \forall (\exists (\text{dog, likes}), \text{see}), \text{cat}).$$

That is, in our model it is true that everything which sees everything bigger than some dog is a cat.
$p$ and $q$ range over unary atoms, 
c over set terms, and $t$ over binary atoms or their negations.

\[
\begin{array}{c}
\exists(p, q) \quad \forall(q, c) \\
\frac{\exists(p, c)}{} \\
\forall(p, q) \quad \exists(p, c) \\
\frac{\exists(q, c)}{} \\
\forall(q, \bar{c}) \quad \exists(p, c) \\
\frac{\exists(p, \bar{q})}{\forall(p, \forall(n, t))} \\
\forall(p, \forall(n, t)) \quad \exists(q, n) \\
\frac{\forall(p, \exists(q, t))}{\exists(p, \exists(q, t))} \\
\forall(p, \exists(q, t)) \quad \forall(q, n) \\
\frac{\exists(p, \exists(n, t))}{\forall(p, \exists(n, t))} \\
\forall(p, \exists(q, t)) \quad \forall(q, n) \\
\frac{\exists(p, \exists(n, t))}{\forall(p, \exists(n, t))} \\
\forall(p, \exists(q, t)) \quad \forall(q, n) \\
\frac{\exists(p, \exists(n, t))}{\forall(p, \exists(n, t))} \\
\end{array}
\]
Most are **monotonicity principles**

\[
\exists (p^\uparrow, q^\uparrow) \quad \forall (p^\downarrow, q^\uparrow) \\
\exists (p^\uparrow, \forall (q^\downarrow, t)) \quad \exists (p^\uparrow, \exists (q^\uparrow, t)) \\
\forall (p^\downarrow, \forall (q^\downarrow, t)) \quad \forall (p^\downarrow, \exists (q^\uparrow, t))
\]

Plus also

\[
\begin{array}{c}
\forall (p, p) \\
\exists (p, c) \\
\forall (p, \bar{p}) \\
\exists (p, \exists (q, t))
\end{array}
\quad
\begin{array}{c}
\exists (p, p) \\
\forall (p, c) \\
\forall (p, \bar{q}) \\
\exists (q, q)
\end{array}
\]

\[
\begin{array}{c}
\forall (q, \bar{c}) \\
\exists (p, c) \\
\forall (p, \forall (n, t)) \\
\exists (q, n)
\end{array}
\quad
\begin{array}{c}
(\star) \\
\forall (p, \exists (q, t)) \\
\forall (p, \forall (n, t)) \\
\exists (q, n)
\end{array}
\]

Of these, (\star) is the most interesting.
Every porter recognizes every porter
No quarterback recognizes any quarterback
No porter is a quarterback

The derivation in the logical system for $\mathcal{R}$ is

\[
\forall(p, \forall(p, r)) \quad [\exists(p, q)] \\
\forall(p, \exists(q, r)) \quad [\exists(p, q)] \\
\forall(q, \forall(q, \overline{r})) \quad \exists(q, \exists(q, r)) \\
\exists(q, \overline{q}) \quad (\star) \\
\forall(p, \overline{q}) \quad \text{RAA}
\]
The system that we have is complete. There are no purely syllogistic logical systems complete for $\mathcal{R}$, that is no systems without RAA.
**Theorem**

The system that we have is complete. There are no purely syllogistic logical systems complete for $\mathcal{R}$, that is no systems without RAA.

**Theorem**

There are no finite, complete syllogistic logical systems for $\mathcal{R}^\dagger$, even ones which allow RAA.
Now let’s add comparative adjectives, interpreted as transitive relations.
**Example**

Every hyena is taller than some jackal

Everything taller than some jackal is not heavier than any warthog

Everything which is taller than some hyena—

is not heavier than any warthog
<table>
<thead>
<tr>
<th>expression</th>
<th>variables</th>
<th>syntax</th>
</tr>
</thead>
<tbody>
<tr>
<td>unary atom</td>
<td>$p, q$</td>
<td></td>
</tr>
<tr>
<td>adjective atom</td>
<td>$a$</td>
<td></td>
</tr>
<tr>
<td>tv atom</td>
<td>$v$</td>
<td></td>
</tr>
<tr>
<td>binary atom</td>
<td>$r$</td>
<td>$v</td>
</tr>
<tr>
<td>positive set term</td>
<td>$c^+, d^+$</td>
<td>$p</td>
</tr>
<tr>
<td>set term</td>
<td>$c, d$</td>
<td>$c^+</td>
</tr>
<tr>
<td>sentence</td>
<td>$\varphi$</td>
<td>$\forall(d^+, c)</td>
</tr>
</tbody>
</table>
A few more examples

∀(boy, see)  those who see all boys
∃(girl, taller)  those who are taller than some girl(s)
∀(boy, see)  those who fail-to-see all boys
= those who see no boys
= those who don’t see any boys
∃(girl, see)  those who fail-to-see some girl
= those who don’t see some girl
The set terms in this language are a recursive construct.

We may embed set terms.

So we have set terms like

\[ \exists(\forall (\text{cat}, \text{sees}), \text{taller}) \]

which may be taken to denote the individuals who are taller than someone who sees no cat.

We should note that the relative clauses which can be obtained in this way are all “missing the subject”, never “missing the object”.

The language is too poor to express predicates like \( \lambda x. \text{all boys see } x \).
A model (for this language $\mathcal{L}$) is a pair $\mathcal{M} = (M, [\ [\ ] \ ]]$, where $M$ is a non-empty set, $[\ [p] \ ]] \subseteq M$ for all $p \in P$, $[\ [r] \ ]] \subseteq M^2$ for all binary atoms $r \in R$.

The only requirement is that for all adjectives $a$, $[\ [a] \ ]]$ should be a transitive relation: if $a(x, y)$ and $a(y, z)$, then $a(x, z)$. 
Given a model $\mathcal{M}$, we extend the interpretation function $[ ]$ to the rest of the language by setting

$[\overline{p}] = M \setminus [p]$

$[\overline{r}] = M^2 \setminus [r]$

$[\exists (l, t)] = \{ x \in M : \text{for some } y \text{ such that } [l](y), [t](x, y) \}$

$[\forall (l, t)] = \{ x \in M : \text{for all } y \text{ such that } [l](y), [t](x, y) \}$

We define the truth relation $\models$ between models and sentences by:

$\mathcal{M} \models \forall (c, d)$ \quad \text{iff} \quad [c] \subseteq [d]$

$\mathcal{M} \models \exists (c, d)$ \quad \text{iff} \quad [c] \cap [d] \neq \emptyset$

If $\Gamma$ is a set of formulas, we write $\mathcal{M} \models \Gamma$ if for all $\varphi \in \Gamma$, $\mathcal{M} \models \varphi$. 
The proof system for logic of $\mathcal{RCA}$. In it, $p$ and $q$ range over unary atoms, $b$ and $c$ over set terms, $d$ over positive set terms, $r$ over binary atoms, and $a$ over adjective atoms.
The following “inference” is invalid:

Every giraffe sees every gnu
Some gnu sees every lion
Some lion sees some zebra
Every giraffe sees some zebra

To see this, consider the model shown below

\[
\text{giraffe} \rightarrow \text{gnu} \rightarrow \text{lion} \rightarrow \text{zebra}
\]

The interpretations of the unary atoms are obvious, and the interpretation of the verb is the relation indicated by the arrow.
(J) If all watches are gold items, then everyone who owns all gold items owns all watches.

(K) If all watches are gold items, then everyone who owns some watch owns some gold item.

(L) If some watches are gold items, then everyone who owns all watches owns some gold item.

(II) If someone owns a watch, then there is a watch.

(TR1) If all watches are bigger than some pencil, then everthing bigger than some watch is bigger than some pencil.

(TR2) If all watches are bigger than all pencils, then everthing bigger than some watch is bigger than all pencils.

(TR3) If some watch is bigger than all pencils, then everthing bigger than all watches is bigger than all pencils.

(TR4) If some watch is bigger than some pencil, then everthing bigger than all watches is bigger than some pencil.
Here is a derivation for an example from early on in this course:

\[
\begin{align*}
\forall (\text{skunk}, \text{mammal}) \\
\forall (\forall (\text{mammal}, \text{respect}), \forall (\text{skunk}, \text{respect})) & \quad (J) \\
\forall (\forall (\forall (\text{skunk}, \text{respect}), \text{fear}), \forall (\forall (\text{mammal}, \text{respect}), \text{fear})) & \quad (J)
\end{align*}
\]

Note that inference using (J) is antitone each time: skunk and mammal have switched positions.
Every hyena is taller than some jackal
Everything taller than some jackal is not heavier than any warthog
Everything which is taller than some hyena
  is not heavier than any warthog

\[ \forall (\exists (\text{hyena, taller}), \exists (\text{jkl, taller})) \quad \text{(tr1)} \quad \forall (\exists (\text{jkl, taller}), \forall (\text{warthog, heavier})) \]
\[ \forall (\exists (\text{hyena, taller}), \forall (\text{warthog, heavier})) \]
Here is a proof of the (ZERO) rule of $S^\dagger$: $\forall(p, \overline{p}) \vdash \forall(p, q)$:

$$
\begin{align*}
[\exists(p, \overline{q})]_1 & \quad \exists(p, p) \quad (I) \\
\exists(p, p) & \quad \forall(p, \overline{p}) \quad (D1) \\
\exists(p, \overline{p}) & \quad \forall(p, q) \quad (RAA)_1
\end{align*}
$$