The Relational Syllogistic (and beyond)

Ian Pratt-Hartmann

(Joint work with Lawrence S. Moss)

School of Computer Science
Manchester University,
Manchester, M13 9PL
email: ipratt@cs.man.ac.uk

NLCS, New Orleans: June, 2013

Outline

- The classical syllogistic, denoted $\mathcal{S}$, is the logic corresponding to the following fragment of English:

Every p is a q	$\forall x(p(x) \rightarrow q(x))$	$\forall(p,q)$
Some p is a q	$\exists x(p(x) \wedge q(x))$	$\exists(p,q)$
No p is a q	$\forall x(p(x) \rightarrow \neg q(x))$	$\forall(p,\bar{q})$
Some p is not a q	$\exists x(p(x) \wedge \neg q(x))$	$\exists(p,\bar{q})$

- Syntax: non-logical signature of unary atoms $p \in \mathbf{P}$;

$$\ell =: p \mid \bar{p} \qquad \text{(literals)}$$
$$\varphi =: \exists(p,\ell) \mid \forall(p,\ell) \qquad \text{(formulas of } \mathcal{S}\text{)}$$

- Semantics: an interpretation is a set $A \neq \emptyset$ with a function $p \mapsto p^{\mathfrak{A}} \subseteq A$;

$$(\bar{p})^{\mathfrak{A}} = A \setminus p^{\mathfrak{A}}$$
$$\mathfrak{A} \models \forall(p,\ell) \text{ iff } p^{\mathfrak{A}} \subseteq \ell^{\mathfrak{A}}$$
$$\mathfrak{A} \models \exists(p,\ell) \text{ iff } p^{\mathfrak{A}} \cap \ell^{\mathfrak{A}} \neq \emptyset.$$

- The classical syllogisms are proof-rules for $\mathcal{S}$:

$$\frac{\forall(p,q) \quad \exists(o,p)}{\exists(o,q)}\ \text{Darii} \qquad\qquad \frac{\forall(p,\bar{q}) \quad \exists(o,p)}{\exists(o,\bar{q})}\ \text{Ferio}$$

- They may be combined to yield derivations in the expected way:

$$\cfrac{\cfrac{\exists(\text{artst},\text{bkpr}) \quad \forall(\text{artst},\text{crpntr})}{\exists(\text{crpntr},\text{bkpr})}\ \text{Darii} \qquad \forall(\text{bkpr},\overline{\text{dntst}})}{\exists(\text{crpntr},\overline{\text{dntst}})}\ \text{Ferio}$$

- Any set X of such rules then induces a derivability relation, for instance:

$$\{\exists(\text{artst},\text{bkpr}),\forall(\text{artst},\text{crpntr}),\forall(\text{bkpr},\overline{\text{dntst}})\} \vdash_{\mathsf{X}} \exists(\text{crpntr},\overline{\text{dntst}})$$

- A sequent with premises Φ and conclusion ψ is valid just in case, for any interpretation $\mathfrak{A}$, $\mathfrak{A} \models \Phi$ implies $\mathfrak{A} \models \psi$ (more briefly: just in case $\Phi \models \psi$.)
- An absurdity, $\bot$ is a formula of the form $\exists(p, \bar{p})$.
- We say that a derivation relation $\vdash$ is
  - sound if $\Phi \vdash \psi$ entails $\Phi \models \psi$
  - complete if $\Phi \models \psi$ entails $\Phi \vdash \psi$
  - refutation-complete if $\Phi \models \bot$ entails $\Phi \vdash \bot$
- For the classical syllogistic, the notions of validity and derivability can be made to coincide:

Theorem

There is a finite set of rules X *in* $\mathcal{S}$ *such that* $\vdash_{\mathsf{X}}$ *is sound and complete.*

- These rules suffice:

$$\frac{\forall(q,\ell) \qquad \exists(p,q)}{\exists(p,\ell)}\ (\mathrm{D1}) \qquad \frac{\forall(p,q) \qquad \forall(q,\ell)}{\forall(p,\ell)}\ (\mathrm{B})$$

$$\frac{\exists(p,\ell) \qquad \forall(p,q)}{\exists(q,\ell)}\ (\mathrm{D2}) \qquad \frac{\psi \quad \bar{\psi}}{\varphi}\ (\mathrm{X}) \qquad \frac{\forall(p,\bar{p})}{\forall(p,\ell)}\ (\mathrm{A})$$

$$\frac{\forall(q,\bar{\ell}) \qquad \exists(p,\ell)}{\exists(p,\bar{q})}\ (\mathrm{D3}) \qquad \frac{}{\forall(p,p)}\ (\mathrm{T}) \qquad \frac{\exists(p,\ell)}{\exists(p,p)}\ (\mathrm{I})$$

- There are some differences from the syllogistic as understood in classical times:
  - $\forall(p,q)$ does not entail $\exists(p,q)$
  - $\forall(p,p)$, $\exists(p,p)$ are allowed

- Completeness was first shown by (Corcoran 72) and (Smiley 73).
- Important qualification: these authors allowed reductio-ad-absurdum (as a final step).

$$\dfrac{\dfrac{\exists(q,o) \quad \forall(o,p)}{\exists(q,p)}\,\text{Darii} \qquad \forall(p,\bar{q})}{\underline{\exists(q,\bar{q})}}\,\text{Ferio}$$

- We write

$$\{\forall(o,p), \forall(p,\bar{q})\} \Vdash_X \forall(o,\bar{q})$$

  where X is any rule-set containing Ferio and Darii.
- However, reductio-ad-absurdum is not actually needed for $\mathcal{S}$ (P-H and Moss 09).
- A completeness theorem was provided by Łukasiewicz 39/50 for the classical syllogistic together with Boolean sentence-connectives.

- Completeness was first shown by (Corcoran 72) and (Smiley 73).
- Important qualification: these authors allowed reductio-ad-absurdum (as a final step).

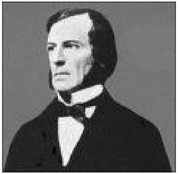
$$\dfrac{\dfrac{\dfrac{[\exists(q,o)] \quad \forall(o,p)}{\exists(q,p)}\,\text{Darii} \qquad \forall(p,\bar{q})}{\exists(q,\bar{q})}\,\text{Ferio}}{\forall(o,\bar{q})}\,\text{RAA}$$

- We write

$$\{\forall(o,p), \forall(p,\bar{q})\} \Vdash_{X} \forall(o,\bar{q})$$

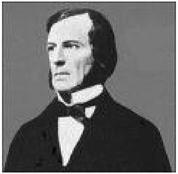
  where X is any rule-set containing Ferio and Darii.
- However, reductio-ad-absurdum is not actually needed for $\mathcal{S}$ (P-H and Moss 09).
- A completeness theorem was provided by Łukasiewicz 39/50 for the classical syllogistic together with Boolean sentence-connectives.

- It is natural to ask what happens if we extend the language of the classical syllogistic
  - Transitive verbs: Every artist admires every beekeeper
  - Numerical determiners: At least three artists are beekeepers
  - Adjectives: Every clever beekeeper is a carpenter
- Such sentences are considered throughout the history of logic, but determined efforts only really began in the 19th Century:

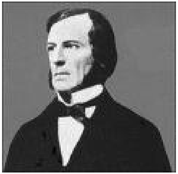

- It is natural to ask what happens if we extend the language of the classical syllogistic
  - Transitive verbs: Every artist admires every beekeeper
  - Numerical determiners: At least three artists are beekeepers
  - Adjectives: Every clever beekeeper is a carpenter
- Such sentences are considered throughout the history of logic, but determined efforts only really began in the 19th Century:

Augustus De Morgan



- It is natural to ask what happens if we extend the language of the classical syllogistic
  - Transitive verbs: Every artist admires every beekeeper
  - Numerical determiners: At least three artists are beekeepers
  - Adjectives: Every clever beekeeper is a carpenter
- Such sentences are considered throughout the history of logic, but determined efforts only really began in the 19th Century:

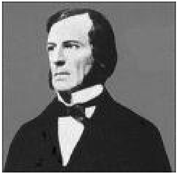
Augustus De Morgan



George Boole

- It is natural to ask what happens if we extend the language of the classical syllogistic
  - Transitive verbs: Every artist admires every beekeeper
  - Numerical determiners: At least three artists are beekeepers
  - Adjectives: Every clever beekeeper is a carpenter
- Such sentences are considered throughout the history of logic, but determined efforts only really began in the 19th Century:

Augustus De Morgan



George Boole

William Stanley Jevons

Outline

- The relational syllogistic, denoted $\mathcal{R}$, extends the classical syllogistic with transitive verbs.

Some artist hates every beekeeper	$\exists(\text{art}, \forall(\text{bkpr}, \text{hate}))$,
No beekeeper hates every artist	$\forall(\text{bkpr}, \exists(\text{artst}, \overline{\text{hate}}))$
Some artist is not a beekeeper	$\exists(\text{art}, \overline{\text{bkpr}})$.

- Syntax: add a non-logical signature of binary atoms $r \in \mathbf{R}$;

$$t =: r \mid \bar{r} \qquad \text{(binary literals)}$$
$$c =: \ell \mid \forall(p, t) \mid \exists(p, t) \qquad \text{(c-terms)}$$
$$\varphi =: \exists(p, c) \mid \forall(p, c) \qquad \text{(formulas of } \mathcal{R}\text{)}$$

- Semantics: $\mathfrak{A}$ also defines a function $r \mapsto r^{\mathfrak{A}} \subseteq (A \times A)$.

$$(\bar{r})^{\mathfrak{A}} = (A \times A) \setminus r^{\mathfrak{A}}$$
$$(\forall(p, t))^{\mathfrak{A}} = \{a \in A \mid \langle a, b \rangle \in t^{\mathfrak{A}} \text{ for all } b \in p^{\mathfrak{A}}\}$$
$$(\exists(p, t))^{\mathfrak{A}} = \{a \in A \mid \langle a, b \rangle \in t^{\mathfrak{A}} \text{ for some } b \in p^{\mathfrak{A}}\}$$

- We can write syllogism-like rules for $\mathcal{R}$ just as for $\mathcal{S}$:

$$\frac{\forall(o, \exists(p, r)) \quad \forall(p, q)}{\forall(o, \exists(q, r))} \; (\exists\forall)$$

- Then we have the derivation

$$\cfrac{\exists(\text{artst}, \forall(\text{bkpr}, \text{hate})) \qquad \cfrac{\cfrac{\forall(\text{bkpr}, \exists(\text{artst}, \overline{\text{hate}})) \quad [\forall(\text{art}, \text{bkpr})]^1}{\forall(\text{bkpr}, \exists(\text{bkpr}, \overline{\text{hate}}))} \; (\exists\forall) \qquad [\forall(\text{art}, \text{bkpr})]^1}{\forall(\text{artst}, \exists(\text{bkpr}, \overline{\text{hate}}))}}{\cfrac{\exists(\text{artst}, \overline{\text{artst}})}{\exists(\text{art}, \overline{\text{bkpr}})} \; (\text{RAA})^1}.$$

- It can be shown (P-H and Moss 09):

Theorem

There is a finite set of rules X in $\mathcal{R}$ such that $\vdash_X$ is sound and refutation-complete.

- A completeness theorem was provided by (Nishihara, Morita and Iwata 90) for $\mathcal{R}$ together with Boolean sentence-connectives.

- RAA is required for $\mathcal{R}$ (P-H and Moss 09):

Theorem

There exists no finite set X *of syllogistic rules in* $\mathcal{R}$ *such that* $\vdash_{\mathrm{X}}$ *is sound and complete.*

Dowód.

Let Γ^n be the set of sentences

$$\begin{array}{ll} \forall(p_i, \exists(p_{i+1}, r)) & (1 \leq i < n) \\ \forall(p_1, \forall(p_n, r)) & \\ \forall(p_i, \bar{p}_j) & (1 \leq i < j \leq n) \end{array}$$

Then $\Gamma^n \models \forall(p_1, \exists(p_n, r))$, but $\Gamma^n \setminus \{\forall(p_i, \exists(p_{i+1}, r))\}$ has no non-trivial $\mathcal{R}$-consequences! □

- The above notation suggests natural extensions of $\mathcal{S}$ and $\mathcal{R}$:

$\forall(\bar{p}, q)$	Every non-p is a q	
$\exists(\bar{p}, \bar{q})$	Some non-p is not a q	
$\forall(p, \exists(\bar{q}, r))$	Every p rs some non-q	$\ldots$

- Formally, we define the languages $\mathcal{S}^\dagger$ and $\mathcal{R}^\dagger$ as follows:

$$e =: \ell \mid \forall(\ell, t) \mid \exists(\ell, t) \qquad \text{(e-terms)}$$

$$\varphi =: \exists(\ell, m) \mid \forall(\ell, m) \qquad \text{(formulas of } \mathcal{S}^\dagger\text{)}$$

$$\varphi =: \exists(\ell, e) \mid \forall(\ell, e) \qquad \text{(formulas of } \mathcal{R}^\dagger\text{)}$$

- Here is a valid argument in $\mathcal{R}^\dagger$:

Every artist hates every beekeeper	$\forall(\text{art}, \forall(\text{bkpr}, \text{hate}))$
Every artist hates every non-beekeeper	$\forall(\text{art}, \forall(\overline{\text{bkpr}}, \text{hate}))$
Every artist hates every carpenter	$\forall(\text{art}, \forall(\text{cptr}, \text{hate}))$

- The following can be shown (PH and Moss 09)

  Theorem

  There exists a finite set X *of syllogistic rules in* $\mathcal{S}^\dagger$ *such that* $\vdash_X$ *is sound and complete.*

  Theorem

  There exists no finite set X *of syllogistic rules in* $\mathcal{R}^\dagger$ *such that* $\Vdash_X$ *is sound and complete.*

- Thus adding 'noun-level' negation does not change the proof-theoretic situation for the classical syllogistic, but it does for the relational syllogistic.

- It is worth noting the complexity of satisfiability for the languages considered so far:
- The existence of syllogistic proof-systems of various kinds imposes upper bounds on complexity:
  - The satisfiability problem for any syllogistic language for which there exists a sound and (refutation-) complete syllogistic system $\vdash_X$ is in PTime.
  - The satisfiability problem for any syllogistic language for which there exists a sound and complete syllogistic system $\Vdash_X$ is in NPTime.
- The following are straightforward to establish:
  - The satisfiability problems for $\mathcal{S}$, $\mathcal{S}^\dagger$, and $\mathcal{R}$ are all NLogSpace-complete
  - The satisfiability problem for $\mathcal{R}^\dagger$ is ExpTime-complete.
- Thus, the existence of a sound and complete $\Vdash_X$ for $\mathcal{R}^\dagger$ would entail NPTime = ExpTime.

Outline

- The numerical syllogistic, denoted $\mathcal{N}$, extends the classical syllogistic with counting determiners

At most C ps are qs	$\exists_{\leq C}(p, q)$
More than C ps are qs	$\exists_{> C}(p, q)$
At most C ps are not qs	$\exists_{\leq C}(p, \bar{q})$
More than C ps are not qs	$\exists_{> C}(p, \bar{q})$

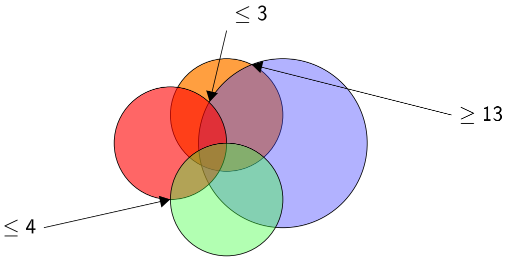
- Syntax:

$$\varphi =: \exists_{>C}(p, \ell) \mid \exists_{\leq C}(p, \ell) \qquad \text{(formulas of } \mathcal{N}\text{)}$$

- Unlike the languages considered above, $\mathcal{N}$ has infinitely many sentence-forms.
- We denote by $\mathcal{S}_k$ the fragment of $\mathcal{N}$ in which all numerical subscripts are bounded by k.

- A valid argument in $\mathcal{N}$ (actually, in $\mathcal{S}_{12}$):

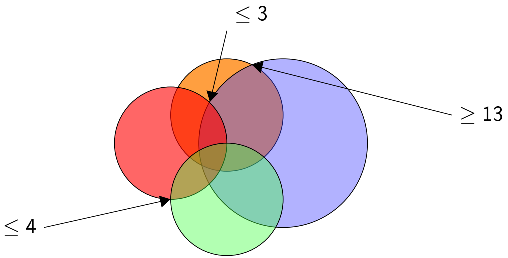
More than twelve artists are beekeepers
At most three beekeepers are carpenters
At most four gardeners are not carpenters



- A valid argument in $\mathcal{N}$ (actually, in $\mathcal{S}_{12}$):

More than twelve artists are beekeepers
At most three beekeepers are carpenters
At most four gardeners are not carpenters

More than five artists are not gardeners.



- Evidently,

$$\forall(p,\ell) \equiv \exists_{\leq 0}(p,\bar{\ell})$$
$$\exists(p,\ell) \equiv \exists_{>0}(p,\ell).$$

- Thus:
  - $\mathcal{S}$ notational variant of $\mathcal{S}_0$;
  - $\mathcal{N}$ is the union of all the $\mathcal{S}_k$.
- De Morgan made a concerted effort to work out a set of syllogism-like rules for $\mathcal{N}$.
- So have a number of twentieth-century authors ...

- The following can be shown (PH09, PH 13)

Theorem
There exists no finite set X *of syllogistic rules in* $\mathcal{N}$ *such that* $\Vdash_X$ *is sound and complete.*

Theorem
There exists no finite set X *of syllogistic rules in* $\mathcal{S}_1$ *such that* $\Vdash_X$ *is sound and complete.*

- The satisfiability problem for $\mathcal{S}_1$ is NPTime-hard, and for $\mathcal{N}$ is in NPTime under binary coding (Eisenbrand and Shmonin 07).
- Adding 'noun-level' negation makes no difference here.

- Of course, we could add numbers to $\mathcal{R}$ and $\mathcal{R}^{\dagger}$, as well, to formalize such arguments as

At most one artist admires at most seven beekeepers
At most two carpenters admire at most eight dentists
At most three artists admire at least seven electricians
At most four beekeepers are not electricians
At most five dentists are not electricians
<u>At most one beekeeper is a dentist</u>

- Of course, we could add numbers to $\mathcal{R}$ and $\mathcal{R}^{\dagger}$, as well, to formalize such arguments as

At most one artist admires at most seven beekeepers
At most two carpenters admire at most eight dentists
At most three artists admire at least seven electricians
At most four beekeepers are not electricians
At most five dentists are not electricians
At most one beekeeper is a dentist

At most six artists are carpenters

- Of course, we could add numbers to $\mathcal{R}$ and $\mathcal{R}^{\dagger}$, as well, to formalize such arguments as

  At most one artist admires at most seven beekeepers
  At most two carpenters admire at most eight dentists
  At most three artists admire at least seven electricians
  At most four beekeepers are not electricians
  At most five dentists are not electricians
  At most one beekeeper is a dentist
  ———
  At most six artists are carpenters

- This logic is NExpTime-complete (under binary coding), and again lacks any sound and complete (indirect) syllogistic proof-system.

Outline

- We have seen that $\mathcal{R}$ admits no sound and complete direct syllogistic rule-system.
- But does it have any extensions which do?
- The answer is yes. We consider statements of the form

$$\text{If there are } p\text{s, then there are } q\text{s} \qquad \exists\!\!\!/(p, q)$$

- Syntax of $\mathcal{RE}$:

$$\varphi =: \exists(p, \ell) \mid \forall(p, \ell) \mid \exists\!\!\!/(p, q)$$

- Semantics:

$$\mathfrak{A} \models \exists\!\!\!/(p, q) \text{ iff } p^{\mathfrak{A}} \neq \emptyset \Rightarrow q^{\mathfrak{A}} \neq \emptyset.$$

Theorem

- *There is a finite set of rules* X *in* $\mathcal{RE}$ *such that* $\vdash_{\mathsf{X}}$ *is sound and complete.*
- The rules set is BIG. The best I could do has 70 rules including

$$\frac{\forall(o', \forall(q, t)) \quad \forall(q, \forall(o, \bar{t})) \quad \not\exists(q, o') \quad \forall(o', p) \quad \not\exists(q, o) \quad \forall(o, p) \quad \exists(p, p)}{\exists(p, \bar{q})}$$

- Various philosophers since (including maybe Aristotle) have wondered why you cannot say

Every p is every q	Some p is every q
No p is every q	Some p is not every q,

- or even

Every p is some q	Some p is some q
No p is some q	Some p is no q,

- all of which sound extremely strange.

- Things become a little clearer if we reformulate the classical syllogistic, thus:

Every p is identical to some q	Some p is identical to some q
No p is identical to any q	Some p is not identical to any q.

- because the second quantifier here can be meaningfully dualized:

Every p is identical to every q	Some p is identical to every q
No p is identical to every q	Some p is not identical to every q.

- This suggests an unnatural extension of $\mathcal{S}$, say $\mathcal{H}$, featuring forms such as

$$\forall(p, \forall q) \qquad \exists(p, \forall q)$$

- Similarly, we could have $\mathcal{H}^\dagger$, allowing noun-negation:

$$\forall(p, \forall \bar{q}) \qquad \exists(p, \forall \bar{q}) \qquad \text{etc.}$$

- Quick test: what does

  Every artist is every beekeeper

  mean?

- This suggests an unnatural extension of $\mathcal{S}$, say $\mathcal{H}$, featuring forms such as

$$\forall(p, \forall q) \qquad \exists(p, \forall q)$$

- Similarly, we could have $\mathcal{H}^\dagger$, allowing noun-negation:

$$\forall(p, \forall \bar{q}) \qquad \exists(p, \forall \bar{q}) \qquad \text{etc.}$$

- Quick test: what does

  Every artist is every beekeeper

  mean?
- Quick answer:

  Either there are no artists or no beekeepers or there is a unique artist and a unique beekeeper and they are identical.

- Example validity in $\mathcal{H}^{\dagger}$:

$$\frac{\begin{array}{l}\text{Every artist is every artist}\\ \text{Every non-artist is every non-artist}\\ \text{Some beekeeper is not a carpenter}\\ \text{Some carpenter is not a dentist}\end{array}}{\text{Every dentist is a beekeeper}}$$

- By the way, $\mathcal{H}$ stands for Hamiltonian syllogistic, after Sir William Hamilton (Bart.) who got into a big argument with De Morgan about quantification of the predicate.

Theorem

There is no finite set of rules that is sound and complete for $\mathcal{H}$ without reductio-ad-absurdum.

Theorem

There is a finite set of rules that is sound and complete for $\mathcal{H}$, as long as reductio-ad-absurdum is allowed as a last step.

Theorem

Unless PTime= NPTime, *there is no finite set of rules that is sound and complete for $\mathcal{H}^\dagger$, with reductio-ad-absurdum allowed (as a last step).*

Theorem

There is a finite set of rules that is sound and complete for $\mathcal{H}^\dagger$ with reductio-ad-absurdum allowed unconditionally.

Outline

- We have investigated the existence of sound and complete syllogistic proof-systems

$\mathcal{S}$	$\vdash_X$	NLogSpace
$\mathcal{S}^\dagger$	$\vdash_X$	NLogSpace
$\mathcal{R}$	"$\Vdash_X$"	NLogSpace
$\mathcal{RE}$	$\vdash_X$	$\leq$ PTime
$\mathcal{R}^\dagger$	No $\Vdash_X$	ExpTime
$\mathcal{S}_z$ $(z \geq 1)$	No $\Vdash_X$	NPTime
$\mathcal{N}$	No $\Vdash_X$	NPTime
$\mathcal{H}$	"$\Vdash_X$"	$\leq$ PTime
$\mathcal{H}^\dagger$	$\Vdash_X$	NPTime

- The classical syllogistic is a logic whose salience derives from the syntax of certain natural languages.
- By considering larger fragments of natural languages, we obtain more expressive logics whose properties we can investigate.
  - ditransitive verbs, anaphora, relative clauses, conjunctions, ellipsis
  - comparatives and expressions of quantity
  - generalized quantification
  - tense and temporal expressions
- Some pointers to people who have worked on similar logics:
  - F.B. Fitch, P. Suppes, D. McAllister, R. Givan, W. Purdy
  - N. Francez, Y. Fyodorov, S. Winter,
  - L. Moss, B. MacCartney, T. Icard
  - J. Szymanik, M. Sevenster.