Quantum logic on finite dimensional Hilbert spaces

Tobias Hagge

U.T. Dallas

May 11, 2009

Joint work with J. Michael Dunn, Lawrence S. Moss, Zhenghan Wang
Inspiration

Hilbert Lattices, axiomatization, and decidability

Dimension in $QL(L_{H^n})$

Finite submodel property
Outline

Inspiration

Hilbert Lattices, axiomatization, and decidability

Dimension in $QL(L_{H^n})$

Finite submodel property
Some personal desires

- Logical foundation for quantum computation
- Logical foundation for representation theory?
- Classical logic as emergent phenomenon?
Outline

Inspiration

Hilbert Lattices, axiomatization, and decidability

Dimension in $QL(L^n_H)$

Finite submodel property
Hilbert ortholattices

The set of closed subspaces $S_H$ of a Hilbert space $H$ give rise to an ortholattice $L_H = (S_H, \wedge, \vee, \neg, \{0\}, H)$ with

- $\wedge =$ intersection,
- $\vee =$ closure of span of union,
- $\neg =$ orthogonal complement.

Axioms for an ortholattice $(S, \wedge, \vee, 0, 1)$:

- $(S, \wedge, 1)$ and $(S, \vee, 0)$ are commutative, idempotent monoids,
- $a \wedge b = a \iff a \vee b = b$ for all $a, b \in L$ (say $a \leq b$ if $a \vee b = b$),
- $\neg : L \to L$ is an involution such that $\neg(a \vee b) = \neg a \wedge \neg b$ for all $a, b \in L$,
- $a \vee \neg a = 1$. 
(Ortho)modularity and distributivity

Hilbert lattices also satisfy the orthomodular law, i.e.:

\[ y \leq x \implies y = x \land (y \lor \neg x) \]

Note: \( P(y, x) := x \land (y \lor \neg x) \) is the projection of \( y \) onto \( x \).

The \( n \)-dimensional Hilbert space \( H^n \) is modular, i.e.:

\[ b \leq x \implies x \land (a \lor b) = (x \land a) \lor b \]

\( L = (S, \land, \lor, 0, 1) \) is not necessarily distributive. One only has:

\[ (a \lor b) \land c \geq (a \land c) \lor (b \land c) \]

\( L_{H^n} \) is distributive iff \( n = 1 \).

A counterexample in \( H^2 \): \( a = \langle (1, 0) \rangle \), \( b = \langle (0, 1) \rangle \), \( c = \langle (1, 1) \rangle \).
Validity

A formula $\phi(a_1, \ldots a_n)$ is a valid in $L$ if $\phi(a_1, \ldots a_n) = 0$ for all choices of $a_1, \ldots, a_n \in L$. Let $QL(L)$ denote the set of formulas valid in $L$.

Note: $a = b$ iff $(a \lor b) \land (\neg a \lor \neg b) = 0$ iff $(a \land b) \lor (\neg a \land \neg b) = 1$.

$QL(H^n)$ becomes weaker as $n$ increases. If $H^{n+1} = H^n \oplus v$, with $v = \langle v_1 \rangle$ and $\langle h, v_1 \rangle = 0$ for all $h \in H^n$, then:

$$\phi_{L_{H^{n+1}}}(a_1 \lor v, \ldots a_n \lor v) = \phi_{L_{H^n}}(a_1, \ldots, a_n) \lor \phi_{L_{H^1}}(v, \ldots, v).$$

However, $QL(L_{H^\infty}) \neq \bigcap_{n \in \mathbb{N}} QL(L_{H^n})$ since $QL(L_{H^\infty})$ is not modular.
Axiomatization of $QL(L_{H^n})$ and $QL(L_{H^\infty})$

- We don’t know how to write a nice set of axioms for $QL(L_{H^n})$ (or $QL(L_{H^\infty})$).
- Does modular + ortholattice + $n$-distributive axiomatize $QL(L_{H^n})$?
- Does $QL(L_{H^n})$ have a finite axiomatization?
Decidability of $QL(H^n)$

∃ undecidable MOLs (Roddy 1989), but $QL(L_{H^n})$ is decidable.
Reduces to decidability of FOL of $\mathbb{R}$ (decidable by (Tarski, 1948)).
Sketch:

- For each $a \in L_{H^n}$, assign a matrix $M_a$ with kernel $a$.
- Build $M_\phi(\bar{a})$ by structural induction, introducing a new matrix variable at each stage
- $\phi$ is valid iff $M_\phi(\bar{a})$ is always the zero matrix.

<table>
<thead>
<tr>
<th>$A$</th>
<th>$M_A \in M_n(\mathbb{C})$: A matrix with kernel $A$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C = A \land B$</td>
<td>$\forall d(M_A d = 0 \land M_B d = 0 \iff M_C d = 0)$</td>
</tr>
<tr>
<td>$B = \neg A$</td>
<td>$\forall c(M_B c = 0 \iff (\forall d(M_A d = 0 \Rightarrow \langle c, d \rangle = 0)))$</td>
</tr>
</tbody>
</table>
Decidability of $QL(L_{H^\infty})$

It is not known whether $QL(L_{H^\infty})$ is decidable.

Everything in this talk depends on the notion of dimension, which is less useful in $H^\infty$. 
Outline

Inspiration

Hilbert Lattices, axiomatization, and decidability

Dimension in $QL(L_{H^n})$

Finite submodel property
Dimensional dependence part 1

\[ QL(L_{H^n}) \neq QL(L_{H^{n+1}}) \] (Dunn, H., Moss, Wang 2004, H. 2007):

Ideas:

1. Given \( s \leq 1 \), and a formula \( \phi \), one may construct a formula \( \phi|_s \) which expresses the proposition that \( \phi \) is valid in the sublattice generated below \( s \).

2. Failure of distributive law is not arbitrary:
   If \( a = p \lor (q \land r), b = (p \lor q) \land (p \lor r) \) then
   \[ \dim((a \lor b) \land (\neg a \lor \neg b)) \leq \left\lfloor \frac{n}{2} \right\rfloor. \]

3. Projections obey dimensional laws:
   \[ \dim(P(P(P(a, b), a), \neg b)) \leq \min(\dim(b), \dim(\neg b)) \leq \left\lfloor \frac{n}{2} \right\rfloor. \]

This was, however, not the first (or nicest) proof of this result.
Michael Roddy - René Mayet

n-DISTRIBUTIVITY IN ORTHOLATTICES

We introduce an ortholattice equation which is equivalent to
n-distributivity in the variety of modular ortholattices, and
discuss it.

Introduction. The basic tool of von Neumann's coordinatization
theorem [10] is the homogeneous n-frame, and the importance of
this configuration in the variety of modular lattices has been
well documented. In particular, Hühn showed that the exclusion
of the n-frame for a given n defines a variety of modular
lattices, the (n-1)-distributive modular lattices. He then
proceeded to develop the theory of these and related lattices
(in the several papers listed in the bibliography).

The definitions and the theory pertain to any class of
algebras each of whose members has a modular lattice reduct, in
particular to MOL (the modular ortholattices). Moreover, these
ideas have natural "ortho"-generalizations. The purpose of
this paper is to present some of this theory in the ortholattice
setting.

One motivation for presenting this material is that the
existence of complements in MOL's makes the ideas even more
powerful than in the variety of modular lattices. We think
that this is illustrated by their use in the proof of the
following two theorems.

Theorem (0.1). ([11]). Every variety of MOL's is comparable
to the variety generated by M0u.

Theorem (0.2). ([12]).
(a) There exists a finitely presented three generated MOL
An ortholattice is \emph{n-distributive} if

\[
x \lor \bigwedge_{i} y_i = \bigwedge_{j}(x \lor \bigwedge_{i \neq j} y_i)
\]

for all \(x, y_1, \ldots, y_n \in L\). (This equation may be replaced by an equation in three variables.)

(Roddy, Mayet 1986) showed that \emph{n-distributivity} holds in a subirreducible MOL iff (among other things) \(L\) has height \(\leq n\). Thus \(D_n \in QL(L_{H^n})\) but \(D_n \notin QL(L_{H^{n+1}})\).
Dimensional dependence part 2

Work in progress by Giuntini and Freytes puts the result in a general framework.

A lattice is *atomic* if for every $a \in L$, $a = \bigvee_i a_i$, where for each $a_i$, $b < a_i \implies b = 0$. Atomic lattices are the most general framework for making dimension arguments.

Let $L_a = \{ b \in L | b \leq a \}$. Then $a < b \implies QL(L_a) \supset QL(L_b)$ in an atomic MOL $L$ iff $L$ is irreducible.
Outline

Inspiration

Hilbert Lattices, axiomatization, and decidability

Dimension in $QL(L_{H^n})$

Finite submodel property
Finite submodel property

(We’ll say) QL(L) has the finite submodel property if for any formula $\phi \not\in QL(L)$, L has a finite sublattice $L'$ such that $\phi \not\in QL(L')$.

**Theorem**

QL($L_{H^n}$) does not have the finite submodel property. Furthermore, for $n \geq 3$ there exists $\phi \not\in QL(L_{H^n})$ such that $\phi \in QL(L)$ for any sublattice L which is not dense in QL($L_{H^n}$).

I do not know the full generality in which this theorem holds.
“Classical” conjunctions

Given a formula $\phi(\vec{a})$, we can construct $\tilde{\phi}(\vec{a}, \vec{b})$ such that:

1. If $\phi(\vec{a}) = 0$ then $\tilde{\phi}(\vec{a}, \vec{b}) = 0$ for all $\vec{b}$,

2. If $\phi(\vec{a}) \neq 0$, then for some $\vec{b}$, $\tilde{\phi}(\vec{a}, \vec{b}) = 1$.

Then $\tilde{\phi} \land \chi$ is valid in $L_{H^n}$ iff for every $\vec{a}$, either $\phi(\vec{a}) = 0$ or $\chi(\vec{a}) = 0$.

Construction sketch:

- If $\text{dim}(\phi(\vec{a})) = m > 0$ we can choose $b, c$ so that $P(P(\phi(\vec{a}), b), c)$ is any arbitrary subspace of dimension $\leq m$.

- $P(P(\phi(\vec{a}), b_1), c_1) \lor \ldots \lor P(P(\phi(\vec{a}), b_n), c_n)$ can take any value with appropriate choices of $b_1, \ldots, b_n$ and $c_1 \ldots c_n$.

A weaker statement is possible in $QL(L_{H\infty})$. 
Dimensional restriction

\[ \overline{D_k|_a(b, c)} = 0 \text{ if } \dim(a) \leq k, \text{ gives arbitrary values from choices of } b \text{ and } c \text{ otherwise.} \]

Let
\[
R_k(a, b_1, c_1, b_2, c_2) = \overline{D_{k-1}|_a(b_1, c_1)} \land \overline{D_{n-k-1}|_{\neg a}(b_2, c_2)}. 
\]

\[ R_k(a, \ldots) = 0 \text{ unless } \dim(a) = k, \text{ arbitrary values if } \dim(a) = k. \]

\[ R_k(a, \ldots) \land \phi(a, d) \text{ is valid iff } \phi(a, d) \text{ is valid when } \dim(a) = k. \]
Proof of theorem

By dimensional restriction, let \( a, b_1, b_2, b_3, c_1, c_2 \) be two dimensional subspaces of \( H_3 \) such that for \( i, j \in [1 \ldots 3] \) and \( k, l \in [1, 2] \) the following hold:

1. \( \dim(a \wedge b_i) = \dim(a \wedge c_k) = 1, \)
2. \( \dim(b_i \wedge b_j) = 1, \)
3. \( \dim(b_i \wedge b_j \wedge a) = 1, \)
4. \( \dim(c_k \wedge c_l) = 1, \)
5. \( \dim(c_k \wedge c_l \wedge a) = 1, \)
6. \( \dim(b_i \wedge c_k \wedge a) = 0. \)

Let \( \bar{b}_i \) and \( \bar{c}_k \) denote the intersections of the \( b_i \) and \( c_k \) respectively with an affine \( \mathbb{C} \)-plane \( a' \) parallel to \( a \). Then

1. \( \bar{b}_1, \bar{b}_2 \) and \( \bar{b}_3 \) are parallel complex lines,
2. \( \bar{c}_1 \) and \( \bar{c}_2 \) are parallel complex lines,
3. Each \( \bar{b}_i \) and \( \bar{c}_k \) intersect in a single point \( p_{i,k} \).
Proof of theorem (continued)

One gets the following:

\[
c_1 \leq c_2
\]
Proof of theorem (continued)

One gets the following:

\[ b_1 \quad b_2 \quad b_3 \quad c_1 \quad c_2 \]

\[ c_1 \quad b_2 \quad b_3 \]
Proof of theorem (continued)

One gets the following:

\[
\begin{align*}
&b_3 \\
&b_1 \\
&b_2 \\
&c_1 \\
&c_2 \\
\end{align*}
\]
Proof of theorem (continued)

One gets the following:
Proof of theorem (continued)

One gets the following:

\[
\begin{align*}
&b_2 \\
&b_3 \\
\end{align*}
\]
Proof of theorem (continued)

One gets the following:

\[
\begin{align*}
&b_1 \quad b_2 \quad b_3 \\
&c_1 \quad c_2
\end{align*}
\]
Proof of theorem (continued)

One gets the following:

Continuing, one gets a dense set of points in $R^3$. 
Proof of theorem (conclusion)

Let $\phi$ be a formula which is not valid in $QL(L_{H^3})$. Let

$$\chi(\ldots) = R_2(a, \overrightarrow{d_a}) \bigwedge_i R_2(b_i, \overrightarrow{d_{b_i}}) \bigwedge_k R_2(c_k, \overrightarrow{d_{c_k}})$$

$$\bigwedge_{i \neq j} R_1(b_i \land b_j, \overrightarrow{d_{b_i \land b_j}}) \bigwedge_{i \neq j} R_1(b_i \land b_j \land a, \overrightarrow{d_{b_i \land b_j \land a}})$$

$$\land R_1(c_1 \land c_2, \overrightarrow{d_{c_1 \land c_2}}) \land R_1(c_1 \land c_2 \land a, \overrightarrow{d_{c_1 \land c_2 \land a}})$$

$$\bigwedge_{i, k} R_0(b_i \land c_k \land a, \overrightarrow{d_{b_i \land c_k \land a}}) \land \phi(\overrightarrow{e})$$

Then $\chi$ is not valid in $QL(L_{H^3})$, but is valid in any non-dense sublattice. By restricting to three dimensional subspaces with appropriate intersection properties, one may construct similar formulas for $L_{H^n}$. 
Thanks!

References:


- Michael Roddy, René Mayet. n-distributivity in ortholattices. Contributions to general algebra 5, Mai 29 - June 1, 1986.