Topological quantum computation and quantum logic

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Microsoft Station Q
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Microsoft Project Q:

Search for non-abelian anyons in topological phases of matter, and build a topological quantum computer.

Theory:

MS station Q

Experiments:

Bell lab, Harvard, Columbia, U Chicago, Caltech, Princeton, Weizmann
Microsoft Station Q
http://stationq.ucsb.edu/

• Michael Freedman (math)
• Chetan Nayak (physics)
• Matthew Fisher (physics)
• Kevin Walker (math)
• Matthew Hastings (cs)
• Simon Trebst (computational physics)
• Parsa Bonderson (physics)
Topological Computation

- Computation
  - initialize
  - apply operators
  - output

- Physics
  - braid
  - create particles
  - measure
Statistics of Identical Particles
non-local or topological interaction

Given a collection of **n identical particles** in a space X, at each moment the state of the n particles is given by a wavefunction $|\psi\rangle$,
Suppose at a later time, the n particles return to the same positions as a set, how does $|\psi\rangle$ change?
Answers depend on dimensions of X.
Statistics of Particles

In $\mathbb{R}^3$, particles are either bosons or fermions.

Worldlines (curves in $\mathbb{R}^3 \times \mathbb{R}$) exchanging two identical particles depend only on permutations.

\[
\begin{array}{c}
\begin{array}{c}
\downarrow \quad \downarrow
\end{array}
\end{array}
= \begin{array}{c}
\begin{array}{c}
\downarrow \quad \downarrow
\end{array}
\end{array}
\]

Statistics is $\lambda: S_n \to \mathbb{Z}_2$
Braid statistics

In $\mathbb{R}^2$, an exchange is of infinite order

Braids form groups $B_n$

Statistics is $\lambda: B_n \rightarrow U(1)$

If not 1 or -1, but $e^{i\theta}$, anyons
Suppose the ground state of \( n \) identical particles is degenerate, and has a basis \( \psi_1, \psi_2, \ldots, \psi_k \). Then after braiding some particles:

\[
\psi_1 \rightarrow a_{11} \psi_1 + a_{12} \psi_2 + \ldots + a_{k1} \psi_k
\]

Particle statistics is \( \lambda : B_n \rightarrow U(k) \).

Particles with \( k > 1 \) are called non-abelian anyons.
Topological phases of matter or anyonic quantum systems

A quantum system whose lowest energy states are effectively described by a topological quantum field theory (TQFT)

Given a theory, put it a surface $Y$, Hilbert space $H(Y) \cong \bigoplus V_i(Y)$---energy $\lambda_i$

Assume energy gap $\lambda_1 > \lambda_0 = 0$,

$Y \rightarrow V^{\text{top}}(Y)$ (part of $V_0(Y)$) is a TQFT
Some features

1) Ground states degeneracy---\( \dim V^{\text{top}} \geq 1 \)
   (memory)
2) No non-trivial continuous evolutions
   (fault-tolerant or deaf)
3) Elementary excitations are “anyons”
   (braiding statistics are gates)
A ribbon category is a braided fusion category with compatible duality=charge conjugation, which yields link invariants such as Jones poly and representations of braid groups.

A ribbon tensor category with finitely many isomorphism classes of simple objects and a non-singular s-matrix.

Simple objects represent anyons. Tensor product is fusion.
Non-abelain anyons, mathematically?

Are non-abelian anyons possible, ie, are there unitary braid group representations?

Jones reps through Temperley-Lieb algebras labeled by $r=3,4,5,…$ (1981)

Jones polynomials at $r$-th root of unity
---computationally hard if $r\neq 3,4,6$
Non-abelian anyons, physically?

- If there were non-abelian anyons, then they can be used to build universal fault-tolerant quantum computers.
- Do they exist in Nature?
- There is evidence and numerical “proof” that they do exist in fractional quantum Hall liquids.
Classical Hall effect

E. H. Hall, 1879
On a new action of the magnet on electric currents

“It must be carefully remembered that the mechanical force which urges a conductor carrying across the lines of the magnetic force, acts, not on the electric current, but on the conductor which carries it”

Maxwell, Electricity and Magnetism
Quantum Hall Effect

1982 H. Stormer, D. Tsui --- FQHE
R. Laughlin (1998 Nobel)

quasi-particle with 1/3 electron charge and braiding statistics (anyons)
Electrons in a flatland

Energy levels for electrons are called Landau levels, the filling fraction $\nu = \# \text{ of electrons} / \# \text{ flux lines}$
Non-Abelian anyons in real life: FQHE?

Fig. 1, Pan et al.
Read-Rezayi conjecture:

\[ \nu = \frac{1}{3} \text{ or } \frac{2}{3} \quad \leftrightarrow \quad \text{Jones rep at } r=3 \]

\[ \nu = \frac{5}{2} \quad \leftrightarrow \quad \text{Jones rep at } r=4 \]

\[ \nu = \frac{12}{5} \text{ or } \frac{13}{5} \quad \leftrightarrow \quad \text{Jones rep at } r=5 \]

(Universal QC)
Experimental Progress

• For $\nu=5/2$, the charge of $e/4$ particles is confirmed

• No conclusive experiments to prove any anyonic statistics, but progress has been made for the last 4 years (Goldman for abelian, and Willet for 5/2)
TQC to Quantum Logic?

Is it possible to address the “touchy and complicated” issue: (von Neumann)

What is a physical proposition?
Quantum Logics

• Birkhoff-von Neumann (1936):
  Continous geometry

• 1960---1970’s:
  Orthomodular lattice

• Third life (Dunn): ?
Continuous Geometries (CGs)

A continuous geometry of von-Neumann: orthcomplemented complete modular lattice (Kaplanski)

Is the word problem decidable in CGs?
In general, they should be very similar to quantum logics of finite dimensional vector spaces.
Qubit continuous geometry

- $\text{PG}(2^n) =$ subspaces of $n$-qubits
  - $\text{PG}(2^n)$ embeds isomorphically in $\text{PG}(2^{n+1})$
    - $p \in \text{PG}(2^n)$, $p \rightarrow p \in \mathbb{C}^2$

- Normalized dimension $\delta(p) = d(p)/d(1)$, metrically completed by
  - $|p-q| = \delta(p \lor q) - \delta(p \land q)$
Type II$_1$ factors

- A von Neumann algebra $M$ is a unital *-algebra of bounded operators on Hilbert space $H$ such that $M=M^\prime$. $M$ is a factor if its center $Z(M)=\mathbb{C}$.

- A factor $N$ is II$_1$ if it has a unique trace $\text{tr}: N \to \mathbb{C}$ s.t. $\{\text{tr}(p): p \text{ a projector}\}=[0,1]$.

- The lattice of projectors=lattice of invariant subspaces is a CG.
Qubit $\mathbb{II}_1$ factor

- $M_2(\mathbb{C})=\text{all } 2 \times 2 \text{ matrices, inclusion of } M_2(\mathbb{C}) \text{ to } M_4(\mathbb{C})$
  by $A \mapsto A \quad I$

- Define a normalized trace $\text{tr}(I)=1$, and then complete the union of $M_{2n}(\mathbb{C})$ to a $\mathbb{II}_1$ factor
Jones towers

Given \( \text{II}_1 \) factors \( N \subset M \), Jones construct a tower

\[
N \subset M \subset M_2 \subset \ldots
\]

\( \text{II}_1 \) factor \( M_i \) (\( M_0 = N \), \( M_1 = M \)) is obtained from \( M_{i-1} \) by adjoining a projector

\[
e_i : L^2(M_i, \text{tr}) \rightarrow L^2(M_{i-1}, \text{tr}).
\]

The \( e_i \)'s form the Temperley-Lieb algebras.
Temperley-Lieb algebras

Fix $d$, $TL_n(d)$ is the finite dimensional algebra generated by $1, e_1, \ldots, e_{n-1}$

$$e_i^2 = e_i = e_i^*$$

$$e_i e_j = e_j e_i \text{ if } |i-j| \geq 2$$

$$e_i e_{i \pm 1} e_i = \frac{1}{d} e_i$$
Geometry of TL algebras

- $e_i$’s are projectors

- Images of $e_i$ and $e_j$ are orthogonal modulo their intersection if $|i-j| \geq 2$

- “Angle” between $i$th and $(i+1)$th are determined by $d$. 
Jones Rep of the Braid Groups

The braid group $B_n$ has a presentation:

$$\{1, \sigma_1, \ldots, \sigma_{n-1}\}$$

$$\sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| \geq 2$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

Fix $q = e^{2\pi i/r}$, Jones rep: $\sigma_i \rightarrow q-(1+q)e_i$
Type II$_1$ factors are behind modular tensor categories describing statistics of nonabelian anyons in topological phases of matter, which are pursued as hardware for topological quantum computers. It is also known Type II$_1$ factors are determined by their modular lattices.

What can we learn about the “touchy and complicated” (von Neumann) issue through II$_1$ factors:

What is a physical proposition?
Can we axiomatize projectors of computable traces?
1. Can quantum logics help the construction of a universal quantum computer?

2. Will the interaction of quantum logics and quantum computation result in a more physical quantum framework?
Topological models:

A topological model can be constructed using any Jones representation for any $r$:

Fix $r=5$,

For 1-qubit gates, $\rho_5 : B_4 \rightarrow U(2)$ or $U(3)$

For 2-qubits gates, $\rho_5 : B_8 \rightarrow U(13)$ or $U(21)$
For $n$ qubits, consider the $4n$ punctured disk $D_{4n}$ and
\[ \rho_5 : \mathcal{B}_{4n} \to \mathbb{U}(N_{4n}) \]

Given a quantum circuit on $n$ qubits:

\[ U_L : (C^2)^n \to (C^2)^n \]

Ideally to find a braid $b \in \mathcal{B}_{4n}$ so that the following diagram commutes (almost FKW):

\[ \begin{array}{ccc}
(C^2)^n & \xrightarrow{U_L} & V(D_{4n}) \\
\rho_{CS5}(b) & \downarrow & \downarrow \\
(C^2)^n & \xrightarrow{\rho_{CS5}(b)} & V(D_{4n})
\end{array} \]