

Things I know, but sometimes forget

1 Every finite group acts freely on a product of spheres

Let G be a finite group.

Lemma 1. *For every element $g \in G$, there exists a G -action on a product of spheres X_g where $\langle g \rangle$ acts freely.*

Proof. If Y is an H -space define the G -space $i_!Y = \text{map}_H(G, Y)$. Then if Y is free over H then $i_!Y$ is free over H \square

2 free \times anything = free \times anything else

If X is a G -set, let X_t be the same underlying set with the trivial G -action. Then there is a bijection of G -sets $G \times X_t \rightarrow G \times X$ given by $(g, x) \mapsto (g, gx)$. (Note: $(e, x) \mapsto (e, x)$ and the rest follows by equivariance. More generally, if F is a free G -set, choose a set of orbit representatives $B \subset F$ and define

$$\begin{aligned} F \times X_t &\rightarrow F \times X \\ (gb, x) &\mapsto (gb, gx) \end{aligned}$$

with inverse

$$\begin{aligned} F \times X &\rightarrow F \times X \\ (gb, x) &\mapsto (gb, g^{-1}x) \end{aligned}$$

3 Virtually cyclic groups come in three types

A virtually cyclic group is a group with a cyclic subgroup of finite index. They come in three types: finite, groups which surject to \mathbb{Z} ($F \rtimes \mathbb{Z}$ with F finite), and groups which surject to D_∞ ($G_0 *_F G_1$ with F finite and of index two).

Theorem 2. *Let Γ be an infinite virtually cyclic group.*

1. *If there is a central element of infinite order, then there is an epimorphism $\Gamma \rightarrow \mathbb{Z}$.*
2. *If there is a central element of finite order, then there is an epimorphism $\Gamma \rightarrow D_\infty$.*

Proof. By intersecting the conjugates of an infinite cyclic subgroup, we may find a normal infinite cyclic subgroup C . Let G be the finite quotient group.

1) In this case G acts trivially on C . Embed C as an index $|G|$ subgroup of an infinite cyclic subgroup C' . Let $\Gamma' = C' \times_C \Gamma$. The image of the obstruction cocycle under the map $H^2(G; C) \rightarrow H^2(G; C')$ is trivial, so there exists a splitting $s : \Gamma \rightarrow C'$ of the inclusion $C' \hookrightarrow \Gamma$. Then $s|_\Gamma : \Gamma \rightarrow s(\Gamma)$ is the desired epimorphism.

2) Let $G_0 = \ker(G \rightarrow \text{Aut}C)$ (the map is by lifting to Γ and using that conjugation preserves the normal subgroup.) Let $\Gamma_0 = \pi^{-1}G_0 < \Gamma$. Then there exists an epimorphism $\phi : \Gamma_0 \rightarrow \mathbb{Z}$ by 1). Likewise, $\Phi : \Gamma \rightarrow G \rightarrow G/G_0 \cong \mathbb{Z}_2$ is an epimorphism. Choose $\gamma \in \Gamma$ so that $\phi(\gamma) = 1$. Then $\Gamma = \Gamma_0 \amalg \Gamma_0\gamma$. Define an epimorphism $\varphi : \Gamma \rightarrow \mathbb{Z} \times \mathbb{Z}_2$ by $\varphi(g) = (g, 0)$ and $\varphi(g\gamma) = (g, 1)$ for $g \in \Gamma_0$. □

4 Inner automorphisms often induce identities

4.1 Groups

Recall a group is a category with one object.

Lemma 3. *Let $F : \text{Groups} \rightarrow \text{Ab}$ be a functor. Suppose $F(f) = F(g)$ for any natural transformation $T : f \rightarrow g$ of morphisms of groups. Then for an inner automorphism $c_\gamma : G \rightarrow G$ of a group, $F(c_\gamma) = \text{Id}_{F(G)}$.*

Proof. There is a natural transformation $T : \text{Id}_G \rightarrow c_\gamma$ given by the morphism γ . \square

Corollary 4. *An inner automorphism induces the identity on the homology of a group.*

Proof. Let $(0 \rightarrow 1)$ be the category with two objects and three morphisms, including a morphism from 0 to 1. A natural transformation T of functors $F, F' : \mathcal{C} \rightarrow \mathcal{D}$ induces a functor $(0 \rightarrow 1) \times \mathcal{C} \rightarrow \mathcal{D}$ and conversely.

Let $T : f \rightarrow g$ be a natural transformation of morphisms of groups $f, g : G \rightarrow G'$. This induces a functor $(0 \rightarrow 1) \times G \rightarrow G'$ and hence a homotopy $B(0 \rightarrow 1) \times BG \rightarrow BG'$ from Bf to Bg .

Thus we can apply the Lemma above with $F(G) = H_n(BG)$. \square

4.2 Rings

Proposition 5. *An inner automorphism of a ring R induces the identity on $K_n R$.*

Proof. Let $\gamma \in R^\times$. Consider the functor $c_{\gamma*} : \mathcal{P}(R) \rightarrow \mathcal{P}(R)$ given by $c_{\gamma*}(P) = R \otimes_{c_\gamma} P$. There is an exact natural transformation $T : \text{Id} \rightarrow c_{\gamma*} : \mathcal{P}(R) \rightarrow \mathcal{P}(R)$ given by $P \rightarrow c_{\gamma*}P \quad x \mapsto \gamma^{-1}x$. It induces a functor

$$(0 \rightarrow 1) \times Q(\mathcal{P}(R)) \rightarrow Q(\mathcal{P}(R))$$

and hence a homotopy between the identity and $BQ(c_{\gamma*})$. \square

5 A souped-up Hurewicz Theorem

A space X is n -connected if every map $S^i \rightarrow X$ for $i \leq n$ is null-homotopic. The classical Hurewicz Theorem says that for an n -connected space, $\pi_i X \xrightarrow{\sim} H_i X$ for $i \leq n + 1$.

Theorem 6. *Let $k > 1$. If X is $(k - 1)$ -connected, the Hurewicz map $\pi_{k+1} X \rightarrow H_{k+1} X$ is onto.*

The theorem is not true when $k = 1$. A counterexample is given by $\mathbb{R}P^2 \times \mathbb{R}P^2$.

Proof. First assume X is an Eilenberg-MacLane space $K(G, k)$ with G an abelian group and $k > 1$. There is a short exact sequence of abelian groups

$$0 \rightarrow F' \rightarrow F \rightarrow G \rightarrow 0$$

where F and F' are free abelian groups. (Indeed, find a surjection $\phi : F \rightarrow G$ with F a free abelian group and note that the subgroup $\ker \phi < F$ is itself free abelian.) By choosing bases for F and F' , build a CW complex Y with only a 0-cell, k -cells, and $(k + 1)$ -cells, with $\pi_k Y = G$, and with $H_{k+1} Y = 0$. Build a $K(G, k)$ by adding on cells of dimension $k + 2$ and higher. Then $H_{k+1} K(G, k)$ is a quotient of $\ker(\partial : C_{k+1} Y \rightarrow C_k Y) = \ker(F' \rightarrow F) = 0$.

Now we prove the theorem for a general $(k - 1)$ -connected space X where $k > 1$. Let $G = \pi_k X$. Choose a map $X \rightarrow K(G, k)$ which is the identity on π_k . Let F be the homotopy fiber. Then the Serre exact sequence (which follows from the Serre spectral sequence) gives a long exact sequence

$$\begin{aligned} H_{2k} F \rightarrow H_{2k} X \rightarrow H_{2k} K(G, k) \rightarrow \cdots \\ \rightarrow H_{k+1} F \rightarrow H_{k+1} X \rightarrow H_{k+1} K(G, k) \rightarrow \cdots \end{aligned}$$

There is a commutative diagram

$$\begin{array}{ccc} \pi_{k+1} F & \xrightarrow{\sim} & H_{k+1} F \\ \simeq \downarrow & & \downarrow \\ \pi_{k+1} X & \longrightarrow & H_{k+1} X \end{array}$$

The left map is an isomorphism because of the homotopy exact sequence and the top map is an isomorphism by the Hurewicz Theorem. Since $H_{k+1} K(G, k) = 0$ by our above arguments, the Serre exact sequence shows the right hand map is onto. Thus the bottom map is onto as desired. \square