

Things I should know, but sometimes forget

1 Every finite group acts freely on a product of spheres

Let G be a finite group.

Lemma 1. *For every element $g \in G$, there exists a G -action on a product of spheres X_g where $\langle g \rangle$ acts freely.*

Proof. If Y is an H -space define the G -space $i_!Y = \text{map}_H(G, Y)$. Then if Y is free over H then $i_!Y$ is free over H \square

2 free \times anything = free \times anything else

If X is a G -set, let X_t be the same underlying set with the trivial G -action. Then there is a bijection of G -sets $G \times X_t \rightarrow G \times X$ given by $(g, x) \mapsto (g, gx)$. (Note: $(e, x) \mapsto (e, x)$ and the rest follows by equivariance. More generally, if F is a free G -set, choose a set of orbit representatives $B \subset F$ and define

$$\begin{aligned} F \times X_t &\rightarrow F \times X \\ (gb, x) &\mapsto (gb, gx) \end{aligned}$$

with inverse

$$\begin{aligned} F \times X &\rightarrow F \times X_t \\ (gb, x) &\mapsto (gb, g^{-1}x) \end{aligned}$$

3 Virtually cyclic groups come in three types

A virtually cyclic group is a group with a cyclic subgroup of finite index. They come in three types: finite, groups which surject to \mathbb{Z} ($F \rtimes \mathbb{Z}$ with F finite), and groups which surject to D_∞ ($G_0 *_F G_1$ with F finite and of index two).

Theorem 2. *Let Γ be an infinite virtually cyclic group.*

1. *If there is a central element of infinite order, then there is an epimorphism $\Gamma \rightarrow \mathbb{Z}$.*
2. *If there is no central element of infinite order, then there is an epimorphism $\Gamma \rightarrow D_\infty$.*

Proof. By intersecting the conjugates of an infinite cyclic subgroup, we may find a normal infinite cyclic subgroup C . Let G be the finite quotient group.

1) In this case G acts trivially on C . Embed C as an index $|G|$ subgroup of an infinite cyclic subgroup C' . Let $\Gamma' = C' \times_C \Gamma$. The image of the obstruction cocycle under the map $H^2(G; C) \rightarrow H^2(G; C')$ is trivial, so there exists a splitting $s : \Gamma \rightarrow C'$ of the inclusion $C' \hookrightarrow \Gamma$. Then $s|_\Gamma : \Gamma \rightarrow s(\Gamma)$ is the desired epimorphism.

2) Let $G_0 = \ker(G \rightarrow \text{Aut}C)$ (the map is by lifting to Γ and using that conjugation preserves the normal subgroup.) Let $\Gamma_0 = \pi^{-1}G_0 < \Gamma$. Then there exists an epimorphism $\phi : \Gamma_0 \rightarrow \mathbb{Z}$ by 1). Likewise, $\Phi : \Gamma \rightarrow G \rightarrow G/G_0 \cong \mathbb{Z}_2$ is an epimorphism. Choose $\gamma \in \Gamma$ so that $\phi(\gamma) = 1$. Then $\Gamma = \Gamma_0 \amalg \Gamma_0\gamma$. Define an epimorphism $\varphi : \Gamma \rightarrow \mathbb{Z} \times \mathbb{Z}_2$ by $\varphi(g) = (g, 0)$ and $\varphi(g\gamma) = (g, 1)$ for $g \in \Gamma_0$. □

4 The fibration underlying the Lyndon-Hochschild-Serre Spectral Sequence

Let $1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1$ be a short exact sequence of groups. Consider the principal Q -fibration,

$$\begin{array}{ccc} Q & \longrightarrow & EQ \\ & & \downarrow \\ & & BQ \end{array}$$

Note that Q acts on $(EG)/N$. Thus we can change the fiber and obtain the fibration

$$\begin{array}{ccc} (EG)/N & \longrightarrow & EQ \times_Q (EG)/N \\ & & \downarrow \\ & & BQ \end{array}$$

Finally note that $(EG)/N$ is a model for BN and that $EQ \times_Q (EG)/N = EQ \times_G EG$ is a model for BG . Hence there is a fibration

$$\begin{array}{ccc} BN & \longrightarrow & BG \\ & & \downarrow \\ & & BQ \end{array}$$

whose Serre Spectral Sequence is the Lyndon-Hochschild-Serre Spectral Sequence.