THE SURGERY SEMICHARACTERISTIC

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1. Introduction

Given a degree-one normal map \((f, \tilde{f}): (M^n, v_M) \to (X, \xi)\), C. T. C. Wall defined the associated surgery obstruction \(\sigma(f, \tilde{f}) \in L_n(\mathbb{Z}_1 X)\). This obstruction vanishes if and only if \((f, \tilde{f})\) is normally bordant to a simple homotopy equivalence. Using Wall's approach, one must perform preliminary surgery to make \(f\) highly connected before the obstruction can be calculated. It is natural to ask for invariants which can be calculated without preliminary surgery. In the even-dimensional case the multisignature gives such an invariant.

The purpose of this paper is to present an odd-dimensional normal bordism invariant defined without preliminary surgery. This invariant is analogous to the semicharacteristic bordism invariant introduced by Lee [13]. A justification for the normal bordism invariant is that it gives strong restrictions on the homology of a manifold with a free action of a finite group. In particular, it gives a good explanation of Lee's results on free actions of finite groups on spheres.

Let \(X\) be a \((2n+1)\)-dimensional Poincaré complex with fundamental group \(\pi\). Suppose \(A\) is a semisimple ring with involution \(a \mapsto a^\ast\), for \(a \in A\). Then \(L^n_{2n+1}(A)\) can be identified as a quotient of a subgroup of \(K_0(A)\) (see §2). Given a homomorphism \((\mathbb{Z}_n, \omega) \to (A, -)\), we can regard \(A\) as a local coefficient system (that is, a \(\mathbb{Z}_n\)-module). The surgery semicharacteristic is

\[
\chi_4(\tilde{X}; A) = \sum_{i=0}^n (-1)^i [H_i(X; A)] \in L^n_{2n+1}(A).
\]

Theorem A. Let \((f, \tilde{f}): (M^{2n+1}, v_M) \to (X, \xi)\) be a degree-one normal map. Then \(\text{im} \sigma(f, \tilde{f}) \in L^n_{2n+1}(A)\) is \(\chi_4(\tilde{M}; A) - \chi_4(\tilde{X}; A)\) where \(\tilde{M}\) is the induced \(\pi\) cover of \(M\).

The proof of Theorem A (see §3) proceeds in two stages. First we show that the difference of the semicharacteristics is a normal bordism invariant. Thus \((f, \tilde{f})\) can be assumed to be highly connected. Then the difference of the semicharacteristics is identified with the image of the Wall surgery obstruction.

If \(R = \mathbb{Q}\) or \(\mathbb{Z}_p\) for some prime \(p\) and \(\pi\) is a finite group, there are local surgery theories which measure whether an \(R\)-normal map is \(R\)-normally bordant to an \(R\)-homology equivalence. The surgery obstruction in \(L^n_{2n+1}(R\pi)\) is completely determined as a difference of semicharacteristics (see §4).

The case where \(p = 2\) is of particular interest. Let \(\pi\) be a finite group and let \(F_2\) be the field with two elements. Let \(F_2\pi/\text{rad}\) be the largest semisimple quotient of the group ring \(F_2\pi\).

A theorem of Wall [31] states that the surgery obstruction of a degree-one normal map between two closed manifolds depends only on the 2-Sylow subgroup \(\pi_2\) of \(\pi\).
$F_2 \pi_2$ is a local ring, $\tilde{K}_2(F_2 \pi_2/\text{rad}) = 0$. This implies that $L_{2n+1}^2(F_2 \pi_2/\text{rad}) = 0$. In §6 these facts are used to deduce

**Theorem B.** Let $(f, \tilde{f}) : (M^{2n+1}, v_M) \to (X, \tilde{X})$ be a degree-one normal map. Suppose $X$ has the homotopy type of a closed manifold. If $\tilde{X}$ is a regular cover of $X$ ($|\pi| < \infty$) then

$$\chi_4(\tilde{M} ; F_2 \pi/\text{rad}) = \chi_4(\tilde{X} ; F_2 \pi/\text{rad}).$$

Thus Theorem B gives a necessary condition for a degree-one normal map to exist between two manifolds. Namely, the manifolds must have equal semicharacteristics. But an even more important use is the following. Suppose $X$ is an odd-dimensional Poincaré complex and $(f, \tilde{f}) : (M, v_M) \to (X, \tilde{X})$ is a degree-one normal map. If

$$\chi_4(\tilde{M} ; F_2 \pi/\text{rad}) \neq \chi_4(\tilde{X} ; F_2 \pi/\text{rad})$$

then $X$ cannot have the homotopy type of a manifold.

In §6 we show that Theorem B leads to:

**Theorem C.** Let $\pi$ be a finite group which acts freely on a closed manifold $M$. Then

$$\chi_4(M ; F_2 \pi/\text{rad}) = \chi_4(\pi \times_{\pi_2} M ; F_2 \pi/\text{rad}).$$

In §7 Theorem C is applied to give $L$-theoretic proofs of J. Milnor's and R. Lee's results on the topological spherical space form problem.

**Theorem (Milnor [15]).** If a finite group acts freely on a closed manifold $M$ which is an $F_2$-homology sphere then every element of order 2 is central.

**Theorem (Lee [13]).** The groups $Q(2^n, p, 1)$, with $n > 3$, do not act freely on any closed manifold $M$ such that $H_4(M ; F_2) = H_4(S^{8k+3} ; F_2)$. In particular, they are not the fundamental groups of closed 3-manifolds.

In §8 rational semicharacteristics are used to show:

**Theorem D (Davis, Weinberger).** If a finite group acts freely on a closed manifold $M^{4k+1}$, and trivially on $H_4(M ; \mathbb{Q})$, then either

$$\sum_{i=0}^{2k} (-1)^i \text{rank}_\mathbb{Q} H_i(M ; \mathbb{Q}) \equiv 0 \pmod{2}$$

or the group is the direct product of a cyclic 2-group and a group of odd order.

In particular this applies when $M$ is a rational homology sphere. Thus there are severe restrictions on the local spherical space form problem in dimension $4k+1$.

Much of the material presented here grew out of my doctoral dissertation. It is a pleasure to thank my thesis advisor R. James Milgram. I would also like to thank Andrew Ranicki for useful conversations. Finally, I am grateful to Shmuel Weinberger for convincing me that rational semicharacteristics can be useful.
2. Preliminaries

Let $\Lambda$ be a ring with involution and let $M$ be a left $\Lambda$-module. The dual module $M^* = \text{Hom}_\Lambda(M, \Lambda)$ is a $\Lambda$-module via $(\lambda \phi)(m) = \phi(m)\lambda$. A quadratic form on $M$ is a $\Lambda$-homomorphism $\theta: M \to M^*$. Equivalently, it is a $\mathbb{Z}$-bilinear pairing $\theta: M \times M \to \Lambda$ satisfying

(a) $\theta(m_1, \lambda m_2) = \lambda \theta(m_1, m_2)$,
(b) $\theta(\lambda m_1, m_2) = \theta(m_1, m_2)\lambda$.

If $\theta = (-1)^n \theta^*$, then $\theta$ is $(-1)^n$-symmetric. If, in addition, for all $m \in M$, $\theta(m, m) = \lambda + (-1)^n \lambda$, then $\theta$ is a $(-1)^n$-symmetric even form. If $M$ is projective, this is equivalent to the existence of a quadratic refinement, a quadratic form $\phi: M \to M^*$ such that $\theta = \phi + (-1)^n \phi^*$. If a finitely generated projective module $P$ admits a nonsingular $(-1)^n$-symmetric even form, we call $P$ $(-1)^n$-even. Note that $P \oplus P^*$ is always $(-1)^n$-even as it admits the standard hyperbolic form.

Let $L^n_0(\Lambda)$ denote the free $L$-groups of $\Lambda$. They are a sequence of abelian groups which are 4-periodic, that is, $L^n_n(\Lambda) = L^n_{n+4}(\Lambda)$. Here $L^n_2(M, \Lambda)$ is the Witt group of nonsingular $(-1)^n$-quadratic forms on free $\Lambda$-modules, while $L^n_{2n+1}(\Lambda)$ is the Witt group of $(-1)^n$-quadratic formations on free $\Lambda$-modules. The projective $L$-groups $L^n_0(\Lambda)$ will also be of use. For the precise definition see Ranicki [20] where he writes $U_n(\Lambda)$ for $L^n_0(\Lambda)$ and $W_n(\Lambda)$ for $L^n_0(\Lambda)$.

Suppose $(X, \partial X)$ is a finite Poincaré pair of dimension $n$ with orientation morphism $\omega(X): \pi_1 X \to \mathbb{Z}/2 = \{ \pm 1 \}$. Let $\tilde{X}$ be a regular cover of $X$, let $\pi$ be the group of covering translations, and let $\varphi: \pi, X \to \pi$ be the characteristic map of the covering. Suppose $\omega: \pi \to \mathbb{Z}/2$ is a group morphism such that $\omega(X) = \omega \circ \varphi$. Let $\tilde{\partial}X$ be the $\pi$-cover of $\partial X$ induced by $X$. Then we say $(\tilde{X}, \tilde{\partial}X)$ is oriented with data $(\pi, \omega)$. The case where $\partial X = \emptyset$ is of particular importance.

**Definition 2.1.** The map $(f, \tilde{f}): (M^n, \partial M, v_M) \to (X^n, \partial X, \tilde{v}_M)$ is a surgery problem with data $(\pi, \omega)$ if

(a) $(X, \partial X)$ is a finite Poincaré pair of dimension $n$ with regular cover $(\tilde{X}, \tilde{\partial}X)$ oriented with data $(\pi, \omega)$,
(b) $M$ is a compact manifold of dimension $n$ with boundary $\partial M$ such that the orientation map $\omega(M): \pi_1 M \to \mathbb{Z}/2$ factors as $\omega(M) = \omega(X) \circ f_*$ (we let $M$ and $\partial M$ denote the covers induced by $f$),
(c) $v_M$ is the stable normal bundle of $M$ and $(f, \tilde{f})$ is a degree-one normal map.

Given a surgery problem $(f, \tilde{f}): (M, \partial M, v_M) \to (X, \partial X, \tilde{v}_M)$ with data $(\pi, \omega)$ such that $f: \partial M \to \partial X$ is a homotopy equivalence, there is the associated surgery obstruction $\sigma(f, \tilde{f}) \in L^n_0(\mathbb{Z}/2)$. For its definition see [29]. Here $\mathbb{Z}/2$ has the involution

$$\sum_{g \in \pi} n_g g \to \sum_{g \in \pi} n_g \omega(g) g^{-1}.$$ 

The obstruction $\sigma(f, \tilde{f})$ is an invariant of the normal bordism class rel $\partial X$ of $(f, \tilde{f})$. If $(f, \tilde{f})$ is a homotopy equivalence then $\sigma(f, \tilde{f}) = 0$. If $\tilde{X}$ is the universal cover of $X$ and $n \geq 5$, then $\sigma(f, \tilde{f}) = 0$ if and only if $(f, \tilde{f})$ is normally bordant rel $X$ to a homotopy equivalence. The surgery obstruction is useful for seeing whether $X$ has the homotopy type of a closed manifold and for counting the number of manifolds in a homotopy type.
Let $A$ be a semisimple ring with involution. (Note that we include Artinian in the definition of semisimple. Thus $A$ is a finite direct product of matrix rings over division rings.) From the exact sequence of Ranicki [20],

$$
\cdots \to L^{h}_{2n+2}(A) \to H^{1}(F_2; \hat{K}_{\delta}(A)) \to L^{h}_{2n+1}(A) \to L^{h}_{2n+3}(A) \to \cdots
$$

and the fact that $L^{h}_{2n+1}(A) = 0$ [21], we see that

$$
L^{h}_{2n+1}(A) = \left\{ \left[ P \right] \in \hat{K}_{\delta}(A) \mid \left[ P \right] = \left[ P^{*} \right] \right\} \left( \left[ P \right] \in \hat{K}_{\delta}(A) \mid P \text{ is } (-1)^{n+1}-\text{even} \right) .
$$

If $(H, \phi; K, L) \in L^{h}_{2n+1}(A)$ is a formation, the corresponding projective is $[K \cap L]$. Note that $L^{h}_{2n+1}(A)$ is a finite group of exponent 2, since $P \oplus P^{*}$ is $(-1)^{n+1}$-even.

We wish to compute $L^{h}_{2n+1}(A)$. Now $\{ \left[ P \right] \in \hat{K}_{\delta}(A) \mid \left[ P \right] = \left[ P^{*} \right] \}$ is generated by (a) simple projectives $P$ such that $\left[ P \right] = \left[ P^{*} \right]$, and (b) $P \oplus P^{*}$ where $P$ is a simple projective such that $\left[ P \right] \neq \left[ P^{*} \right]$.

The projectives in Case (b) are $(-1)^{n+1}$-even. The projectives in Case (a) are the simple projectives of involution-invariant simple rings occurring in the Wedderburn decomposition of $A$.

Let $B$ be a simple Artinian ring with involution $\bar{\cdot}$. Let $F$ be the centre of $B$, and suppose $\dim_{F} B = k^{2} < \infty$. If $\bar{\cdot}: F \to F$ is non-trivial, we say $B$ is of Type II. If $\bar{\cdot}: F \to F$ is the identity then the dimension of the $+1$-eigenspace of the involution $\bar{\cdot}: B \to B$ is $\frac{1}{2}k(k+1)$ or $\frac{1}{2}k(k-1)$. (Proof. Tensor over a splitting field and use the Skolem-Noether theorem.) We call these Type $I^{+}$ and Type $I^{-}$ respectively. Append the letter $F$ or $D$ to $I^{+}$ or $I^{-}$ according to whether $B$ is a matrix ring over $F$ or not.

**Theorem 2.3.** Let $B$ be a simple finite-dimensional $F$-algebra with involution. Then simple projectives are $(-1)^{n+1}$-even if and only if

(a) $\text{Char } B \neq 2$ and $B$ is not of Type $I^{\pm}_{-1}F$, or

(b) $\text{Char } B = 2$ and $B$ is not of Type $I^{1}F$.

**Proof.** The proof consists of using Morita theory to reduce to the case where $B$ is a division algebra. See [8] for details.

If $\dim_{F} B = \infty$ then $B$ is a matrix ring over a division ring. Simple projectives are $(-1)^{n+1}$-even.

**Lemma 2.4.** Let $V$ be a finitely generated $K\pi$ module, where $K$ is a subfield of $\mathbb{R}$ and $\pi$ is a finite group. Then $V$ is $(+1)$-even.

**Proof.** Let $\overline{\theta}: V \times V \to K$ be a positive definite symmetric form on the $K$-vector space $V$. Then

$$
\theta(x, y) = \sum_{g \in G} \sum_{h \in G} h^{-1} \overline{\theta}(gx, hy) g
$$

is a symmetric non-singular $K\pi$ form.

**Corollary 2.5.** (a) $L^{0}_{2}(K\pi) = 0$.

(b) A Type $I^{1}F$ simple factor of $K\pi$ is actually Type $I^{1}F$. 
3. The semicharacteristic

Let $X$ be a Poincaré complex with regular cover $	ilde{X}$ oriented with data $(\pi, \omega)$. The set of cellular chains $C_\ast(\tilde{X})$ is a free $\mathbb{Z}\pi$-module. Let $(\mathbb{Z}\pi, \omega) \to (A, -)$ be a mapping of rings with involution. Define

$$H_\ast(X; A) = H_\ast(A \otimes_{\mathbb{Z}\pi} C(\tilde{X})), \quad H^\ast(X; A) = H^\ast(\text{Hom}_{\mathbb{Z}\pi}(C(\tilde{X}), A)).$$

Note that $\text{Hom}_{\mathbb{Z}\pi}(C(\tilde{X}), A)$ is a left $A$-module via $(\varphi)c = \varphi(c)a$. Thus both $H_\ast(X; A)$ and $H^\ast(X; A)$ are left $A$-modules. Duality holds; if $\dim X = n$ then $H^\ast(X; A) \cong H_{n-\ast}(X; A)$.

For the rest of this section let $A$ be a semisimple ring with involution. Then

(a) $H_\ast(X; A)$ and $H^\ast(X; A)$ are projective $A$-modules, and

(b) cohomology is dual to homology, $H^i(X; A) = H_i(X; A)^\ast$.

Let $X$ be a Poincaré complex of dimension $2n + 1$. Note that in $K_0(A)$,

$$\sum_{i=0}^{n} (-1)^i[H_i(X; A)] + \sum_{i=0}^{n} (-1)^i[H_i(X; A)]^\ast = \sum_{i=0}^{2n+1} (-1)^i[H_i(X; A)]$$

$$= \sum_{i=0}^{2n+1} (-1)^i[A \otimes_{\mathbb{Z}\pi} C(\tilde{X})] = 0.$$

**DEFINITION 3.1.** If $X$ is a Poincaré complex of dimension $2n + 1$ then the surgery semicharacteristic is

$$\chi_s(\tilde{X}; A) = \sum_{i=0}^{n} (-1)^i[H_i(X; A)] \in L_{2n+1}^s(A).$$


**PROPOSITION 3.2.** Let $(f, \tilde{f})$: $(W^{2n+2}, \partial W, v_w) \to (Y, \partial Y, \xi)$ be a surgery problem with data $(\pi, \omega)$. Let $(\mathbb{Z}\pi, \omega) \to (A, -)$ be a homomorphism with $A$ semisimple. Then

$$\chi_s(\tilde{W}; A) - \chi_s(\tilde{Y}; A) = 0 \in L_{2n+1}^s(A).$$

**Proof.** Let $K_i$ denote the surgery kernel groups. Consider the exact sequence

$$K_{n+1}(W) \xrightarrow{h} K_{n+1}(W, \partial W) \to K_n(\partial W) \to K_0(\partial W) \to K_0(W, \partial W).$$

Coefficients in $A$ are understood. Let

$$\chi_s = \sum_{i=0}^{n} (-1)^i[K_i(\partial W)] = \chi_s(\tilde{W}; A) - \chi_s(\tilde{Y}; A).$$

A standard argument shows that the Euler characteristic of an exact sequence of projective modules is zero in $K_0(A)$. Thus

$$0 = \sum_{i=0}^{n+1} (-1)^i[K_i(W, \partial W)] - \sum_{i=0}^{n} (-1)^i[K_i(W)] + \chi_s + (-1)^0[\text{im } h]$$

$$= -\chi_s(K(W)) + \chi_s + (-1)^0[\text{im } h].$$
This last equality holds in $K_0(A)/[[P \oplus P^*]] [P] \in K_0(A)$] and follows since

$$K_\ast(W, \partial W) \cong K^{2n+2}_{\ast - i}(W) \cong K_{2n+2 - i}(W)^\ast.$$ 

Now since $K_\ast(W)$ is the homology of a free complex (the algebraic mapping cone of $C(\overline{M}) \to C(\overline{X})$), $\chi(K_\ast(W)) = 0$ in $K_0(A)$. Thus $\chi_\ast = (-1)^{\ast + 1} [\text{im } h]$ in $L_{2n+1}^h(A)$.

We shall see below that the intersection pairing

$$\lambda: K_{n+1}(W; \mathbb{Z} \pi) \times K_{n+1}(W; \mathbb{Z} \pi) \to \mathbb{Z} \pi$$

gives a $(-1)^{\ast + 1}$-symmetric even form. Assume this for the moment. Consider

$$\begin{array}{ccc}
K_{n+1}(W) & \to & K_{n+1}(W)^\ast \\
\downarrow \lambda & & \downarrow \lambda \\
K_{n+1}(W, \partial W) & \phi & \end{array}$$

where $\phi$ is given by Poincaré duality. Write $K_{n+1}(W) = \ker h \oplus \text{im } h$. Note that $\lambda(\ker h) = 0$. Since $\lambda$ is symmetric, $\lambda(\text{im } h) \subset (\text{im } h)^\ast$ and $\lambda: \text{im } h \to (\text{im } h)^\ast$ is in fact an isomorphism since $\phi$ is injective. This shows that $[\text{im } h]$ is $(-1)^{\ast + 1}$-even and hence zero in $L_{2n+1}^h(A)$.

To complete the proof of Proposition 3.2 we need:

**Lemma 3.4.** The intersection pairing

$$\lambda: K_{n+1}(W; \mathbb{Z} \pi) \times K_{n+1}(W; \mathbb{Z} \pi) \to \mathbb{Z} \pi$$

is a $(-1)^{\ast + 1}$-symmetric even form.

**Proof.** We work with the isomorphic pairing

$$\lambda: K^{n+1}(W, \partial W; \mathbb{Z} \pi) \times K^{n+1}(W, \partial W; \mathbb{Z} \pi) \to \mathbb{Z} \pi.$$ 

The lemma follows from Ranicki's algebraic theory of surgery [24, p. 46]. A more elementary proof is sketched below.

Now $K^{n+1}(W, \partial W; \mathbb{Z} \pi)$ is a subgroup of $H^{n+1}_{\text{comp}}(\overline{W}, \partial \overline{W}; \mathbb{Z} \pi)$. (If $\pi$ is infinite, cohomology with compact supports is needed.) Then

$$\lambda(x, y) = \sum_{\text{gen}} (gx \cup y)[\overline{W}]g.$$ 

It is clear that $\lambda$ is $(-1)^{\ast + 1}$-symmetric. To show that $\lambda$ is even it suffices to show that $(gx \cup x)[\overline{W}]$ is even for all $g \in \pi$ such that $g = g^{-1}$.

**Case 1:** $g = e$. The reduction of $(x \cup x)[\overline{W}]$ is $(S^q x)[\overline{W}] = (\nu_{n+1} \cup x)[\overline{W}]$, where $\nu_{n+1}$ is the $(n+1)$th Wu class. But $\nu_{n+1}$ is a characteristic class of the normal bundle, so $\nu_{n+1} = f^*v$ for some $v \in H^{n+1}(\overline{X}, \partial \overline{X}; \mathbb{Z}/2)$. Since $x \in K^{n+1}(\overline{W}, \partial \overline{W}; \mathbb{Z} \pi)$, $(f^*v \cup x)[\overline{W}] = 0$. (Compare Browder [3].)

**Case 2:** $g^2 = e, g \neq e$. In fact, for all $x \in H^{n+1}_{\text{comp}}(\overline{W}, \partial \overline{W}; \mathbb{Z} \pi), (gx \cup x)[\overline{W}] = 0$. One can prove this by modifying the argument of Brumfiel and Milgram [4] to the bounded case with compact coefficients. (See also Lee [13].)

Thus the difference of semicharacteristics is a normal bordism invariant. We now identify $\chi_\lambda(\overline{W}, A) - \chi_\lambda(\overline{Y}; A)$ with the image of the surgery obstruction $\sigma(f, \overline{f})$ under the change of rings map $L_{2n+1}^h(\mathbb{Z} \pi) \to L_{2n+1}^h(A)$. 


THEOREM 3.5. Let \((f, \bar{f}): (M^{2n+1}, v_M) \to (X, \xi)\) be a surgery problem with data \((\pi, \omega)\). Let \((\mathbb{Z}, \omega) \to (A, -)\) be a homomorphism with \(A\) semisimple. Then

\[
\chi_4(M; A) - \chi_4(X; A) = \text{im} \sigma(f, \bar{f}).
\]

Proof. By preliminary surgeries, we can assume \(f\) is \(n\)-connected. Then \(\sigma(f, \bar{f})\) is represented by the formation \((K_d(\partial U), \lambda, \mu; K_{n+1}(U, \partial U), K_{n+1}(M_0, \partial U))\). From \([29, \S 6]\) comes the braid diagram:

\[
\begin{array}{ccc}
0 & \longrightarrow & K_{n+1}(M, M_0; A) = K_{n+1}(U, \partial U; A) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & K_n(M; A) = K_n(M_0, \partial U; A)
\end{array}
\]

Thus

\[
\text{im} \sigma(f, \bar{f}) = K_{n+1}(U, \partial U; A) \cap K_{n+1}(M_0, \partial U; A) = K_{n+1}(M; A) = K_d(M; A)^*.
\]

In \(L_{2n+1}^*(A)\),

\[
K_d(M; A)^* = K_d(M; A) = \chi_4(M; A) - \chi_4(X; A).
\]

4. Local surgery theory

If \(R = \mathbb{Q}\) or \(\mathbb{Z}/p\) (the localization of \(\mathbb{Z}\) at the prime \(p\)), there are local surgery theories which measure whether an \(R\)-normal map is \(R\)-normally cobordant to an \(R\pi\)-homology equivalence. In more detail, an \(R\)-Poincaré complex is a finite \(CW\)-complex with fundamental class \([X] \in H_d(X; R)\) inducing Poincaré duality

\[
\bigcap[X]: H^d(X; R\pi_1 X) \xrightarrow{\sim} H_{d-}(X; R\pi_1 X).
\]

An \(R\)-normal map is a normal map \((f, \bar{f}): (M^*, v_M) \to (X, \xi)\) from a closed manifold \(M^*\) to an \(R\)-Poincaré complex \(X\) inducing an isomorphism

\[
f_*: H_d(M^*; R\pi_1 X) \xrightarrow{\sim} H_d(X; R\pi_1 X).
\]

THEOREM 4.1. If \((f, \bar{f}): (M^*, v_M) \to (X, \xi)\) is an \(R\)-normal map, there is a surgery obstruction \(\sigma(f, \bar{f}) \in L^*_n(R\pi_1 X)\) which vanishes if and only if \((f, \bar{f})\) is \(R\)-normally cobordant to an \(R\pi\)-homology equivalence.

The above theorem is due independently to Pardon [17] and Geist [9] in the case where \(R = \mathbb{Q}\) and \(\pi_1 X\) finite. For the general case see Anderson [1] and the remarks of Pardon [18].

Let \(F_p\) denote the finite field with \(p\) elements. Let \(\pi\) be a finite group, and let \(F_p\pi/\text{rad}\) be the largest semisimple quotient of \(F_p\pi\).

PROPOSITION 4.2. The map \(L_{2n+1}^b(\mathbb{Z}/p\pi) \to L_{2n+1}^b(F_p\pi/\text{rad})\) is injective.

Proof. See Pardon [19, 3.10 and 3.8].

Thus in odd dimensions, the local surgery obstruction lies in a semisimple ring. We now interpret this as a difference of semicharacteristics.
THEOREM 4.3. Let \((f, \tilde{f}) : (M^{2n+1}, \nu_{M}) \to (X, \xi)\) be an \(R\)-normal map with \(\pi = \pi_{1}X\) finite. Then \((f, \tilde{f})\) is \(R\)-normally cobordant to an \(R\pi\)-homology equivalence if and only if 
\[
\chi_{4}(\tilde{M} ; R\pi/\rad) - \chi_{4}(\tilde{X} ; R\pi/\rad) = 0 \text{ in } L_{2n+1}^{2}(R\pi/\rad).
\]

Proof. Indeed \(L_{2n+1}^{2}(R\pi) \to L_{2n+1}^{2}(R\pi/\rad)\) is injective and \(\im(\sigma(f, \tilde{f}))\) can be identified exactly as in §3.

The above theorem was known to Pardos [17] in the case where \(R = \mathbb{Q}\).

5. An algebraic interlude

This section develops some of the algebra needed for the computations in §8. We also show that for odd primes the \(p\)-local semicharacteristics are zero for integral surgery problems.

DEFINITION 5.1. Let \(\Lambda\) be a ring. The Jacobson radical of \(\Lambda\) is

\[
\rad \Lambda = \bigcap_{M} \text{Ann}(M),
\]

where the intersection is taken over all simple left \(\Lambda\)-modules \(M\). Here \(\text{Ann}(M) = \{r \in \Lambda \mid rM = 0\}\).

Other definitions equivalent to Definition 5.1 are given in [25]. If \(\Lambda\) is a finitely generated algebra over a commutative local ring, \(\Lambda/\rad\) is the largest semisimple quotient of \(\Lambda\). We use the shorthand notation \(\Lambda/\rad\) for \(\Lambda/\rad\Lambda\).

There are two immediate consequences of Definition 5.1. First, if \(\varphi : A \to B\) is a surjective map of rings, \(\varphi(\rad A) \subset \rad B\). Second, if \(I\) is a two-sided ideal of \(\Lambda\) contained in \(\rad \Lambda\), then \(\Lambda/\rad = (\Lambda/I)/\rad\).

Let \(\pi\) be a finite group and \(p\) a prime number. Let \(\mathbb{Z}_{p}\) denote the \(p\)-adic integers. Nakayama's lemma [25] implies that \(p \in \rad \mathbb{Z}_{p}\pi\) and \(p \in \rad \mathbb{Z}_{(p)}\pi\). Thus \(\mathbb{Z}_{p}\pi/\rad = F_{p}\pi/\rad = \mathbb{Z}_{(p)}\pi/\rad\).

LEMMA 5.2. If \(H\) is a normal subgroup of \(\pi\) with \(|H| = p^{r}\), then \(F_{p}\pi/\rad = F_{p}[\pi/H]/\rad\).

Proof. Let \(I = \ker(F_{p}\pi \to F_{p}[\pi/H])\). Here \(I\) is the two-sided ideal of \(F_{p}\pi\) generated by \(\{h-1 \mid h \in H\}\). Let \(M\) be a simple left \(F_{p}\pi\)-module. Then \(M\) is an \(F_{p}H\)-module with \(H\) a \(p\)-group, so there is a \(v \in M\) such that \(hv = v\) for all \(h \in H\). For a proof see [26, Proposition 26]. So

\[
((h-1) \sum_{g \in \pi} a_{g}g)v = \sum_{g \in \pi} a_{g}g(g^{-1}hg - 1)v = 0,
\]

since \(H\) is normal. It follows that \(I \subset \text{Ann}(M)\). Hence \(I \subset \rad F_{p}\pi\). Hence \(F_{p}\pi/\rad = (F_{p}\pi/I)/\rad = F_{p}[\pi/H]/\rad\).

COROLLARY 5.3. If \(|H| = p^{r}\) and \((|\pi/H|, p) = 1\), then \(F_{p}\pi/\rad = F_{p}[\pi/H]\).

Proof. By Maschke's theorem \(F_{p}[\pi/H]\) is semisimple.
THEOREM 5.4 (Wall [30]). The maps \( L_n^h(\mathbb{Q}_p) \rightarrow L_0^h(F_p\pi) \rightarrow L_n^h(F_p\pi/\text{rad}) \) are isomorphisms.

Recall that the maps \( \bar{K}_0(\mathbb{Z}\pi) \rightarrow \bar{K}_0(\mathbb{Q}\pi) \) and \( \bar{K}_0(\mathbb{Z}\pi) \rightarrow \bar{K}_0(F_p\pi/\text{rad}) \) are zero [28].

THEOREM 5.5. If \( A = \mathbb{Q}\pi \) or \( F_p\pi/\text{rad} \) with \( p \) odd, then \( L_{2n+1}^h(\mathbb{Z}\pi) \rightarrow L_{2n+1}^h(A) \) is zero. Also \( L_n^h(\mathbb{Z}\pi) \rightarrow L_n^h(F_p\pi/\text{rad}) \) is zero.

Proof. For the case where \( A = \mathbb{Q}\pi \) see Carlsson and Milgram [5]. Wall's arithmetic square

\[
\begin{array}{ccc}
\mathbb{Z}\pi & \rightarrow & \mathbb{Q}\pi \\
\downarrow & & \downarrow \\
\mathbb{Z}\pi & \rightarrow & \mathbb{Q}\pi \\
\end{array}
\]

gives rise to a long exact sequence

\[ \cdots \rightarrow L_1(\mathbb{Z}\pi) \rightarrow L_1^h(\mathbb{Z}\pi) \oplus L_1^h(\mathbb{Q}\pi) \rightarrow L_1^h(\mathbb{Q}\pi) \rightarrow L_2^h(\mathbb{Z}\pi) \rightarrow \cdots. \]

See [22] for example. Here \( \mathbb{Z} = \bigcup_p \mathbb{Z}_p \) and \( \mathbb{Q} = \mathbb{Q} \otimes \mathbb{Z} \). From the result where \( A = \mathbb{Q}\pi \), we get exact sequences

\[ L_{2n+1}^h(\mathbb{Z}\pi) \rightarrow L_{2n+1}^h(\mathbb{Z}_p\pi) \rightarrow L_{2n+1}^h(\mathbb{Q}_p\pi) \]

for all primes \( p \). The right-hand map is injective for \( p \) odd or for \( n = 0 \) and \( p = 2 \) [5].

COROLLARY 5.7. The maps \( L_{2n+1}^h(\mathbb{Z}\pi) \rightarrow L_{2n+1}^h(\mathbb{Q}\pi) \), \( L_{2n+1}^h(\mathbb{Z}\pi) \rightarrow L_{2n+1}^h(\mathbb{Z}_p\pi) \) (with \( p \) odd), and \( L_n^h(\mathbb{Z}\pi) \rightarrow L_n^h(\mathbb{Z}_p\pi) \) are all zero.

Thus for integral surgery problems (i.e. over \( \mathbb{Z}\pi \)), semicharacteristics are only interesting when \( p = 2 \). Even in this case they give limited information about \( L_{2n+1}^h(\mathbb{Z}\pi) \). For example, if the 2-Sylow subgroup \( \pi_2 \) of \( \pi \) is normal, then there is the following commutative diagram:

\[
\begin{array}{ccc}
L_{2n+1}^h(\mathbb{Z}\pi) & \rightarrow & L_{2n+1}^h(\mathbb{Z}[\pi/\pi_2]) \\
\downarrow & & \downarrow \\
L_{2n+1}^h(F_2\pi/\text{rad}) & = & L_{2n+1}^h(F_2[\pi/\pi_2])
\end{array}
\]

Since \( \pi/\pi_2 \) is of odd order [2], \( L_{2n+1}^h(\mathbb{Z}[\pi/\pi_2]) = 0. \)

6. The case where \( p = 2 \)

This case is especially interesting due to its relation to surgery problems on closed manifolds.

THEOREM 6.1. Let \( X \) be a closed manifold of dimension \( 2n+1 \). Let \((f, \bar{f}): (M, \nu_M) \rightarrow (X, \xi)\) be a surgery problem with data \((\pi, \omega)\). If \( \pi \) is finite, \( \text{im} \sigma(f, \bar{f}) \in L_{2n+1}^h(F_2\pi) \) is zero.
Proof. Now \((f, \tilde{f})\) gives rise to a map \(\phi: \tilde{X} \to B\pi \times G/TOP\) where \(B\pi = K(\pi, 1)\). The surgery obstruction \(\sigma(f, \tilde{f})\) depends only on the bordism class of \(\phi\) in \(\Omega_{2n+1}(B\pi \times G/TOP)\). (See Wall [29].) Let \(\pi_2\) be a Sylow \(2\)-subgroup of \(\pi\). There is the commutative diagram

\[
\begin{array}{ccc}
\mathfrak{M}_{2n+1}(B\pi \times G/TOP) & \longrightarrow & L^h_{2n+1}(F_2) \\
\downarrow \text{res} & & \downarrow \text{res} \\
\mathfrak{M}_{2n+1}(B\pi_2 \times G/TOP) & \longrightarrow & L^h_{2n+1}(F_2\pi) \\
\downarrow \text{ind} & & \downarrow \text{ind} \\
\mathfrak{M}_{2n+1}(B\pi \times G/TOP) & \longrightarrow & L^h_{2n+1}(F_2) \\
\end{array}
\]

Bordism theory shows that

\[\text{ind} \circ \text{res}: \mathfrak{M}_{2n+1}(B\pi \times G/TOP) \to \mathfrak{M}_{2n+1}(B\pi \times G/TOP)\]

is the identity. Since

\[L^h_{2n+1}(F_2\pi) = L^h_{2n+1}(F_2) = 0,\]

the map \(\mathfrak{M}_{2n+1}(B\pi \times G/TOP) \to L^h_{2n+1}(F_2)\) is zero.

Remark 6.2. The above theorem is due essentially to Wall [31], where he gives more general results.

Corollary 6.3. Let \((f, \tilde{f}): (M^{2n+1}, v_M) \to (X, \xi)\) be a surgery problem with data \((\pi, \omega)\). If \(\pi\) is finite and \(X\) has the homotopy type of a closed manifold, then

\[\chi_4(M; F_2\pi/\text{rad}) = \chi_4(\tilde{X}; F_2\pi/\text{rad}).\]

Before we can state our next theorem, some notation is needed. Suppose \(H\) is a subgroup of \(\pi\) and \(H\) acts on a space \(Y\). Let

\[\pi \times_H Y = \frac{\pi \times Y}{(gh, y) \sim (g, hy)}\]

for \(g \in \pi, h \in H,\) and \(y \in Y\). Then \(\pi\) acts on \(\pi \times_H Y\).

The importance of the following theorem is that it gives a necessary homology condition for the existence of a free group action on a manifold.

Theorem 6.4. Suppose a finite group \(\pi\) acts freely on a closed manifold \(M\) of odd dimension. Then

\[\chi_4(M; F_2\pi/\text{rad}) = \chi_4(\pi \times M; F_2\pi/\text{rad}).\]

Proof. Let \(\pi_p\) denote the Sylow \(p\)-subgroup of \(\pi\). Let \(|\pi| = 2^n p_1^{a_1} \ldots p_r^{a_r}\) be the prime factorization of \(|\pi|\). Write \(q_i = p_i^{b_i}\) and \(q_1 \ldots q_i \ldots q_r\) for \(q_1 \ldots q_i \ldots q_{i+1} \ldots q_r\). The numbers \(q_1 \ldots q_r, 2^{a_1} q_1 \ldots q_i \ldots q_r, (i = 1, \ldots, r)\) have no common factor. Thus there exist integers \(b, b_1, b_2, \ldots, b_r\) with the \(b_i\) even, such that
We then construct a degree-one normal map

\[ b(M/\pi_j) \prod_{i=1}^r b(M/\pi_{p_i}) \to M/\pi, \]

where \( b_i(M/\pi_{p_i}) \) denotes \( b_i \) disjoint copies of \( M/\pi_{p_i} \) with the orientation reversed if \( b_i \) is negative. The map is the covering map. It has degree 1. Since a covering map is covered by a map of tangent bundles, it must also be covered by a map of stable normal bundles. Now \( L_{2n+1}^3(F_n/\pi) \) has exponent 2. Since \( b \) is odd and the \( b_i \) are even, Corollary 6.3 gives

\[ \chi(M ; F_n/\pi) = \chi(\pi \times n; M ; F_n/\pi). \]

One last thing to note about the case in which \( p = 2 \) is that the difference of the semicharacteristics in \( L_{2n+1}^3(F_n/\pi) \) is in fact an invariant of projective surgery theory \( (L_{2n+1}^3(\mathbb{Z} \pi)) \).

7. Application to the space form problem

The 'topological spherical space form problem' or, more briefly, 'space form problem' is the study of fixed-point free actions of finite groups on spheres. Equivalently, it is the study of space forms, i.e. manifolds whose universal cover is the sphere. In dimension 3, the space form problem is closely related to the classification of all closed 3-manifolds with finite fundamental group, as any such manifold is the quotient of a free action on a homotopy sphere. A survey of the space form problem is given in [7].

A spectral sequence argument gives

**Theorem 7.1** [6]. If a finite group \( \pi \) acts freely on \( S^{n-1} \), then \( H^n(\pi ; \mathbb{Z}) = \mathbb{Z}/|\pi| \).

If \( H^n(\pi ; \mathbb{Z}) = \mathbb{Z}/|\pi| \) for some \( n \), then \( \pi \) is called a periodic group. The period of \( \pi \) is \( \min\{m > 0 | H^n(\pi ; \mathbb{Z}) = \mathbb{Z}/|\pi| \} \}. The converse of Theorem 7.1 is of interest: given a group \( \pi \) of period \( n \), does \( \pi \) act freely on \( S^{n-1} \)?

The basic negative results are the theorems of Milnor and Lee mentioned in the introduction. They involve the dihedral groups

\[ D_{2n} = \langle x, z | x^2 = z^n = 1, xzx^{-1} = z^{-1} \rangle \]

and the groups

\[ Q(2^n a_1, a_2, a_3) = \langle x, y, z_1, z_2, z_3 | x^{2n^2} = (xy)^2 = y^2, z_1^{a_1} = 1, \]

\[ xz_1 x^{-1} = z_1, xz_2 x^{-1} = z_2^{-1}, \]

\[ xz_3 x^{-1} = z_3^{-1}, yz_1 y^{-1} = z_1^{-1}, \]

\[ yz_2 y^{-1} = z_2, yz_3 y^{-1} = z_3^{-1} \rangle. \]

Here \( n \geq 3 \) and \( a_1, a_2, a_3 \) are relatively prime odd integers. The notation \( Q(2^n a_1, a_2, a_3) \) is due to Milnor [15]. We note the following properties. If \( b_i \) \( \mid a_i \), then \( Q(2^n b_1, b_2, b_3) \) is a subgroup of \( Q(2^n a_1, a_2, a_3) \). Also \( Q(2^n b_1, b_2, b_3) \cong Q(2^n b_1, b_2, b_3) \). When \( n = 3 \), \( Q(2^n b_1, b_2, b_3) \cong Q(2^n b_2, b_3, b_1) \).
The following theorem was a triumph of surgery theory.

**Theorem 7.2** (Madsen, Thomas, and Wall [14]). Let π be a group of period n. If π contains no dihedral subgroups, π acts freely on $S^{2n-1}$. If, in addition, π contains no subgroups of the form $Q(2^n, p, 1)$, with $n > 3$, and none of the form $Q(8p, q, 1)$, then π acts freely on $S^{n-1}$.

The 2-period of $D_{2r}$ (r odd) and $Q(2^n, a_1, a_2, a_3)$ is 4. Thus they cannot act freely on an $F_2$-homology sphere of dimension n if $n \equiv 3 \mod 4$. The groups $Q(2^n, a_1, a_2, a_3)$ actually act freely and orthogonally on $S^{8k+7}$, where $k \geq 0$.

The purpose of this section is to use the surgery semicharacteristic to prove:

**Theorem 7.3.** The dihedral groups $D_{2p}$, with p an odd prime, cannot act freely on a closed manifold $M$ if $H_*(M; F_2) = H_*(S^{2k+1}; F_2)$.

**Theorem 7.4.** The groups $Q(2^n, p, 1)$, with $n > 3$ and p an odd prime, cannot act freely on a closed manifold $M$ if $H_*(M; F_2) = H_*(S^{2k+3}; F_2)$, for $k \geq 0$.

**Corollary 7.5** (Milnor). If a finite group π acts freely on a closed manifold which is an $F_2$-homology sphere then π has a unique element of order 2.

*Proof.* If $x$ and $y$ are distinct elements of order 2, the $x$ and $xy$ generate a dihedral subgroup. (I thank Frank Connolly for this argument.)

The method of proof of Theorems 7.3 and 7.4 is clear. Assume there exists a free action, and calculate the semicharacteristics $\chi_4(M; F_2)$ and $\chi_4(\pi \times_{(\pi; F_2)}; F_2(\pi/\text{rad})$. If they are not equal, then Theorem 6.4 gives a contradiction to the existence of a free action. The calculations are quite similar to those of Lee [13], and the reader may wish to compare the two. We have

$$Q[D_{2p}] = Q \oplus Q \oplus M_2(Q[\lambda_p]),$$

where $\lambda_p$ is a primitive pth root of unity and $\lambda_p = \zeta_p + \zeta_p^{-1}$. Define

$$F_2[\lambda_p] = \mathbb{Z}[\lambda_p]/2.$$

Then $F_2[\lambda_p]$ is a direct product of fields, depending on the factorization of the minimal polynomial of $\lambda_p$ in the field $F_2$. The central idempotent $(1 + z + \ldots + z^{p-1})/p$ gives a decomposition

$$F_2[D_{2p}] = F_2[\mathbb{Z}/2] \oplus M_2(F_2[\lambda_p]).$$

Now $M_2(F_2[\lambda_p])$ is semisimple, so

$$F_2[D_{2p}] / \text{rad} = F_2 \oplus M_2(F_2[\lambda_p]).$$

Let $A = F_2[D_{2p}] / \text{rad}$. The involution on the centre of $A$ is trivial, so there are no even non-singular forms on the simple summands of $A$. Thus

$$L_{\alpha \text{odd}}(A) = \tilde{K_0}(A)/2\tilde{K_0}(A).$$

Let

$$V = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \mid a, b \in F_2[\lambda_p] \right\}.$$
Then \( A = F_2 \oplus V \oplus V \) as a left \( A \)-module. In particular, \([F_2] = 0 \in \text{L}^h_{\text{odd}}(A)\) and \([V] \neq 0 \in \text{L}^h_{\text{odd}}(A)\).

Now \( F_2[D_{2p}] \) deviates from semisimplicity only at the base summand \( F_2[\mathbb{Z}/2] \); that summand is irrelevant for dealing with \( \text{L}^h_{\text{odd}} \). If \( W \) is a finitely generated \( F_2[D_{2p}] \)-module, write \([W]\) for \([M_2(F_2[\mathbb{Z}/2])W] \in \text{L}^h_{\text{odd}}(A)\). If \( X \) is a finite complex with a free \( D_{2p} \)-action, then \([H(X/D_{2p}; A)] = [H(X; F_2)]\).

Suppose \( D_{2p} \) acts freely on \( M^{2k+1} \), an \( F_2 \)-homology sphere. Then
\[
\chi_4(M; A) = [H_0(M; F_2)] = 0 \in \text{L}^h_{2k+1}(A).
\]

Thus
\[
\chi_4(D_{2p} \times \mathbb{Z}/2 M; A) = [H_0(D_{2p} \times \mathbb{Z}/2 M; F_2)]
= [F_2[D_{2p}] \otimes_{F_2[\mathbb{Z}/2]} H_0(M; F_2)]
= [F_2 \oplus V]
= [V] \neq 0 \in \text{L}^h_{2k+1}(A).
\]

This completes the proof of Theorem 7.3.

Consider the exact sequence
\[
1 \to Q(2^{n-1}) \to Q(2^n, p, 1) \to D_{2p} \to 1.
\]
Here \( \langle x^2, y \rangle \simeq Q(2^{n-1}) \), the generalized quaternion group of order \( 2^{n-1} \). We are in luck; for \( Q(2^{n-1}) \) is a normal 2-group and so \( F_2[Q(2^n, p, 1)]/\text{rad} = F_2[D_{2p}]/\text{rad} = A \). If \( Q(2^n, p, 1) \) acts freely on a finite complex \( X \), then
\[
[H_i(X/Q(2^n, p, 1); A)] = [H_i(X/Q(2^{n-1}); F_2)] \in \text{L}^h_{\text{odd}}(A).
\]

Suppose \( Q(2^n, p, 1) \) acts freely on \( M \) with \( H_i(M; F_2) = H_i(S^{2k+1}; F_2) \). Then \( H_i(M/Q(2^{n-1}); F_2) = H_i(Q(2^{n-1}); F_2) \) for \( i < 8k+3 \). From [6],
\[
H_i(Q(2^{n-1}); F_2) = \begin{cases} F_2 & \text{if } i \equiv 0, 3 \pmod{4}, \\ (F_2)^2 & \text{if } i \equiv 1, 2 \pmod{4}. \end{cases}
\]

Thus
\[
\chi_4(M; F_2) = \sum_{i=0}^{4k+1} [H_i(M/Q(2^{n-1}); F_2)] = 0 \in \text{L}^h_{3}(A).
\]

Also
\[
\chi_4(Q(2^n, p, 1) \times Q(2^{n-1}); M; A) = \chi_4(D_{2p} \times \mathbb{Z}/2(M/Q(2^{n-1}); A))
= \sum_{i=0}^{4k+1} [F_2[D_{2p}] \otimes_{F_2[\mathbb{Z}/2]} H_i(M/Q(2^{n-1}); F_2)]
= (6k+3)[F_2[D_{2p}] \otimes_{F_2[\mathbb{Z}/2]} F_2]
= (6k+3)[V] \neq 0 \in \text{L}^h_{3}(A).
\]

This completes the proof of Theorem 7.4.

The group \( Q(8, p, 1) \simeq Q(8p, 1, 1) \) is a binary dihedral group and does act freely on \( S^1 \). In fact, there is no semicharacteristic obstruction to a free action of \( Q(8a_1, a_2, a_3) \) on a \( \mathbb{Z}/2 \)-homology sphere of dimension \( 8k+3 \). Indeed such actions exist [18].
When re-proving an old result, it is necessary to comment on the differences between the old and new proof. In [13], an odd-dimensional bordism invariant was defined, the Lee semicharacteristic:

\[ \chi_4 : \mathfrak{H}_{2n+1}(B\pi) \to \tilde{K}_{G,ad}(\pi). \]

Lee also considered \( M/\mathbb{Q}(2^{n-1}) \), but its appearance in his proof seemed totally ad hoc. However, in the formulation given here this is quite natural. Namely, since \( \mathbb{Q}(2^{n-1}) \) is a normal 2-subgroup of \( G = \mathbb{Q}(2^n, p, 1) \), \( F_2G/\text{rad} = F_2[G/\mathbb{Q}(2^{n-1})]/\text{rad}. \)

8. Free actions on manifolds of dimension \( 4k + 1 \)

This section contains joint work with Shmuel Weinberger. Its aim is to prove:

**Theorem 8.1.** Let \( \pi \) be a finite group acting freely on a closed manifold \( M \) of dimension \( 4k + 1 \). If the induced action of \( \pi \) on \( H_*(M; \mathbb{Q}) \) is trivial, then either

\[ \hat{\chi}_4(M; \mathbb{Q}) = \sum_{i=0}^{2k} (-1)^i \text{rank}_\mathbb{Q} H_i(M; \mathbb{Q}) \]

is even, or \( \pi \) is the direct product of a cyclic 2-group and a group of odd order.

**Lemma 8.2.** If \( \mathbb{Z}/2 \times \mathbb{Z}/2 \) acts freely on a closed manifold \( M \) of dimension \( 4k + 1 \), then \( \hat{\chi}_4(M; \mathbb{Q}) \) is even.

**Proof.** Let \( g_1 \) and \( g_2 \) be generators of \( \pi = \mathbb{Z}/2 \times \mathbb{Z}/2 \). Consider the degree-one normal map

\[ M/\pi \sqcup M/\langle g_1 \rangle \sqcup M/\langle g_2 \rangle \to M/\pi. \]

Since \( L^*_1(\mathbb{Z}\pi) \to L^*_1(\mathbb{Q}\pi) \) is the zero map, we have

\[ \chi_4(\pi \times \langle g_1 \rangle M; \mathbb{Q}\pi) - \chi_4(\pi \times \langle g_2 \rangle M; \mathbb{Q}\pi) = \sum_{i=0}^{2k} (-1)^i [\mathbb{Q}\pi \times \mathbb{Q}\langle g_i \rangle H_i(M; \mathbb{Q})] \]

\[ - \sum_{i=0}^{2k} (-1)^i [\mathbb{Q}\pi \otimes \mathbb{Q}\langle g_i \rangle H_i(M; \mathbb{Q})] \]

\[ = 0 \in L^*_1(\mathbb{Q}\pi). \]

The theory of \( \S 2 \) shows that

\[ L^*_1(\mathbb{Q}\pi) = \bar{K}_0(\mathbb{Q}\pi)/2\bar{K}_0(\mathbb{Q}\pi). \]

It follows that \( \hat{\chi}_4(M; \mathbb{Q}) \equiv 0 \) (mod 2).

**Corollary 8.3.** If \( \pi \) is a 2-group which acts freely on a closed manifold \( M \) of dimension \( 4k + 1 \) with \( \hat{\chi}_4(M; \mathbb{Q}) \) odd, then \( \pi \) is cyclic.

**Proof.** We will prove this by induction on the order of \( \pi \). Let \( H \) be a normal subgroup of \( \pi \) with \( \pi/H \cong \mathbb{Z}/2 \). By the induction hypothesis, \( H = \langle g \rangle \) is cyclic. Let \( \rho \in \pi - H \). Then \( \rho^2 = g \), for some \( i \). Now \( \pi/\langle g^i \rangle \) acts freely on \( M/\langle g \rangle \). A transfer argument shows that \( \text{rank}_\mathbb{Q} H_i(M; \mathbb{Q}) \equiv \text{rank}_\mathbb{Q} H_i(M/\langle g \rangle; \mathbb{Q}) \) (mod 2). Thus Lemma 8.2 implies that \( \pi/\langle g \rangle \cong \mathbb{Z}/2 \) and hence \( \pi \) is cyclic with generator \( \rho \).
Remark 8.4. A close look at the proof of Lemma 8.2 shows that the corollary holds even when the action of $\pi$ on $H_a(M; \mathbb{Q})$ is non-trivial.

Lemma 8.5. If a dihedral group $D_{2p}$ acts freely on a closed manifold $M$ of dimension $4k+1$, and acts trivially on rational homology, then $\hat{\chi}_4(M; \mathbb{Q})$ is even.

Proof. Without loss of generality we may assume that $p$ is an odd prime. Now $$\mathbb{Q}[D_{2p}] = \mathbb{Q} \oplus \mathbb{Q} \oplus M_2(\mathbb{Q}(\lambda_p)).$$ Let $V$ be a simple projective summand of $M_2(\mathbb{Q}(\lambda_p))$. Then $$L^3_1(\mathbb{Q}[D_{2p}]) = \bar{K}_0(\mathbb{Q}[D_{2p}])/\langle 2\bar{K}_0(\mathbb{Q}[D_{2p}]), 2[V] \rangle,$$ and hence is $(\mathbb{Z}/2)^2$. By considering the degree-one normal map $$M/\mathbb{Z}/2 \to (p-1)/2 M/\mathbb{Z}/p \to M/D_{2p},$$ one sees that $\hat{\chi}_4(M; \mathbb{Q})[V] = 0 \in L^3_1(\mathbb{Q}[D_{2p}]).$

Remark 8.6. The conclusion of the lemma is not necessarily true if $\mathbb{Z}/p \subset D_{2p}$ acts non-trivially on $H_a(M; \mathbb{Q})$. An example is given by $D_{2p} 	imes \mathbb{Z}/2 S^5$ where $\mathbb{Z}/2$ acts on $S^5$ via the antipodal action.

Proof of Theorem 8.1. Let $\pi_2$ be the Sylow 2-subgroup of $\pi$. First note that if $\pi_2$ is cyclic normal, then $\pi = \pi_2 \times H$ with $H$ of odd order. Indeed we have $$1 \to \pi_2 \to \pi \to H \to 1.$$ The map $\pi \to H$ splits since $(2, |H|) = 1$. The map $H \to \text{Aut}(\pi_2)$ is zero since $\text{Aut}(\pi_2)$ is of even order.

If $\pi_2$ is not cyclic, then $\hat{\chi}_4(M; \mathbb{Q})$ is even by Corollary 8.3. Now assume $\pi_2 = \mathbb{Z}/2^a = \langle g \rangle$, but is not normal. Let $$i = \min \{ j | \langle g^2^j \rangle \text{ is normal in } \pi \}.$$ Let $h$ be an element of $\pi$ which does not normalize $\langle g^{2i} \rangle$. Then $hg^{2i-1}h^{-1}, g^{2i-1}$ generate a dihedral subgroup of $\pi/\langle g^{2^i} \rangle$ which acts freely on $M/\langle g^{2^i} \rangle$. Hence by Lemma 8.5, $\hat{\chi}_4(M; \mathbb{Q}) = \hat{\chi}_4(M/\langle g^{2^i} \rangle; \mathbb{Q})$ is even.

Remark 8.7. Weinberger [32] has proved a converse to Theorem 8.1. Let $M^{4k+1}$ be a closed manifold whose symmetric signature $\sigma^s(M) \in L^1_1(\mathbb{Q} \pi)$ is zero. Then if $\hat{\chi}_4(M; \mathbb{Q})$ is even or if $\pi = \mathbb{Z}/2 \times H_{\text{odd}}$, he shows that there is a free $\pi$-action on a closed manifold $N$, which is $\mathbb{Q}[\pi_1 M]$-homology equivalent to $M$. He also gives alternative derivations of Lemma 8.2 based on previous work of Stong [27].

9. Connections with the Lee semicharacteristic

Let $\pi$ be a finite group. Quadratic forms $$\theta: V \times V \to F_2 \pi$$ on an $F_2 \pi$-module $V$ are in one-to-one correspondence with bilinear pairings $$\hat{\theta}: V \times V \to F_2$$
such that \( \tilde{\theta}(gx, gy) = \tilde{\theta}(x, y) \) for all \( g \in \pi \), and \( x, y \in V \). The correspondence is given by
\[
\tilde{\theta}(x, y) = \sum_{g \in \pi} \tilde{\theta}(gx, y)g.
\]

**Definition 9.1.** A form \( \theta: V \times V \to F_2\pi \) is Lee even if for any \( x \in V \) with \( \tilde{\theta}(x, x) = \sum_{g \in \pi} a_g g \) then \( a_g = 0 \) for all elements \( g \) of order 2 (that is, \( g^2 = e \) with \( g \neq e \)). A finitely generated \( F_2\pi \)-module \( V \) is Lee even if there is a symmetric non-singular Lee even form on \( V \).

**Remark 9.2.** If \( \pi \) acts freely on a compact manifold \( W \) of dimension \( 2n \), then the intersection pairing
\[
\varphi: H_d(W; F_2) \times H_d(W; F_2) \to F_2\pi
\]
is Lee even.

Lee defined the following 'Grothendieck group'.

**Definition 9.3.** The Grothendieck group \( \tilde{R}_{GL, \text{ev}}(\pi) \) is the abelian group with
- Generators: \( [V] \), isomorphism classes of finitely generated \( F_2\pi \)-modules;
- Relations: (a) if \( 0 \to V_1 \to V_2 \to V_3 \to 0 \) is an exact sequence of finitely generated \( F_2\pi \)-modules, then \( [V_2] - [V_1] - [V_3] = 0 \);
- (b) \( [F_2\pi] = 0 \);
- (c) if \( V \) is Lee even then \( [V] = 0 \).

Note that we are not dealing with projective modules.

**Definition 9.4.** If \( M^{2m+1} \) is a closed manifold with a free \( \pi \)-action, the Lee semicharacteristic is
\[
\chi_4(M; F_2) = \sum_{i=0}^{m} (-1)^i [H_i(M; F_2)] \in \tilde{R}_{GL, \text{ev}}(\pi).
\]

Let \( \mathcal{N}_*(B\pi) \) denote the unoriented bordism group of closed manifolds with free \( \pi \)-actions.

**Theorem 9.5** (Lee). The map \( \chi_4: \mathcal{N}_*(B\pi) \to \tilde{R}_{GL, \text{ev}}(\pi) \) is a well-defined homomorphism.

Bordism theory shows that the composition
\[
\mathcal{N}_*(B\pi) \xrightarrow{\text{res}} \mathcal{N}_*(B\pi_2) \xrightarrow{\text{ind}} \mathcal{N}_*(B\pi)
\]
is injective. From this Lee concludes

**Theorem 9.6.** If \( \pi \) acts freely on a closed manifold \( M \), then
\[
\chi_4(\pi \times \pi, M; F_2) = \chi_4(M; F_2).
\]
The Lee semicharacteristic is a bordism invariant, while the surgery semicharacteristic is a normal bordism invariant. The notion of Lee even is weaker than the notion of even, so that $L_{n+1}^b(F_3\pi/rad)$ tends to be larger than $\bar{R}_{GL,even}(\pi)$. Furthermore, it is not clear how to calculate $\bar{R}_{GL,even}(\pi)$ in general. The basic problem is that the notion of Lee even does not respect exact sequences. So given $0 \to V_1 \to V_2 \to V_3 \to 0$ with two modules Lee even, there is no guarantee that the third is.

Lee's formula in Theorem 9.6 is analogous to the previous formula in Theorem 6.4 involving the surgery semicharacteristic. We now explain that analogy and show that Theorem 6.4 is in fact stronger than Theorem 9.6.

**Proposition 9.7.** There is a natural map $s: L_{2n+1}^b(F_3\pi) \to \bar{R}_{GL,even}(\pi)$. If $(f, \bar{f}): (M^{2n+1}, \nu_M) \to (X, \xi)$ is a surgery problem with data $(\pi, \omega)$, then

$$\text{im} \sigma(f, \bar{f}) = \chi_4(M; F_2) - \chi_4(\bar{R}; F_2).$$

**Proof.** The map is $s(H, \nu; K, L) = [K \cap L] \in \bar{R}_{GL,even}(\pi)$.

To show that $s$ is well defined we need to show that $s$ of a graph formation is zero. Let $\theta: F \to F^*$ be a symmetric even form on a free $F_2\pi$-module. Let

$$\Gamma = \{ (x, \theta(x)) \in F \oplus F^* \mid x \in F \}.$$

The graph formation is the formation $(H_4(F); F, \Gamma)$. Also $F \cap \Gamma = \ker \theta$. The induced form

$$\theta: F/\ker \theta \to (F/\ker \theta)^*$$

is a non-singular symmetric even form. Thus $[\ker \theta] = 0$, and $s$ is well defined.

Lee's arguments (or those of § 3) show that the difference of the semicharacteristics in $\bar{R}_{GL,even}(\pi)$ is a normal bordism invariant. The identification with $\text{im} \sigma(f, \bar{f})$ then proceeds as in Theorem 3.5.

Since $L_{2n+1}^b(F_3\pi) = L_{2n+1}^b(F_3\pi/rad)$, it follows that Theorem 6.4 is weaker than Theorem 9.6. For example, a computation which motivated this paper was that the map

$$s: L_{2n+1}^b(F_2[Q(2^n, p, 1)]) \to \bar{R}_{GL,even}(Q(2^n, p, 1))$$

is zero.

In certain cases $s$ is injective. Pardon discussed these in [18].

The key notion of the proof of bordism invariance of the semicharacteristic is that the intersection pairing on an even-dimensional compact manifold is Lee even. This can be extended.

**Lemma 9.8.** If $(X, \partial X)$ is a $\mathbb{Z}/2$-Poincaré pair with a free cellular $\pi$-action, then the intersection pairing

$$\lambda: H_4(X; F_2) \times H_n(X; F_2) \to F_2\pi$$

is Lee even.

A proof is essentially given in [4] or [23]. The geometric notion needed is the geometric transfer map associated to a double covering.

Let $W_{2n+1}(B)\mathbb{Z}/2$ be the unoriented cobordism group of $\mathbb{Z}/2$-Poincaré complexes with free cellular $\pi$-actions. We now note
Corollary 9.9. The Lee semicharacteristic defines a homomorphism \( \chi_n^L: \mathcal{R}_{2n+1}(Bn) \to \tilde{R}_{\mu,\nu}(n) \).

In [23] it was asserted that the Lee semicharacteristic is an invariant of the symmetric \( L \)-group \( L_{2n+1}(\mathbb{Z}n) \). This is not correct. The intersection pairing need not be Lee even.

References


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