PERIODIC KNOTS, SMITH THEORY,
AND MURASUGI'S CONGRUENCE

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A knot $K$ in a homology 3-sphere $\Sigma$ has period $n$ if it is invariant under a homeomorphism $h: \Sigma \to \Sigma$ of order exactly $n$ with fixed set $B$, a circle disjoint from $K$. The quotient space $\tilde{\Sigma} = \Sigma/h$ is a homology sphere containing $\tilde{K}$, the quotient knot. Kunio Murasugi [Mu] discovered the following congruence involving the Alexander polynomials of the two knots. (See also the proof by J. Hillman [H].)

**Theorem A.** Let $K$ be a knot of prime power period $p^r$ in a homology 3-sphere $\Sigma$ with fixed set $B$ and quotient knot $\tilde{K}$. Let $\Delta_K(t)$ and $\Delta_{\tilde{K}}(t)$ be their Alexander polynomials and let $\lambda$ be the linking number of $K$ and $B$. Then

$$\Delta_K(t) \equiv \Delta_{\tilde{K}}(t)^{p^r}(1 + t + \ldots + t^{\lambda-1})^{p^r-1} \pmod{p},$$

where $\equiv$ means congruent up to multiplication by $ut^i$ where $u$ and $i$ are integers and $u$ is relatively prime to $p$.

In another direction it is easily shown that if $G = \mathbb{Z}/p$ acts cellularly on a finite CW complex $X$, then $\chi(X) + (p-1)\chi(X^G) = p\chi(X/G)$. Using Smith theory, E. Floyd [F] gave a proof of this when $X$ is a finite-dimensional CW complex with $\text{rk} H_*(X; \mathbb{Z}/p) < \infty$. The proof can be generalized easily to the case of semifree actions of a $p$-group $G$ on $X$. (An action is semifree if every point in $X$ is either freely permuted by $G$ or fixed by all of $G$. An action of $\mathbb{Z}/p$ is automatically semifree.) We will prove a multiplicative analogue of Floyd’s theorem and use it to deduce Murasugi’s congruence.

If $X$ is a space with an action of the infinite cyclic group $C_\infty = \langle t \rangle$ and $F$ is a field with $\text{rk} H_*(X; F) < \infty$, we define a multiplicative Euler characteristic

$$\chi_m(X; F) \in F(t)^*/F[t, t^{-1}]^*$$

to be the alternating product of the generator of the order ideals of $H_i(X; F)$. 

(See [Mi] or §1 for definitions). We will be most interested in the case $F = \mathbb{F}_p$, the finite field with $p$ elements.

**Theorem B.** Let $G$ be a $p$-group. Suppose $C_\infty \times G$ act on a finite-dimensional CW complex $X$ with $\text{rk} H_*(X; \mathbb{F}_p) < \infty$, so that $G$ acts semifreely and cellularly. Then

\[
\chi_m(X; \mathbb{F}_p)\chi_m(X^G; \mathbb{F}_p)^{|G|^{-1}} = \chi_m(X/G; \mathbb{F}_p)^{|G|}.
\]

Applying this to the case where $X$ is the infinite cyclic cover of $\Sigma - K$ will immediately yield Murasugi’s congruence. One advantage of our approach is that it generalizes to the case of high-dimensional periodic knots.

In §1 we prove Theorem B and derive Theorem A. In §2 we discuss the high-dimensional case and in §3 give the following application of Murasugi’s congruence to links.

**Proposition C.** Let $L$ be a two-component link in a homology 3-sphere. If the $\mathbb{Z}/2 \times \mathbb{Z}/2$—cover branched over the link is also a homology 3-sphere, then the linking number of the two components is congruent to $\pm 1$ modulo 8.

**§1. Murasugi’s Congruence**

We will derive Theorem A from Theorem B and then prove Theorem B, but we first give some homological preliminaries. If $R$ is a commutative Noetherian UFD with quotient field $K$ and $M$ is a finitely generated torsion $R$-module then we define the order of $M$ to be $[M] = E^0(M) \in R/R^*$. Here we take an exact sequence

\[
R^k \overset{A}{\to} R^m \to M \to 0,
\]

and we let $E^0(M)$ be a greatest common divisor of the determinants of the $m \times m$-submatrices of $A$. If $M$ is a torsion f.g. $R$-module then $[M] \neq 0$, and we consider the order $[M]$ as an element of $K^*/R^*$. If

\[
0 \to M' \to M \to M'' \to 0
\]

is an exact sequence of torsion f.g. $R$-modules, then J. Levine [L, lemma 5] shows $[M] = [M'] [M'']$. It follows for formal reasons that if $C_* = \{C_n \to \ldots \to C_0\}$ is a chain complex of torsion f.g. $R$-modules then
\[ \chi_m(C_\ast) := \prod [C_i](-1)^i \]
equals \( \chi_m(H_\ast(C_\ast)) \). In particular if \( C_\ast \) is exact, then \( \chi_m(C_\ast) = 1 \).

Next we turn to Alexander polynomials. By Alexander duality \( H_1(\Sigma - K) \cong \mathbb{Z} \). Let \( \pi: X \to \Sigma - K \) be the infinite cyclic cover of the knot complement. The infinite cyclic group \( C_\infty = \langle t \rangle \) acts on \( X \) and \( H_1(X; \mathbb{Z}) \) is a free \( \mathbb{Z} \)-module over the group ring \( \mathbb{Z}[C_\infty] = \mathbb{Z}[t, t^{-1}] \). The Alexander polynomial \( \Delta_K(t) \) is its associated order. (Note that \( \mathbb{Z}[t, t^{-1}]^* \) consists of \( \pm t^i \) and the quotient field of \( \mathbb{Z}[t, t^{-1}] \) is the field of rational functions \( \mathbb{Q}(t) \).

As usual we normalize so that \( \Delta_K(t) \) is a polynomial with integer coefficients and non-zero constant term.

If \( K \) has period \( p^r \), let \( \tilde{\pi}: \tilde{X} \to \Sigma - \tilde{K} \) be the infinite cyclic cover of the quotient knot. The \( G = \mathbb{Z}/p^r \)-action on \( \Sigma - K \) lifts to a \( G \)-action on \( X \) with quotient \( \tilde{X} \) and fixed set \( \tilde{B} = \pi^{-1}(B) \). Indeed, let \( g \) be a generator of \( G \). Then \( g \circ \pi: X \to \Sigma - K \) induces the trivial map on \( H_1 \) and so lifts to \( \tilde{g}: X \to X \). Since \( g \) has a non-empty, path-connected fixed-point set there is a unique lift \( \tilde{g} \) with fixed points and the fixed point set is \( \tilde{B} \). Since \( \tilde{g}^p \) is a lift of the identity which has fixed points, it itself is the identity and hence \( \tilde{g} \) is a map of period \( p^r \). This gives an action of \( C_\infty \times G \) on \( X \). It further follows that \( X/G \to \Sigma - \tilde{K} \) is an abelian cover inducing the trivial map on \( H_1 \), so that we can identify this cover with \( \tilde{\pi} \) and \( X/G \) with \( \tilde{X} \).

The cover \( \pi \) is classified by a map \( c: \Sigma - K \to S^1 = K(\mathbb{Z}, 1) \) inducing an isomorphism on \( H_1 \). The inclusion map \( B \to \Sigma - K \) induces multiplication by the linking number \( \lambda \) on \( H_1 \). Thus by considering \( c|_B \) which classifies \( \pi: \tilde{B} \to B \), we see \( \tilde{B} \) is homeomorphic to \( \lambda \) disjoint copies of \( \mathbb{R} \), cyclically permuted by the action of \( C_\infty \).

Now \( H_i(X) \) and \( H_i(\tilde{X}) \) are zero for \( i > 1 \) and \( H_0(X) \) and \( H_0(\tilde{X}) \) are isomorphic to \( F_p \cong F_p[t, t^{-1}]/(t - 1)F_p[t, t^{-1}] \), so \( \chi_m(X) = (t - 1)/\Delta_K(t) \) and \( \chi_m(\tilde{X}) = (t - 1)/\Delta_{\tilde{K}}(t) \). Since \( X^G = \tilde{B} \) consists of \( \lambda \) arcs cyclically permuted by \( C_\infty = \langle t \rangle \), \( \chi(X^G) = t^\lambda - 1 \). Putting this together with Theorem B we see

\[ [(t - 1)/\Delta_K(t)] (t^\lambda - 1)^{p - 1} = [(t - 1)/\Delta_{\tilde{K}}(t)]^{p^r} \]
or \( \Delta_K(t) = \Delta_{\tilde{K}}(t)^{p^r}(1 + t + \ldots + t^{\lambda - 1})^{p^r - 1} \) with the equality taking place in \( F_p(t)/F_p[t, t^{-1}]^* \). This gives Murasugi’s congruence.

**Proof of Theorem B.** We prove the theorem by induction on the order of \( G \). Let \( G \) be a group of prime order \( p \) with generator \( g \). Let
\[ \sigma = 1 + g + g^2 + \ldots + g^{p-1} \]
\[ \delta = 1 - g \]
be elements of the group ring \( F_p[G] \). Note that \( \delta \sigma = 0 = \sigma \delta \) and \( \delta^{p-1} = \sigma \).

We consider the following chain complexes of \( F_p[t, t^{-1}] \)-modules (all homology is with \( F_p \)-coefficients).

\[
\begin{align*}
0 & \to C_* (X^G) \to C_* (X) \xrightarrow{\text{tr}} \sigma C_* (X) \to 0 \\
0 & \to \delta C_* (X) \oplus C_* (X^G) \to C_* (X) \xrightarrow{\sigma} \sigma C_* (X) \to 0 \\
0 & \to \sigma C_* (X) \to \delta C_* (X) \xrightarrow{\delta} \delta^2 C_* (X) \to 0 \\
& \quad \vdots \\
0 & \to \sigma C_* (X) \to \delta^{p-2} C_* (X) \xrightarrow{\delta} \delta^{p-1} C_* (X) \to 0.
\end{align*}
\]

These induce long exact sequences in homology. All homology is finitely generated and torsion over the PID \( F_p[t, t^{-1}] \). We use shorthand notation – if \( p \in F_p[G] \), we write \( \chi^p (X) \) instead of \( \chi (H_* (p C_* (X))) \). The above homological considerations show

\[
\begin{align*}
\chi (\overline{X}) & = \chi (X^G) \chi^o (X) \\
\chi (X) & = \chi^\delta (X) \chi (X^G) \chi^o (X) \\
\chi^\delta (X) & = \chi^o (X) \chi^{\delta^2} (X) \\
& \quad \vdots \\
\chi^{\delta^{p-2}} (X) & = \chi^o (X) \chi^o (X).
\end{align*}
\]

Multiplying all equations but the first together and cancelling terms we see

\[
\chi (X) = \chi (X^G) \cdot \chi^o (X)^p.
\]

Using the first equation to substitute for \( \chi^o (X) \) one finds

\[
\chi (X) = \chi (\overline{X})^p / \chi (X^G)^{p-1}.
\]

Finally suppose \( G \) has order \( p^r \). Let \( G_1 \) be a normal subgroup of index \( p \). By the exact sequences above \( \text{rk} H_* (X/G_1; F_p) < \infty \). By applying inductively the result for the \( G_1 \)-action on \( X \) and the \( G/G_1 \) action on \( X/G_1 \), Theorem B follows.

\section{§2. High-dimensional periodic knots}

One advantage of our approach to Murasugi's congruence is that it applies equally well to a more general situation. Higher-dimensional periodic knots
were introduced in the thesis of R. Cruz [C]. He showed that if there is a semifree \( \mathbb{Z}/q \)-action on \( S^n \) with non-empty fixed set and an invariant knot \( K^{n-2} \) disjoint from the fixed set, then the fixed set is \( S^1 \) if \( q \neq 2 \), and is \( S^1 \) or \( S^0 \) if \( q = 2 \).

For our purposes a knot \( K \) in a homology \( n\)-sphere \( \Sigma \) is an embedded \((n-2)\)-dimensional homology sphere. Let \( G \) be a finite group. The knot \( K \) is \( G \)-periodic if it is invariant under a semifree \( G \)-action on \( \Sigma \) with fixed set \( B \equiv S^1 \) disjoint from \( K \). To simplify technicalities we assume the action is smooth. Several complications arise: the group need not be cyclic, the action need not be linear and the quotient \( \tilde{\Sigma} = \Sigma/G \) will not be a manifold. (Even in the linear case the quotient looks like a double suspension of a spherical space form.) However we can still make sense of Alexander polynomials.

**Proposition 2.1.** \( H_*(\tilde{\Sigma} - \bar{K}) \cong H_*(S^1) \).

First we need a lemma.

**Lemma 2.2.** The linking number \( \lambda = \text{lk}(B, K) \) is relatively prime to the order of \( G \).

**Proof.** (See also [C, 2.1.1]). By restricting the action to a subgroup \( \mathbb{Z}/p \) of \( G \), we will assume \( G = \mathbb{Z}/p \), and show \( (\lambda, p) = 1 \). By applying the Lefschetz Fixed-Point Theorem to a generator \( g \) of \( \mathbb{Z}/p \), we see that if \( n \) is odd, the action on \( K \) is orientation-preserving, while if \( n \) is even, then \( p = 2 \) and the action is orientation-reversing. For local coefficients we will use \( \mathbb{Z}' \), the integers with the \( \mathbb{Z}/p \)-module structure given by \( (\Sigma a_i g^i) \cdot k = \Sigma a_i (-1)^{(n+1)k} \).

Let \( \tilde{\Sigma} - B \to K(\mathbb{Z}/p, 1) \) classify the \( G \)-cover. We will consider the commutative diagram:

\[
\begin{array}{cccc}
H_{n-2}(K; \mathbb{Z}) & \xrightarrow{\alpha} & H_{n-2}(\bar{K}; \mathbb{Z}') & \rightarrow & H_{n-2}(K(\mathbb{Z}/p, 1); \mathbb{Z}') \\
\downarrow & & \downarrow & & \parallel \\
H_{n-2}(\Sigma - B; \mathbb{Z}) & \rightarrow & H_{n-2}(\tilde{\Sigma} - B; \mathbb{Z}') & \rightarrow & H_{n-2}(K(\mathbb{Z}/p, 1); \mathbb{Z}')
\end{array}
\]

(*)

The two groups on the left are infinite cyclic and the left vertical map is multiplication by \( \lambda \). A diagram chase shows we will be done if we can show both horizontal exact sequences are isomorphic to the short exact sequence \( 0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/p \to 0 \).

The map \( \alpha \) is isomorphic to \( \mathbb{Z}^{\times p} \to \mathbb{Z} \) because it comes from a \( p \)-fold cover of \((n - 2)\)-dimensional closed manifolds. The map

\[
H_{n-2}(\bar{K}; \mathbb{Z}') \to H_{n-2}(\mathbb{Z}/p; \mathbb{Z}')
\]
we compute algebraically by using a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$ as a substitute for the Eilenberg-MacLane space. By lifting a CW structure on $\bar{K}$ to $K$,

$$C_\ast(K) = \{C_{n-2} \to ... \to C_0\}$$

with the $i$-chains $C_i$ free $\mathbb{Z}G$-modules. By mapping a free $\mathbb{Z}G$-module onto $\ker(C_{n-2} \to C_{n-3})$ and continuing inductively, one constructs a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$

$$D_\ast = \{... \to D_n \to D_{n-1} \to C_{n-2} \to ... \to C_0\}.$$  

It follows that

$$H_{n-2}(\bar{K}; \mathbb{Z}^i) = H_{n-2}(C_\ast(K) \otimes \mathbb{Z}G \mathbb{Z}^i)$$

maps onto $H_{n-2}(D_\ast \otimes \mathbb{Z}G \mathbb{Z}^i) = H_{n-2}(\mathbb{Z}/p; \mathbb{Z}^i)$. Furthermore by using the standard $\mathbb{Z}G$-resolution of $\mathbb{Z}$ (see e.g. [Mac]), one easily computes that $H_{n-2}(\mathbb{Z}/p; \mathbb{Z}^i) \cong \mathbb{Z}/p$.

Choose a $G$-invariant normal disk to $B$ in $\Sigma$ and let $S^{n-2}$ be its boundary. Then the inclusion $S^{n-2} \to \Sigma - B$ is a homology equivalence. By the comparison theorem applied to the spectral sequence of the $G$-coverings (see [Mac]), the bottom row of (*) is isomorphic to

$$H_{n-2}(S^{n-2}; \mathbb{Z}) \to H_{n-2}(S^{n-2}/G; \mathbb{Z}^i) \to H_{n-2}(G; \mathbb{Z}^i),$$

and hence by the previous paragraph to $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/p \to 0$. Thus $(\lambda, p) = 1$.

**Proof of 2.1.** Let $N$ be an equivariant tubular neighborhood of $B$. Then

$$0 = H_*((\Sigma - K, N; \mathbb{Z}[1/\lambda]) = H_*((\Sigma - K - B, N - B; \mathbb{Z}[1/\lambda])$$

where the first equality holds by the definition of linking number and the second by excision. Then

$$0 = H_*((\Sigma - K - B)/G, (N - B)/G; \mathbb{Z}[1/\lambda]) = H_*((\Sigma - K)/G, N/G; \mathbb{Z}[1/\lambda])$$

$$= H_*((\Sigma - K)/G, B; \mathbb{Z}[1/\lambda]),$$

where the first equality follows from the spectral sequence of a covering, the second by excision and the third by the homotopy equivalence $B \to N/G$. Thus $H_*(\bar{\Sigma} - \bar{K})$ looks like $H_*(S^1)$ except possibly for some $\lambda$-torsion. But by 2.1, $\lambda$ is prime to the order of $G$, so for all primes $q$ dividing $\lambda$, the transfer map $tr: H_*(\bar{\Sigma} - \bar{K}; \mathbb{Z}/q) \to H_*(\Sigma - K; \mathbb{Z}/q)$ is injective so there is no extra $\lambda$-torsion.

To state Murasugi's congruence in higher dimensions is it necessary to find a substitute for the Alexander polynomial. Let $X$ and $\bar{X}$ be the infinite cyclic
covers of \( \Sigma - K \) and \( \tilde{\Sigma} - \tilde{K} \) respectively. Let \( \Delta_{K}(t) = \prod_{i>0}[H_{i}(X)]^{(-1)^{i+1}} \) and \( \Delta_{\tilde{K}}(t) = \prod_{i>0}[H_{i}(\tilde{X})]^{(-1)^{i+1}} \). The Wang sequence shows that multiplication by \( t - 1 \) induces an isomorphism on \( H_{i}(X) \) for \( i > 0 \), so that if we take the polynomial represented by \( [H_{i}(X)] \) and plug in \( t = 1 \) we get \( \pm 1 \). (Indeed if we consider the ring homomorphism \( \varphi : \mathbb{Z}[t, t^{-1}] \to \mathbb{Z} \) defined by \( \varphi(t) = 1 \), then \( \varphi([H_{i}(X)]) \) is a divisor of \( [H_{i}(X) \otimes \mathbb{Z}, t^{-1}] \mathbb{Z} = [0] = 1 \in \mathbb{Z}/\mathbb{Z}^{*} \).) Thus \( [H_{i}(X)] \) represented a non-zero element in \( \mathbb{F}_{p}[t, t^{-1}] \), and hence \( \Delta_{K}(t) \) and \( \Delta_{\tilde{K}}(t) \) give well-defined elements of \( \mathbb{F}_{p}(t)^{*}/\mathbb{F}_{p}[t, t^{-1}]^{*} \). Then the considerations of §1 show:

**Theorem 2.3.** Let \( K \) be a \( G \)-periodic knot in a homology \( q \)-sphere \( \Sigma \) with fixed set \( B \), where \( G \) is a group of prime power order \( p^{r} \). Let \( \lambda \) be the linking number of \( K \) and \( B \). Then

\[
\Delta_{K}(t) \equiv \Delta_{\tilde{K}}(t)^{p^{r}}(1 + t + \ldots + t^{\lambda-1})^{p^{r}-1} \pmod{p}.
\]

**§3. An Application of Murasugi’s Congruence**

For any \( \lambda \equiv \pm 1 \pmod{8} \), T. tom Dieck and J. Davis [D-D] constructed a 2-component link with linking number \( \lambda \) in a homology 3-sphere \( \Omega \) whose \( C_{2} \times C_{2} \)-cover branched over the link is a homology 3-sphere \( \Sigma \). We will show that this congruence condition is necessary. Equivalently, we show

**Theorem 3.1.** Suppose the Klein 4-group \( G \times H \cong C_{2} \times C_{2} \) acts on a homology 3-sphere \( \Sigma \) so that the fixed sets \( \Sigma^{G} \) and \( \Sigma^{H} \) are disjoint circles. Then their linking number \( \lambda \) is congruent to \( \pm 1 \pmod{8} \).

**Proof.** We have

\[
\begin{align*}
\Sigma & \to \Sigma/G \\
\downarrow & \downarrow \\
\Sigma/H & \to \Sigma/(G \times H)
\end{align*}
\]

All four of these manifolds are homology 3-spheres and each has two disjoint circles given by the images of the fixed sets. The linking numbers of each pair of circles are all equal.

Let \( K = \Sigma^{G}/G \subset \Sigma/G \) and \( \tilde{K} = K/H \subset \Sigma/(G \times H) \). Then \( K \) is a knot of period 2. Renormalize \( \Delta_{K}(t) \) and \( \Delta_{\tilde{K}}(t) \in \mathbb{Z}[t, t^{-1}] \) so that \( \Delta_{K}(t) = \Delta_{K}(t^{-1}) \), \( \Delta_{\tilde{K}}(t) = \Delta_{\tilde{K}}(t^{-1}) \), and \( \Delta_{K}(1) = 1 = \Delta_{\tilde{K}}(1) \). Murasugi’s congruence shows
\[ \Delta_K(t) = \Delta_K(t)^2(t^{(1-\lambda)/2} + \ldots + 1 + \ldots + t^{(\lambda-1)/2}) + 2f(t), \]

where \( f(t) \in \mathbb{Z}[t, t^{-1}] \) satisfies \( f(t) = f(t^{-1}) \). Writing

\[ f(t) = a_n t^{-n} + \ldots + a_0 + \ldots + a_n t^n, \]

we see \( f(1) \equiv f(-1) \pmod{4} \). Since \( \Sigma \to \Sigma/G \) is a 2-fold cover branched over \( K \), \( |\Delta_K(-1)| = |H_1(\Sigma)| = 1 \). So \( 1 = \Delta_K(1) \equiv \Delta_K(-1) \pmod{4} \), and we see \( \Delta_K(-1) = 1 \). Take equation (**) and plug in \( t = 1 \) and \( t = -1 \):

\[
1 = 1 \cdot \lambda + 2 \cdot f(1)
\]

\[
1 = 1 \cdot (-1)^{(\lambda-1)/2} + 2 \cdot f(-1).
\]

Thus \( \lambda \equiv (-1)^{(\lambda-1)/2} \pmod{8} \) so \( \lambda \equiv \pm 1 \pmod{8} \).

Applying the high-dimensional version of Murasugi’s congruence ones sees that if \( G \times H \cong C_2 \times C_2 \) acts on a homology \( q \)-sphere \( \Sigma \) so that \( \Sigma^G \) is a homology \( q - 2 \) sphere and \( \Sigma^H \) is a circle disjoint from \( \Sigma^G \), then their linking number \( \lambda \) is congruent to \( \pm 1 \) modulo 8. This and considerations from \( L \)-theory lead us to conjecture that if \( G \times H \cong C_2 \times C_2 \) acts on a homology \( q \)-sphere \( \Sigma \) so that \( \Sigma^G \) is a homology \( k \)-sphere and \( \Sigma^H \) is a homology \( q - k - 1 \)-sphere disjoint from \( \Sigma^G \), then their linking number \( \lambda \) is congruent to \( \pm 1 \) modulo 8.

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